1. (a) $3x^2y + xy^2 - 18 = 0$

$$\left(6xy + 3x^2\frac{dy}{dx}\right) + \left(y^2 + 2xy\frac{dy}{dx}\right) = 0 \qquad \text{1M}$$

$$\frac{dy}{dx} = \frac{-6xy - y^2}{3x^2 + 2xy} \qquad \text{1A}$$

(b) Let $(a,\ b)$ be the coordinates of the point of contact.

$$\frac{-6ab - b^2}{3a^2 + 2ab} = -3 \qquad \text{1M}$$

$$-6ab - b^2 = -9a^2 - 6ab$$

$$b = \pm 3a \qquad \text{1M}$$

When $b = -3a$, $3x^2y + xy^2 - 18 = 3a^2(-3a) + a(-3a)^2 - 18 = -18 \neq 0$.

When $b = 3a$,

$$3a^2(3a) + a(3a)^2 - 18 = 0 \qquad \text{1M}$$

$$a^3 = 1$$

$$a = 1$$

The tangent to C at $(1,\ 3)$ is parallel to the straight line $3x + y + 1 = 0$.

There is only one required tangent.

The claim is disagreed. 1A

2. (a) $$f'(x) = \frac{2(x-5) - (2x-4)(1)}{(x-5)^2} \qquad \text{1M}$$

$$= \frac{-6}{(x-5)^2} \qquad \text{1A}$$

(b) The coordinates of A are $(2,\ 0)$. 1A

Slope of tangent $= \dfrac{-6}{(2-5)^2} = -\dfrac{2}{3}$ 1M+1A

Required equation is

$$y - 0 = -\frac{2}{3}(x - 2) \qquad \text{1M}$$

$$2x + 3y - 4 = 0 \qquad \text{1A}$$

3. (a) $$y = \frac{25 - 4x^2}{5 + x^2}$$

$$= -4 + \frac{45}{5 + x^2}$$

$$\frac{dy}{dx} = -\frac{45(2x)}{(5 + x^2)^2} \qquad \text{1M}$$

$$= -\frac{90x}{(5 + x^2)^2} \qquad \text{1A}$$

(b) Let $(a,\ b)$ be the point of contact.

$$\frac{b+4}{a+4} = \frac{-90a}{(5+a^2)^2} \qquad \text{1M}$$

$$\frac{\left(-4+\frac{45}{5+a^2}\right)+4}{a+4} = \frac{-90a}{(5+a^2)^2}$$

$$45(5+a^2) = -90a(a+4)$$

$$135a^2+360a+225=0$$

$$a=-1 \quad \text{or} \quad -\frac{5}{3} \qquad \text{1A}$$

When $a=-1$, $b=\frac{7}{2}$ and $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{5}{2}$.
The equation of tangent is

$$y-\frac{7}{2}=\frac{5}{2}(x+1)$$

$$5x-2y+12=0 \qquad \text{1A}$$

When $a=-\frac{5}{3}$, $b=\frac{25}{14}$ and $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{243}{98}$.
The equation of tangent is

$$y-\frac{25}{14}=\frac{243}{98}\left(x+\frac{5}{3}\right)$$

$$243x-98y+580=0 \qquad \text{1A}$$

4. (a) $10\ln(0-ek)=0^2+10$

$$\ln(-ek)=1$$

$$-ek=e$$

$$k=-1 \qquad \text{1A}$$

(b) $10\ln(x-y)=x^2+10$

$$\frac{10}{x-y}\left(1-\frac{\mathrm{d}y}{\mathrm{d}x}\right)=2x \qquad \text{1M+1A}$$

$$5-5\frac{\mathrm{d}y}{\mathrm{d}x}=x^2-xy$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-x^2+xy+5}{5} \qquad \text{1A}$$

At $A\ (0,\ -e)$, $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{0+5}{5}=1$. 1M
Required equation is

$$y+e=1(x-0)$$

$$x-y-e=0 \qquad \text{1A}$$

5. $\frac{\mathrm{d}}{\mathrm{d}x}\left(4x + y^2\right) = \frac{\mathrm{d}}{\mathrm{d}x}(16)$

$4 + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ 1M

$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2}{y}$

Let the coordinates of the point of contact of L and the curve be $(a,\ b)$.

$\frac{b - 0}{a - 8} = -\frac{2}{b}$ 1M

$b^2 = 16 - 2a$

$(16 - 4a) = 16 - 2a$ 1M

$a = 0$

The coordinates of the point of contact are $(0,\ 4)$ and $(0,\ -4)$. 1A

Equation of tangent to the curve at $(0,\ 4)$ is

$y - 4 = -\frac{2}{4}(x - 0)$

$y = -\frac{x}{2} + 4$ 1A

Equation of tangent to the curve at $(0,\ -4)$ is

$y + 4 = -\frac{2}{(-4)}(x - 0)$

$y = \frac{x}{2} - 4$ 1A

The equation of L is $y = -\frac{x}{2} + 4$ or $y = \frac{x}{2} - 4$.

6. $3x^2 - xy - y^2 = 9$

$6x - \left(y + x\frac{\mathrm{d}y}{\mathrm{d}x}\right) - 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ 1M

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{6x - y}{x + 2y}$

Let $(a,\ b)$ be the point of contact.

$\frac{6a - b}{a + 2b} = \frac{5}{3}$ 1M+1A

$18a - 3b = 5a + 10b$

$13a = 13b$

$a = b$

Put $a = b$ into $3a^2 - ab - b^2 = 9$.

$3a^2 - a^2 - a^2 = 9$ 1M

$a^2 = 9$

$a = \pm 3$

The points of contact are $(3,\ 3)$ and $(-3,\ -3)$.

Required equations are

$$y - 3 = \frac{5}{3}(x - 3) \quad \text{and} \quad y + 3 = \frac{5}{3}(x + 3) \qquad \text{1M}$$

$$5x - 3y - 6 = 9 \qquad 5x - 3y + 6 = 0 \qquad \text{1A}$$

7. $$\frac{dy}{dx} = \frac{1}{2}(x^2 - 16)^{-\frac{1}{2}}(2x) \qquad \text{1M}$$

$$= \frac{x}{\sqrt{x^2 - 16}}$$

Let (a, b) be the point of contact.

$$\frac{b + 2}{a - 2} = \frac{a}{\sqrt{a^2 - 16}} \qquad \text{1M}$$

$$\frac{\sqrt{a^2 - 16} + 2}{a - 2} = \frac{a}{\sqrt{a^2 - 16}}$$

$$(a^2 - 16) + 2\sqrt{a^2 - 16} = a^2 - 2a$$

$$\sqrt{a^2 - 16} = 8 - a$$

$$a^2 - 16 = a^2 - 16a + 64$$

$$16a - 80 = 0$$

$$a = 5 \qquad \text{1A}$$

Required equation is

$$y + 2 = \frac{5}{\sqrt{5^2 - 16}}(x - 2) \qquad \text{1M}$$

$$5x - 3y - 16 = 0 \qquad \text{1A}$$

8. (a) $$x^3 y - 3x + 1 = 0$$

$$\left(3x^2 y + x^3 \frac{dy}{dx}\right) - 3 = 0 \qquad \text{1M}$$

$$\frac{dy}{dx} = \frac{3 - 3x^2 y}{x^3} \qquad \text{1A}$$

(b) Let $(a,\ b)$ be the coordinates of the point of contact.

$$\frac{b-7}{a+\frac{2}{3}} = \frac{3-3a^2b}{a^3} \qquad \text{1M}$$

$$a^3b - 7a^3 = 3a - 3a^3b + 2 - 2a^2b$$

$$4a^3b - 7a^3 = 3a + 2 - 2a^2b$$

$$4(3a-1) - 7a^3 = 3a + 2 - 2\left(\frac{3a-1}{a}\right)$$

$$12a^2 - 4a - 7a^4 = 3a^2 + 2a - 2(3a-1)$$

$$-7a^4 + 9a^2 - 2 = 0$$

$$a^2 = 1 \quad \text{or} \quad \frac{2}{7}$$

$$a = \pm 1 \quad \text{or} \quad \pm\sqrt{\frac{2}{7}} \qquad \text{1M}$$

We have $(a,\ b) = (1,\ 2)$ or $(-1,\ 4)$ or $\left(\sqrt{\frac{2}{7}},\ \frac{21}{2} - \frac{7\sqrt{14}}{4}\right)$ or $\left(-\sqrt{\frac{2}{7}},\ \frac{21}{2} + \frac{7\sqrt{14}}{4}\right)$. 1M

Note that the slopes of tangents at four points are different.

There are four distinct tangents to C which passes through the point $\left(-\frac{2}{3},\ 7\right)$.

The claim is disagreed. 1A

Remarks:

Candidates are expected to mention that the tangents at four points of contact are distinct before arriving at the conclusion.

It may happen that the tangent to the curve at certain point touches the curve again at another point.

9. $x^2 = y^2 - 25$

$$2x = 2y\frac{dy}{dx} \qquad \text{1M}$$

$$\frac{dy}{dx} = \frac{x}{y}$$

Let $P\ (a,\ b)$ be the point of contact.

$$\frac{a}{b} = \frac{b-1}{a+1} \qquad \text{1M}$$

$$a^2 + a = b^2 - b$$

$$(b^2 - 25) + a = b^2 - b$$

$$a = 25 - b \qquad \text{1M}$$

The point $(25-b,\ b)$ lies on C.

$$(25-b)^2 = b^2 - 25$$

$$b^2 - 50b + 625 = b^2 - 25$$

$$b = 13$$

When $b = 13$, $a = 12$ and $\frac{dy}{dx} = \frac{12}{13}$. 1M

Required equation is

$$y - 1 = \frac{12}{13}(x + 1)$$

$12x - 13y + 25 = 0$ 1A

10. (a) $\frac{dy}{dx} = -e^{-3x^{-2}} - xe^{-3x^{-2}}(6x^{-3})$ 1M

$= -e^{-3x^{-2}}(6x^{-2} + 1)$ 1A

$\frac{d^2y}{dx^2} = e^{-3x^{-2}}(6x^{-3})(6x^{-2} + 1) - e^{-3x^{-2}}(-12x^{-3})$

$= 6x^{-3}e^{-3x^{-2}}(1 - 6x^{-2})$ 1A

(b) $\frac{d^2y}{dx^2} = 0$

$6x^{-3}e^{-3x^{-2}}(1 - 6x^{-2}) = 0$

$1 - \frac{6}{x^2} = 0$

$x^2 = 6$

$x = \pm\sqrt{6}$ 1M

x	$x < -\sqrt{6}$	$-\sqrt{6} < x < 0$	$0 < x < \sqrt{6}$	$x > \sqrt{6}$
$\frac{d^2y}{dx^2}$	$-$	$+$	$-$	$+$

1M

The graph of $y = -xe^{-\frac{3}{x^2}}$ has two points of inflexion.
The claim is disagreed. 1A

11. (a) Vertical asymptote is $x = 1$. 1A

$r(x) = 2x - 1 + \frac{x - 2}{(x - 1)^2}$ 1M

Oblique asymptote is $y = 2x - 1$. 1A

(b) $\frac{d}{dx}r(x) = 2 + \frac{(x - 1)^2 - 2(x - 2)(x - 1)}{(x - 1)^4}$ 1M

$= 2 + \frac{-x + 3}{(x - 1)^3}$ 1A

(c) $r''(x) = \frac{-(x - 1)^3 - 3(-x + 3)(x - 1)^2}{(x - 1)^6}$

$= \frac{2x - 8}{(x - 1)^4}$

$= \frac{2(x - 4)}{(x - 1)^4}$

When $r''(x) = 0$, $x = 4$.

x	$x < 1$	$1 < x < 4$	$x > 4$
$r''(x)$	$-$	$-$	$+$

1M

There is only one point of inflexion at $x = 4$.

The claim is agreed. 1A

12. (a) Vertical asymptote is $x = 2$. 1A

$$y = \frac{2x^2 - 3x + 4}{x - 2}$$

$$= 2x + 1 + \frac{6}{x - 2}$$ 1M

Oblique asymptote is $y = 2x + 1$. 1A

(b) C cuts the y-axis at $(0,\ -2)$.

$$\frac{dy}{dx} = 2 - \frac{6}{(x-2)^2}$$ 1M

$$\left.\frac{dy}{dx}\right|_{x=0} = 2 - \frac{6}{(0-2)^2}$$ 1M

$$= \frac{1}{2}$$

Required equation is

$$y + 2 = \frac{1}{2}(x - 0)$$

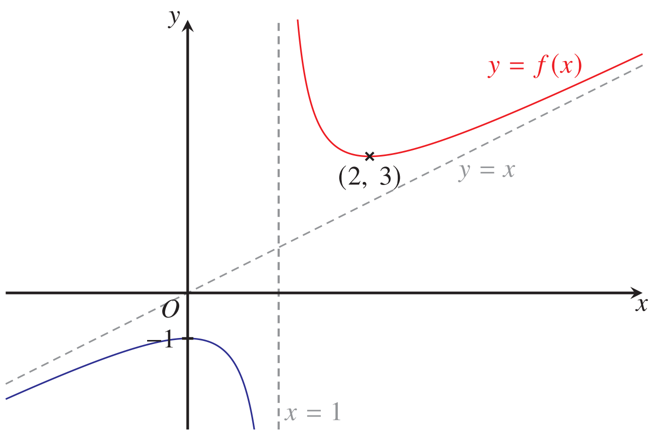
$$x - 2y - 4 = 0$$ 1A

13. (a) Maximum point is $(0,\ -1)$. 1A

Minimum point is $(2,\ 3)$. 1A

(b) (Shape of the graph) 1A

(All correct) 1A



14. (a) $\frac{dy}{dx} = \frac{-\sin(\ln x)}{x}$ 1A

$$\frac{d^2y}{dx^2} = \frac{-x\cos(\ln x)\left(\frac{1}{x}\right) + \sin(\ln x)}{x^2}$$ 1M

$$= \frac{\sin(\ln x) - \cos(\ln x)}{x^2}$$ 1A

(b) When $f''(x) = 0$, $\sin(\ln x) - \cos(\ln x) = 0$.

$$\sin(\ln x) = \cos(\ln x)$$

$$\tan(\ln x) = 0$$

$$\ln x = -\frac{3\pi}{4} \quad \text{or} \quad \frac{\pi}{4}$$

$$x = e^{-\frac{3\pi}{4}} \quad \text{or} \quad e^{\frac{\pi}{4}}$$ 1M

x	$e^{-\pi} < x < e^{-\frac{3\pi}{4}}$	$e^{-\frac{3\pi}{4}} < x < e^{\frac{\pi}{4}}$	$e^{\frac{\pi}{4}} < x < e^{\pi}$
$\frac{d^2y}{dx^2}$	+	−	+

1M

There are two points of inflexion at $x = e^{-\frac{3\pi}{4}}$ and $x = e^{\frac{\pi}{4}}$ respectively.

The claim is agreed. 1A

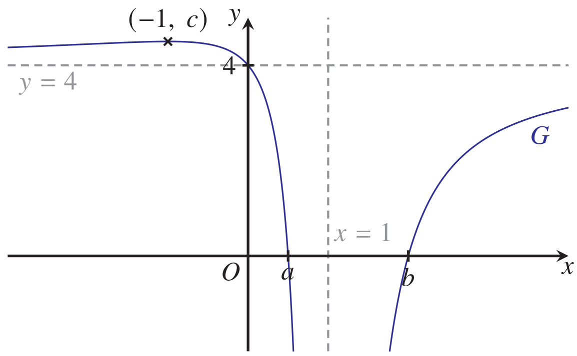
15. (a) Vertical asymptote is $x = 1$. 1A

Horizontal asymptote is $y = 4$. 1A

(b) (Correct shape) 1A

(Maximum point, intercepts) 1A

(All correct) 1A



(c) We have $f(x) \geq 0$.

Thus, $x \leq a$ or $x \geq b$. 1A

From the graph above, we have $f(x) \leq c$.

$$y^2 \leq c$$

$$-\sqrt{c} \leq y\sqrt{c}$$ 1A

16. (a) Vertical asymptotes are $x = 1$ and $x = 4$. 1A

Horizontal asymptote is $y = 0$. 1A

(b) (i) $f'(x) = \dfrac{A(x^2 - 5x + 4) - Ax(2x - 5)}{(x^2 - 5x + 4)^2}$ 1M

$= \dfrac{-Ax^2 + 4A}{(x^2 - 5x + 4)^2}$

The graph of $y = f'(x)$ passes through $\left(3, -\dfrac{5}{2}\right)$.

$-\dfrac{5}{2} = \dfrac{-9A + 4A}{(9 - 15 + 4)^2}$

$A = 2$

Thus, $f'(x) = \dfrac{-2x^2 + 8}{(x^2 - 5x + 4)^2}$. 1A

(ii) $f'(x) = \dfrac{-2(x + 2)(x - 2)}{(x - 1)^2(x - 4)^2}$.

When $f'(x) = 0$, $x = -2$ or 2. 1M

x	$x < -2$	$-2 < x < 1$	$1 < x < 2$	$2 < x < 4$	$x > 4$
$f'(x)$	$-$	$+$	$+$	$-$	$-$

1M

The maximum point is $(2, -2)$. 1A

17. (a) $f'(x) = \dfrac{(x - 4)(2x) - x^2(1)}{(x - 4)^2}$ 1M

$= \dfrac{x^2 - 8x}{(x - 4)^2}$ 1A

(b) $\dfrac{x^2 - 8x}{(x - 4)^2} = 0$

$x = 0 \quad \text{or} \quad 8$ 1A

x	$x < 0$	$0 < x < 4$	$4 < x < 8$	$x > 8$
$f'(x)$	$+$	$-$	$-$	$+$

1M

$f(0) = 3$ and $f(8) = 19$

The local maximum is 3 and the local minimum is 19. 1A

(c) Vertical asymptote is $x = 4$. 1A

$f(x) = 3 + \dfrac{x^2}{x - 4}$

$= x + 7 + \dfrac{16}{x - 4}$

Oblique asymptote is $y = x + 7$. 1A

The claim is disagreed. 1A

18. (a) $f(x) = x^3 + 3x^2 - 4$

$f'(x) = 3x^2 + 6x$ 1M

$= 3x(x + 2)$

$f''(x) = 6x + 6$

$= 6(x + 1)$

When $f'(x) = 0$, $x = 0$ or -2.

x	$x < -2$	$-2 < x < 0$	$x > 0$
$f'(x)$	+	−	+

1M

The maximum point is $(-2,\ 0)$. 1A

The minimum point is $(0,\ -4)$. 1A

When $f''(x) = 0$, $x = -1$.

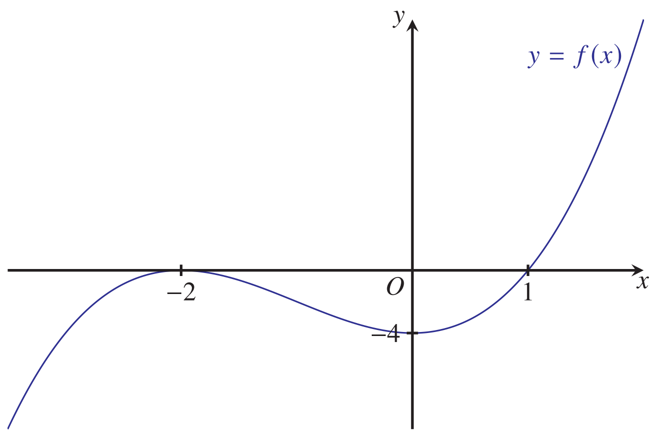
x	$x < -1$	$x > -1$
$f''(x)$	−	+

1M

The point of inflexion is $(-1,\ -2)$. 1A

(b) (Correct shape) 1A

(All correct) 1A



19. (a) $f'(x) = \dfrac{(-2x-3)(x-1)-(-x^2-3x)(1)}{(x-1)^2}$ 1M

$= \dfrac{-(x-3)(x+1)}{(x-1)^2}$

When $f'(x) = 0$, $x = -1$ or 3.

x	$x < -1$	$-1 < x < 1$	$1 < x < 3$	$x > 3$
$f'(x)$	−	+	+	−

1M

The maximum point is $(3,\ -9)$. 1A

(b) The vertical asymptote is $x = 1$. 1A

$f(x) = -x - 4 - \dfrac{4}{x-1}$ 1M

The oblique asymptote is $y = -x - 4$. 1A

20. (a) $\frac{dy}{dx} = (4 - 6x)e^{2x} + (4x - 3x^2)(2e^{2x})$ 1M

$= (4 + 2x - 6x^2)e^{2x}$ 1A

$\frac{d^2y}{dx^2} = (2 - 12x)e^{2x} + (4 + 2x - 6x^2)(2e^{2x})$

$= (10 - 8x - 12x^2)e^{2x}$ 1A

(b) When $\frac{d^2y}{dx^2} = 0$,

$-12x^2 - 8x + 10 = 0$

$6x^2 + 4x - 5 = 0$

$x = \frac{-4 \pm \sqrt{4^2 - 4(6)(-5)}}{2(6)}$ 1M

$= \frac{-2 \pm \sqrt{34}}{6}$

x	$x < \frac{-2 - \sqrt{34}}{6}$	$\frac{-2 - \sqrt{34}}{6} < x < \frac{-2 + \sqrt{34}}{6}$	$x > \frac{-2 + \sqrt{34}}{6}$
$\frac{d^2y}{dx^2}$	$-$	$+$	$-$

1M

There are two points of inflexion at $x = \frac{-2 + \sqrt{34}}{6}$ and $x = \frac{-2 - \sqrt{34}}{6}$ respectively.

The claim is agreed. 1A

21. (a) Vertical asymptote is $x = -2$. 1A

$f(x) = x - 4 + \frac{9}{x + 2}$ 1M

Oblique asymptote is $y = x - 4$. 1A

(b) $\frac{dy}{dx} = 1 - \frac{9}{(x + 2)^2}$ 1M

Put $f'(x) = -8$.

$1 - \frac{9}{(x + 2)^2} = -8$

$(x + 2)^2 = 1$

$x = -1 \quad \text{or} \quad -3$

The points of contact are $(-1,\ 4)$ and $(-3,\ -16)$.

At the point $(-1,\ 4)$, the equation of tangent is

$y - 4 = -8(x + 1)$

$8x + y + 4 = 0$ 1A

At the point $(-3,\ -16)$, the equation of tangent is

$$y + 16 = -8(x + 3)$$

$$8x + y + 40 = 0 \qquad \text{1A}$$

22. (a)

$$f(-4) = -5$$

$$\frac{(-4)^2 + a(-4) + b}{-4 + 2} = -5$$

$$-4a + b = -6$$

$$f'(x) = \frac{(2x + a)(x + 2) - (x^2 + ax + b)(1)}{(x + 2)^2} \qquad \text{1M}$$

$$= \frac{x^2 + 4x + 2a - b}{(x + 2)^2} \qquad \text{1A}$$

$$f'(-4) = 0$$

$$\frac{(-4)^2 + 4(-4) + 2a - b}{(-4 + 2)^2} = 0 \qquad \text{1M}$$

$$2a - b = 0$$

Solving, we have $a = 3$ and $b = 6$. 1A

(b) Vertical asymptote is $x = -2$. 1A

$$f(x) = x + 1 + \frac{4}{x + 2} \qquad \text{1M}$$

Oblique asymptote is $y = x + 1$. 1A

(c) $f'(x) = \dfrac{x^2 + 4x}{(x + 2)^2} = \dfrac{x(x + 4)}{(x + 2)^2}$

When $f'(x) = 0$, $x = 0$ or -4. 1M

x	$x < -4$	$-4 < x < -2$	$-2 < x < 0$	$x > 0$
$f'(x)$	+	−	−	+

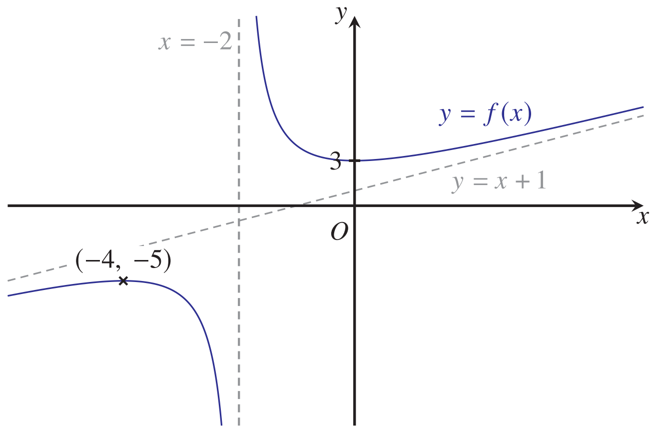
1M

Maximum point is $(-4,\ -5)$. 1A

Minimum point is $(0,\ 3)$. 1A

(d) (Shape) 1A

(All correct) 1A



23. (a) Vertical asymptote is $x = 3$. 1A

$$f(x) = \frac{2(x-1)^3}{(x-3)^2}$$

$$= 2x + 6 + \frac{24}{x-3} + \frac{16}{(x-3)^2}$$ 1M

Oblique asymptote is $y = 2x + 6$. 1A

(b) $$f'(x) = 2 - \frac{24}{(x-3)^2} - \frac{32}{(x-3)^3}$$ 1M

$$= \frac{2(x-1)^2(x-7)}{(x-3)^3}$$

When $f'(x) = 0$, $x = 1$ or 7. 1M

x	$x < 1$	$1 < x < 3$	$3 < x < 7$	$x > 7$
$f'(x)$	+	+	−	+

1M

Minimum point is $(7,\ 27)$. 1A

No maximum point.

(c) $$f''(x) = \frac{48}{(x-3)^3} + \frac{94}{(x-3)^4}$$

$$= \frac{48(x-1)}{(x-3)^4}$$

When $f''(x) = 0$, $x = 1$.

x	$x < 1$	$1 < x < 3$	$x > 3$
$f''(x)$	−	+	+

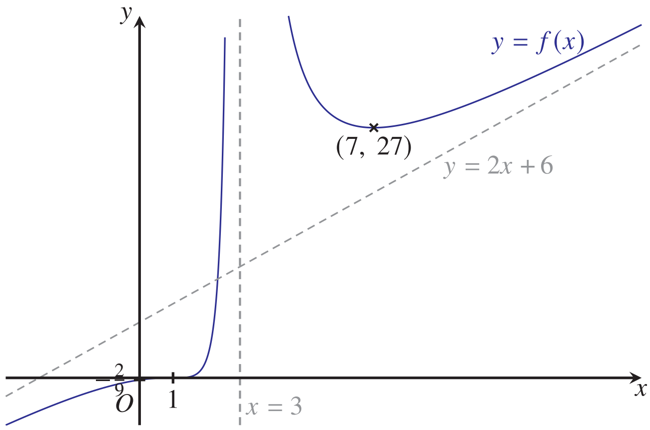
1M

Point of inflexion is $(1,\ 0)$. 1A

(d) (Asymptote) 1A

(Shape) 1A

(All correct) 1A



24. (a) Vertical asymptote is $x = 6$. 1A

$$f(x) = \frac{x^2 - 2x - 15}{x - 6}$$

$$= x + 4 + \frac{9}{x - 6}$$ 1M

Oblique asymptote is $y = x + 4$. 1A

(b) $f(x) = x + 4 + \dfrac{9}{x - 6}$

$$f'(x) = 1 - \frac{9}{(x - 6)^2}$$ 1M

$$= \frac{x^2 - 12x + 27}{(x - 6)^2}$$

$$= \frac{(x - 3)(x - 9)}{(x - 6)^2}$$

When $f'(x) = 0$, $x = 3$ or 9.

x	$x < 3$	$3 < x < 6$	$6 < x < 9$	$x > 9$
$f'(x)$	$+$	$-$	$-$	$+$

1M

The maximum point is $(3,\ 4)$. 1A

The minimum point is $(9,\ 16)$. 1A

(c) $f'(x) = 1 - \dfrac{9}{(x - 6)^2}$

$$f''(x) = \frac{18}{(x - 6)^3}$$ 1M

Note that $f''(x) \neq 0$ for all $x \neq 6$.

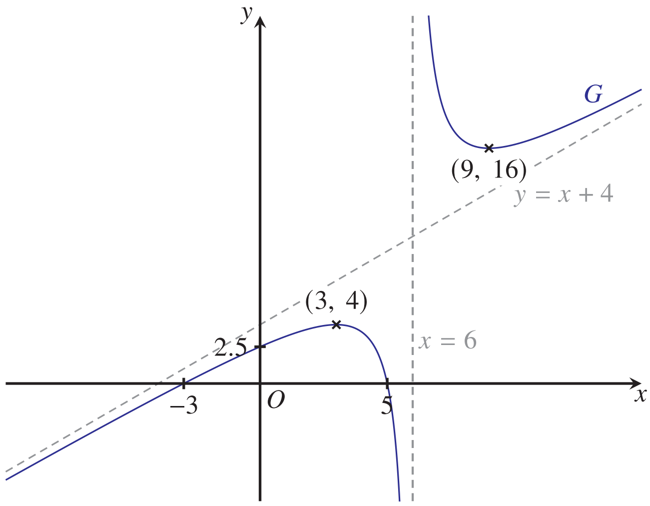
G has no points of inflexion.

The claim is agreed. 1A

(d) (Correct shape) 1A

(Asymptotes) 1A

(All correct) 1A



25. (a) $f'(x) = \dfrac{(2x+k)(x-1)-(x^2+kx-2)(1)}{(x-1)^2}$

$= \dfrac{x^2-2x+3-k}{(x-1)^2}$ 1A

$f'(-1) = 0$

$(-1)^2 - 2(-1) + 2 - k = 0$

$k = 5$ 1A

We have $f'(x) = \dfrac{x^2-2x-3}{(x-1)^2} = \dfrac{(x-3)(x+1)}{(x-1)^2}$.

When $f'(x) = 0$, $x = -1$ or 3.

Thus, $h = 3$. 1A

(b) Maximum point is $(-1,\ 3)$. 1A

Minimum point is $(3,\ 11)$. 1A

No point of inflexion. 1A

(c) Vertical asymptote is $x = 1$. 1A

$f(x) = \dfrac{x^2+5x-2}{x-1}$

$= x + 6 + \dfrac{4}{x-1}$

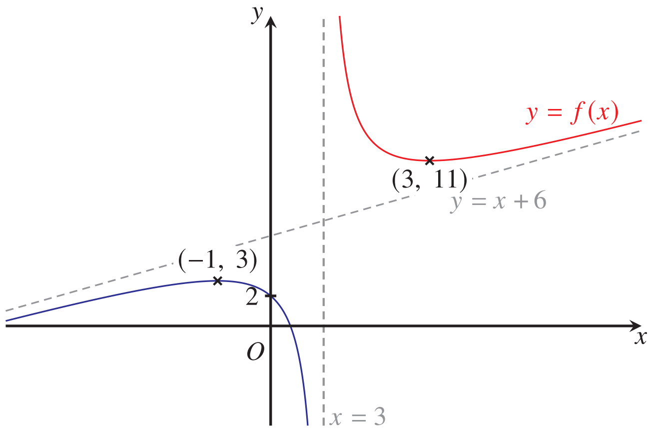
Oblique asymptote is $y = x + 6$. 1A

(d) y-intercept is 2. 1A

(Maximum points, minimum points) 1A

(Asymptotes) 1A

(All correct) 1A



26. (a) $x^2 = AC^2 - 1^2$

$= (1^2 + 1^2 - 2(1)(1)\cos\theta) - 1$ 1M

$x = \sqrt{1 - 2\cos\theta}$ 1

(b) $\dfrac{dx}{dt} = \dfrac{1}{2}(1 - 2\cos\theta)^{-\frac{1}{2}}(2\sin\theta)\dfrac{d\theta}{dt}$ 1M

Put $\theta = \dfrac{2\pi}{3}$ and $\dfrac{d\theta}{dt} = \dfrac{1}{10}$.

$\dfrac{dx}{dt} = \dfrac{1}{2}\left(1 - 2\cos\dfrac{2\pi}{3}\right)^{-\frac{1}{2}}\left(2\sin\dfrac{2\pi}{3}\right)\left(\dfrac{1}{10}\right)$ 1M

$\dfrac{dx}{dt} = \dfrac{\sqrt{6}}{40}$

Required rate is $\dfrac{\sqrt{6}}{40}$ m/s. 1A

27. (a) $v = 10e^{-u}$

$e^{-u} = \dfrac{v}{10}$

$u = -\ln\dfrac{v}{10}$ 1M

The coordinates of D and E are $(0,\ v)$ and $(0,\ 10)$.

Area of $\triangle PDE = \dfrac{1}{2}\left(-\ln\dfrac{v}{10}\right)(10 - v)$ 1M

$= \dfrac{1}{2}(v - 10)\ln\dfrac{v}{10}$ 1A

(b) $v = 10e^{-u}$

$\dfrac{dv}{dt} = -10e^{-u}\dfrac{du}{dt}$

Let S square units be the area of $\triangle PDE$.

$S = \dfrac{1}{2}(v - 10)\ln\dfrac{v}{10}$

$\dfrac{dS}{dt} = \dfrac{1}{2}\left[\dfrac{v - 10}{v} + \ln\dfrac{v}{10}\right]\dfrac{dv}{dt}$ 1M

Put $u = \ln 2$ and $\frac{du}{dt} = \frac{1}{5}$, we have $v = 10e^{-\ln 2} = 5$ and $\frac{dv}{dt} = -10e^{-\ln 2} \times \frac{1}{5} = -1$. 1M

$$\frac{dS}{dt} = \frac{1}{2}\left[\frac{5-10}{5} + \ln\frac{5}{10}\right](-1) \quad \text{1M}$$

$$= \frac{1+\ln 2}{2} \quad \text{1A}$$

Required rate is $\frac{1+\ln 2}{2}$ square units per second.

28. (a) We have $\frac{d}{dx}(\ln x) = \frac{1}{x}$ and $\frac{d}{dx}(e^x) = e^x$. 1A

$$e^x = \frac{1}{u} \quad \text{1M}$$

$$x = -\ln u$$

When $x = -\ln u$, $y = \frac{1}{u}$. 1A

Required equation is

$$y - \frac{1}{u} = \frac{1}{u}(x + \ln u)$$

$$y = \frac{x}{u} + \frac{\ln u + 1}{u} \quad \text{1A}$$

(b) The coordinates of P and Q are $(u,\ \ln u)$ and $(u,\ e^u)$ respectively.

$$OP^2 = u^2 + (\ln u)^2$$

$$\frac{dOP^2}{dt} = \left(2u + \frac{2\ln u}{u}\right)\frac{du}{dt} \quad \text{1M}$$

Put $u = 2$ and $\frac{dOP^2}{dt} = 2$.

$$2 = \left(4 + \frac{2\ln 2}{2}\right)\frac{du}{dt}$$

$$\frac{du}{dt} = \frac{2}{4+\ln 2} \quad \text{1A}$$

Let A be the area of $\triangle OPQ$.

$$A = \frac{1}{2}u(e^u - \ln u) \quad \text{1M}$$

$$\frac{dA}{dt} = \frac{1}{2}\left[(e^u - \ln u) + u\left(e^u - \frac{1}{u}\right)\right]\frac{du}{dt}$$

$$\left.\frac{dA}{dt}\right|_{u=2} = \frac{1}{2}\left[(e^2 - \ln 2) + 2\left(e^2 - \frac{1}{2}\right)\right]\frac{2}{4+\ln 2}$$

$$= \frac{3e^2 - \ln 2 - 1}{4 + \ln 2} \quad \text{1A}$$

Required rate is $\frac{3e^2 - \ln 2 - 1}{4+\ln 2}$ square unit per second.

29. (a) Let r cm be the radius of the water surface.

$$\frac{r}{h} = \frac{10}{\sqrt{26^2 - 10^2}} \qquad \text{1M}$$

$$r = \frac{5h}{12}$$

$$A = \pi\left(\frac{5h}{12}\right)\sqrt{h^2 + \left(\frac{5h}{12}\right)^2} \qquad \text{1M}$$

$$= \pi\left(\frac{5h}{12}\right)\sqrt{\frac{169h^2}{144}}$$

$$\frac{65}{144}\pi h^2 \qquad \text{1}$$

(b) $\frac{1}{3}\pi\left(\frac{5h}{12}\right)^2 h = 100\pi$ 1M

$$h^3 = 1728$$

$$h = 12 \qquad \text{1A}$$

$$A = \frac{65}{144}\pi h^2$$

$$\frac{dA}{dt} = \frac{65}{72}\pi h\frac{dh}{dt} \qquad \text{1M}$$

$$\left.\frac{dA}{dt}\right|_{h=12} = \frac{65}{72}\pi(12)\left(\frac{6}{\pi}\right)$$

$$= 65 \qquad \text{1A}$$

Required rate is 65 cm²/s.

30. (a) $V = \frac{1}{3}\pi(h\tan 60°)^2 h$ 1M

$$= \pi h^3 \qquad \text{1A}$$

(b) $\frac{dV}{dt} = 3\pi h^2\frac{dh}{dt}$ 1M

Put $h = 1$ and $\frac{dV}{dt} = -3$.

$$-3 = 3\pi h^2\frac{dh}{dt} \qquad \text{1M}$$

$$\frac{dh}{dt} = -\frac{1}{\pi}$$

The water level is falling at a rate of $-\frac{1}{\pi}$ cm/s. 1A

31. (a) Let A be the area of the circle.

$$A = \left(\frac{\sqrt{u^2+v^2}}{2}\right)^2 \pi$$

$$= \frac{\pi S}{4} \qquad \text{1M}$$

$$\frac{dA}{dt} = \frac{\pi}{4}\frac{dS}{dt}$$

$$4\pi = \frac{\pi}{4}\frac{dS}{dt}$$

$$\frac{dS}{dt} = 16$$

S increases at a constant rate. 1A

(b) $S = u^2 + 3^{2u-2}$

$$\frac{dS}{dt} = \left[2u + 2(3^{2u-2})\ln 3\right]\frac{du}{dt} \qquad \text{1M}$$

Put $u = 1$.

$$16 = (2 + 2\ln 3)\frac{du}{dt} \qquad \text{1M}$$

$$\frac{du}{dt} = \frac{8}{1+\ln 3}$$

Let B be the area of $\triangle OPQ$.

$$B = \left(\frac{3^u - 3^{u-1}}{2}\right)u \qquad \text{1M}$$

$$= u(3^{u-1})$$

$$\frac{dB}{dt} = \left[3^{u-1} + u(3^{u-1})\ln 3\right]\frac{du}{dt} \qquad \text{1M}$$

Put $u = 1$ and $\frac{du}{dt} = \frac{8}{1+\ln 3}$.

$$\frac{dB}{dt} = (1+\ln 3)\frac{8}{1+\ln 3}$$

$$= 8 \qquad \text{1A}$$

Required rate is 8 square units per second.

32. (a) The coordinates of P are $\left(u,\ \frac{e^{2u}}{2}\right)$.

$$y = \frac{e^{2x}}{2}$$

$$\frac{dy}{dx} = e^{2x} \qquad \text{1M}$$

Let the coordinates of R be $(r,\ 0)$.

$$\frac{0 - \frac{e^{2u}}{2}}{r-u} = e^{2u} \qquad \text{1M}$$

$$r = u - \frac{1}{2} \qquad \text{1A}$$

The coordinates of R are $\left(u - \frac{1}{2},\ 0\right)$.

(b) The coordinates of Q are $\left(u,\ \cos\left(2u + \frac{\pi}{3}\right)\right)$.

$$\cos\left(2u + \frac{\pi}{3}\right) = -\frac{1}{2}$$

$$2u + \frac{\pi}{3} = \frac{2\pi}{3}$$

$$u = \frac{\pi}{6} \qquad \text{1A}$$

Let A square units be the area of $\triangle PQR$ at time t s.

$$A = \frac{1}{2}\left[\frac{e^{2u}}{2} - \cos\left(2u + \frac{\pi}{3}\right)\right]\left[u - \left(u - \frac{1}{2}\right)\right] \qquad \text{1M}$$

$$= \frac{1}{8}\left[e^{2u} - 2\cos\left(2u + \frac{\pi}{3}\right)\right]$$

$$\frac{dA}{dt} = \frac{1}{8}\left[2e^{2u} + 4\sin\left(2u + \frac{\pi}{3}\right)\right]\frac{du}{dt} \qquad \text{1M}$$

$$0.5 = \frac{1}{8}\left[2e^{\frac{\pi}{3}} + 4\sin\frac{2\pi}{3}\right]\frac{du}{dt}$$

$$\frac{du}{dt} = \frac{2}{e^{\frac{\pi}{3}} + \sqrt{3}} \qquad \text{1A}$$

Required rate is $\frac{2}{e^{\frac{\pi}{3}} + \sqrt{3}}$ unit per second.

33. Volume of the pyramid $= \frac{1}{3}(4)^2(6)$

$= 32\,\text{cm}^3$

Consider when the container is half filled with water.

$$16 = 32\left(\frac{h}{6}\right)^3$$

$$h = \sqrt[3]{108} \qquad \text{1M}$$

Consider the relationship between V and h.

Let t s be the time elapsed since the water is poured into the container.

$$V = 32 \times \left(\frac{h}{6}\right)^3$$

$$V = \frac{4h^3}{27} \qquad \text{1A}$$

$$\frac{dV}{dt} = \frac{4h^2}{9} \cdot \frac{dh}{dt} \qquad \text{1M}$$

Put $\frac{dh}{dt} = 0.08$ and $h = \sqrt[3]{108}$.

$$\frac{dV}{dt} = \frac{4(\sqrt[3]{108})^2}{9}(0.08) \qquad \text{1M}$$

$$\approx 0.806 \qquad \text{1A}$$

Required rate is $0.806\,\text{cm}^3/\text{s}$.

34. (a) $V = \frac{1}{3}\pi(9)^2(12) \times \left(\frac{h}{12}\right)^3$ 1M

$= \frac{3}{16}\pi h^3$ 1

(b) (i) Required volume

$= \frac{3}{16}\pi(12^3 - 10^3)$ 1M

$= \frac{273\pi}{2}$ cm^3 1A

(ii) Let H cm be the depth of oil at time t s.

$\frac{3}{16}\pi h^3 + \frac{273\pi}{2} = \frac{3}{16}\pi(h + H)^3$ 1M

$\frac{9}{16}\pi h^2 \frac{dh}{dt} = \frac{9}{16}\pi(h + H)^2 \left(\frac{dh}{dt} + \frac{dH}{dt}\right)$ 1M

Put $h = 1$.

$\frac{3}{16}\pi + \frac{273\pi}{2} = \frac{3}{16}\pi(1 + H)^3$

$(1 + H)^3 = 729$

$H = 8$

Put $h = 1$, $H = 8$ and $\frac{dh}{dt} = -0.2$.

$\frac{9}{16}\pi(1)^2(-0.2) = \frac{9}{16}\pi(1 + 8)^2 \left(-0.2 + \frac{dH}{dt}\right)$

$\frac{dH}{dt} = \frac{16}{81}$ 1A

Required rate is $\frac{16}{81}$ cm/s.

35. (a) Let r cm be the radius of the water surface in the container.

$\frac{r}{h} = \tan 30°$

$r = \frac{h}{\sqrt{3}}$ 1M

$V = \frac{1}{3}\pi\left(\frac{h}{\sqrt{3}}\right)^2 h$

$= \frac{\pi}{9}h^3$ 1

$A = \pi r\sqrt{r^2 + h^2}$

$= \pi \cdot \frac{h}{\sqrt{3}}\left(\frac{2h}{\sqrt{3}}\right)$

$= \frac{2\pi}{3}h^2$ 1

(b) When $V = \frac{64\pi}{9}$,

$\frac{\pi}{9}h^3 = \frac{64\pi}{9}$

$h = 4$ 1A

When $h = 4$,

$$V = \frac{\pi}{9}h^3$$

$$\frac{dV}{dt} = \frac{\pi}{9}(3h^2)\frac{dh}{dt}$$ 1M

$$8\pi = \frac{\pi}{9}(48)\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{3}{2}$$

Also,

$$\frac{dA}{dt} = \left(\frac{4\pi}{3}h\right)\frac{dh}{dt}$$ 1M

$$= \frac{16\pi}{3}\left(\frac{3}{2}\right)$$

$$= 8\pi$$

Required rate is 8π cm²/s. 1A

36. (a) $\tan\theta = \tan(\angle ARQ + \angle BRQ)$

$$= \frac{\tan\angle ARQ + \tan\angle BRQ}{1 - \tan\angle ARQ \tan\angle ABQ}$$ 1M

$$= \frac{\frac{100}{100+x} + \frac{y}{100}}{1 - \left(\frac{100}{100+x}\right)\left(\frac{y}{100}\right)}$$

$$= \frac{10\,000 + 100y + xy}{100(100 + x - y)}$$ 1

(b) (i) When $t = 0$, we have $x = y = 0$.

$$\tan\theta = \frac{10\,000 + 100(0) + (0)(0)}{100(100 + 0 - 0)}$$ 1M

$$= 1$$

The value of θ remains unchanged.

$$\frac{10\,000 + 100y + xy}{100(100 + x - y)} = 1$$

$$10\,000 + 100y + xy = 10\,000 + 100x - 100y$$

$$200y + xy = 100x$$

$$y = \frac{100x}{x + 200}$$ 1

(ii) $$\frac{dy}{dt} = \frac{100(x + 200) - 100x}{(x + 200)^2} \times \frac{dx}{dt}$$ 1M

$$= \frac{20\,000}{(x + 200)^2} \times \frac{dx}{dt}$$

When $t = 50$, $x = 4(50) = 200$.

$$\frac{dy}{dt} = \frac{20\,000}{(200+200)^2} \times \frac{dx}{dt}$$
$$= \frac{1}{8} \times 4$$
$$= \frac{1}{2}$$

Required speed is $\frac{1}{2}$ m/s. 1A

37. (a) $y = 2e^{\frac{x}{2}}$

$$\frac{dy}{dx} = e^{\frac{x}{2}}$$ 1M

Let the coordinates of R be $(r,\ 0)$.

$$\frac{0 - 2e^{\frac{u}{2}}}{r-u} = e^{\frac{u}{2}}$$ 1M

$$-2 = r - u$$

$$r = u - 2$$ 1A

The coordinates of R are $(u-2,\ 0)$.

(b) Let A be the area of $\triangle PQR$.

$$A = \frac{(2e^{\frac{u}{2}} - \ln(2u))(2)}{2}$$ 1M

$$= 2e^{\frac{u}{2}} - \ln(2u)$$

$$\frac{dA}{dt} = \left(e^{\frac{u}{2}} - \frac{1}{u}\right)\frac{du}{dt}$$ 1M

When Q lies on the x-axis, $u = \frac{1}{2}$. 1A

$$0.5 = \left(e^{\frac{1}{4}} - 2\right)\frac{du}{dt}$$ 1M

$$\frac{du}{dt} = \frac{1}{2(e^{\frac{1}{4}} - 2)}$$ 1A

Required rate is $\frac{1}{2(e^{\frac{1}{4}} - 2)}$ unit per second.

38. (a) Let A be the area of the circle.

$$A = (a^2 + b^2)\pi$$ 1M

$$= S\pi$$

$$\frac{dA}{dt} = \pi\frac{dS}{dt}$$

$$4\pi = \pi\frac{dS}{dt}$$

$$\frac{dS}{dt} = 4$$

$$> 0$$

S increases at a constant rate. 1A

(b) Let B be the area of $\triangle OPQ$.

$$B = \frac{1}{2}a(e^{a+1} - e^{a})$$
$$= \frac{ae^{a}(e-1)}{2} \quad \text{1M}$$
$$\frac{dB}{dt} = \frac{ae^{a}(e-1) + e^{a}(e-1)}{2} \cdot \frac{da}{dt}$$

From (a), $\dfrac{dS}{dt} = 4$.

$$\frac{d}{dt}\left(a^2 + b^2\right) = 4$$
$$2a\frac{da}{dt} + 2b\frac{db}{dt} = 4 \quad \text{1M}$$
$$2a\frac{da}{dt} + 2e^{a} \cdot e^{a}\frac{da}{dt} = 4$$
$$\frac{da}{dt} = \frac{2}{a + e^{2a}}$$

Put $t = 3$.

$$\frac{dB}{dt} = \frac{e^{3}(3+1)(e-1)}{2} \cdot \frac{2}{3 + e^{6}} \quad \text{1M}$$
$$= \frac{4e^{3}(e-1)}{3 + e^{6}} \quad \text{1A}$$

Required rate is $\dfrac{4e^{3}(e-1)}{3 + e^{6}}$ square units per second.

39. (a) $y = e^{x+1}$

$$\frac{dy}{dx} = e^{x+1} \quad \text{1M}$$

Let $(q,\ 0)$ be the coordinates of Q.

$$\frac{0 - e^{u+1}}{q - u} = e^{u+1} \quad \text{1M}$$
$$q = u - 1 \quad \text{1A}$$

The x-coordinate of Q is $u - 1$.

(b)
$$OP^2 = u^2 + (e^{u+1})^2$$
$$= u^2 + e^{2u+2}$$
$$\frac{d}{dt}\left(OP^2\right) = \left(2u + 2e^{2u+2}\right)\frac{du}{dt}$$
$$8 = (2u + 2e^{2u+2})\frac{du}{dt} \quad \text{1M}$$

Put $u = 2$, we have $\dfrac{du}{dt} = \dfrac{4}{2 + e^{6}}$.

Let A square units be the area of $\triangle OPQ$.

$$A = \frac{1}{2}(u-1)(e^{u+1}) \qquad \text{1M}$$

$$\frac{dA}{dt} = \frac{1}{2}(e^{u+1} + ue^{u+1} - e^{u+1})\frac{du}{dt} \qquad \text{1M}$$

$$= \frac{1}{2}ue^{u+1}\frac{du}{dt}$$

Put $u = 2$ and $\frac{du}{dt} = \frac{4}{2+e^6}$.

$$\frac{dA}{dt} = \frac{1}{2}(2e^3)\cdot\frac{4}{2+e^6} \qquad \text{1M}$$

$$= \frac{4e^3}{2+e^6} \qquad \text{1A}$$

Required rate is $\frac{4e^3}{2+e^6}$ square units per second.

40. (a) Let A be the area of $\triangle OPQ$.

$$A = \frac{1}{2}ue^{u-1} \qquad \text{1M}$$

$$\frac{dA}{dt} = \frac{1}{2}\left(e^{u-1} + ue^{u-1}\right)\frac{du}{dt} \qquad \text{1M}$$

$$2 = \frac{1}{2}e^{u-1}(1+u)\frac{du}{dt}$$

$$\frac{du}{dt} = \frac{4}{e^{u-1}(1+u)} \qquad \text{1}$$

(b) Let p units be the perimeter of $\triangle OPQ$.

$$p = u + e^{u-1} + \sqrt{u^2 + e^{2(u-1)}} \qquad \text{1M}$$

$$\frac{dp}{dt} = \left[1 + e^{u-1} + \frac{1}{2}(u^2 + e^{2u-2})^{-\frac{1}{2}}(2u + 2e^{2u-2})\right]\frac{du}{dt} \qquad \text{1M}$$

When $u = 1$, $\frac{du}{dt} = \frac{4}{e^0(1+1)} = 2$. 1M

$$\frac{dp}{dt} = \left[1 + e^0 + \frac{1}{2}(1+e^0)^{-\frac{1}{2}}(2 + 2e^0)\right](2) \qquad \text{1M}$$

$$= 2(2+\sqrt{2}) \qquad \text{1A}$$

Required rate is $2(2+\sqrt{2})$ units per second.

41. (a) Let h cm be the depth of water in the vessel.

$$\frac{r}{h} = \tan\frac{\pi}{6} \qquad \text{1M}$$

$$h = \sqrt{3}r$$

$$A = \pi r\sqrt{(\sqrt{3}r)^2 + r^2} \qquad \text{1M}$$

$$= 2\pi r^2 \qquad \text{1A}$$

(b) Let V cm^3 be the volume of water in the vessel.

$$\frac{1}{3}\pi r^2(\sqrt{3}r) = 192\pi \qquad \text{1M}$$

$$r = 4\sqrt{3} \qquad \text{1A}$$

$$A = 2\pi r^2$$

$$\frac{dA}{dt} = 4\pi r \times \frac{dr}{dt}$$

$$-8\pi = 4\pi r \times \frac{dr}{dt}$$

$$\frac{dr}{dt} = -\frac{2}{r}$$

When $r = 4\sqrt{3}$,

$$V = \frac{1}{3}\pi r^2(\sqrt{3}r)$$

$$\frac{dV}{dt} = \sqrt{3}\pi r^2 \frac{dr}{dt} \qquad \text{1M}$$

$$= \sqrt{3}\pi(4\sqrt{3})^2\left(-\frac{2}{4\sqrt{3}}\right)$$

$$= -24\pi \qquad \text{1A}$$

The volume of water is decreasing at a rate of 24π cm^3/s.

42. (a) Let r cm be the radius of the water surface.

$$\frac{r-6}{10-6} = \frac{h}{12} \qquad \text{1M}$$

$$r = \frac{h+18}{3}$$

Consider the volume of the water.

$$V = \frac{\pi}{3}h\left[6^2 + 6\left(\frac{h+18}{3}\right) + \left(\frac{h+18}{3}\right)^2\right] \qquad \text{1M}$$

$$= \frac{\pi}{27}h(h^2 + 54h + 972)$$

$$= \frac{\pi}{27}(h^3 + 54h^2 + 972h) \qquad \text{1}$$

(b) (i) $$V = \frac{\pi}{27}(h^3 + 54h^2 + 972h)$$

$$\frac{dV}{dt} = \frac{\pi}{27}(3h^2 + 108h + 972)\frac{dh}{dt} \qquad \text{1M}$$

$$= \frac{\pi}{9}(h^2 + 36h + 324)\frac{dh}{dt}$$

When the radius of the water surface equals the depth of water,

$$h = \frac{h+18}{3}$$

$$3h = h + 18$$

$$h = 9$$

When $\frac{dV}{dt} = -10\pi$ and $h = 9$,

$$-10\pi = \frac{\pi}{9}(9^2 + 36(9) + 324)\frac{dh}{dt} \qquad \text{1M}$$

$$\frac{dh}{dt} = -\frac{10}{81} \qquad \text{1A}$$

Required rate is $-\frac{10}{81}$ cm/s.

(ii) When $\frac{dV}{dt} = -10\pi$ and $\frac{dh}{dt} = -0.144$,

$$-10\pi = \frac{\pi}{9}(h^2 + 36h + 324)(-0.144) \qquad \text{1M}$$

$$625 = h^2 + 36h + 324$$

$$0 = h^2 + 36h - 301$$

$$h = 7 \quad \text{or} \quad -43 \text{ (rejected)}$$

When $h = 12$, $V = 784\pi$.
When $h = 7$, $V = \frac{9793\pi}{27} < \frac{784\pi}{2}$.
More than half of the water in the container has been pumped out.
The claim is agreed. 1A

43. (a)

$$x^2 + y^2 - 2xy\cos\frac{\pi}{3} = 13 \qquad \text{1M}$$

$$x^2 + y^2 - xy = 13$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} - \left(y\frac{dx}{dt} + x\frac{dy}{dt}\right) = 0 \qquad \text{1M+1A}$$

$$(2y - x)\frac{dy}{dt} = (-2x + y)\frac{dx}{dt}$$

$$\frac{dy}{dt} = \left(\frac{2x - y}{x - 2y}\right)\frac{dx}{dt} \qquad \text{1}$$

(b) When $x = 3$,

$$3^2 + y^2 - 3y = 13 \qquad \text{1M}$$

$$y^2 - 3y - 4 = 0$$

$$y = 4 \quad \text{or} \quad -1 \text{ (rejected)}$$

Put $x = 3$, $y = 4$ and $\frac{dx}{dt} = -\frac{1}{2}$.

$$\frac{dy}{dt} = \left(\frac{2(3) - 4}{3 - 2(4)}\right)\left(-\frac{1}{2}\right) \qquad \text{1M}$$

$$= \frac{1}{5} \qquad \text{1A}$$

Required rate is $\frac{1}{5}$ m/s.

44. (a) $\left(\frac{x}{2}\right)^2 + y^2 = 13^2$

$$x^2 + 4y^2 = 676 \qquad \text{1A}$$

(b) When $y = 12$, we have $x = 10$. 1M

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(x^2 + 4y^2\right) = \frac{\mathrm{d}}{\mathrm{d}t}(676)$$

$$2x\frac{\mathrm{d}x}{\mathrm{d}t} + 8y\frac{\mathrm{d}y}{\mathrm{d}t} = 0$$ 1M

$$x\frac{\mathrm{d}x}{\mathrm{d}t} + 4y\frac{\mathrm{d}y}{\mathrm{d}t} = 0$$

Put $x = 10$, $y = 12$ and $\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{1}{3}$.

$$10\frac{\mathrm{d}x}{\mathrm{d}t} + 4(12)\left(\frac{1}{3}\right) = 0$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{8}{5}$$ 1A

Let S m^2 be the area of $\triangle ABH$.

$$S = \frac{1}{2}xy$$

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \frac{1}{2}\left(y\frac{\mathrm{d}x}{\mathrm{d}t} + x\frac{\mathrm{d}y}{\mathrm{d}t}\right)$$ 1M

$$= \frac{1}{2}\left[12\left(-\frac{8}{5}\right) + 10\left(\frac{1}{3}\right)\right]$$

$$= -\frac{119}{15}$$ 1A

Required rate is $-\frac{119}{15}$ m^2/s.

45. (a) $PQ = e^{\frac{u}{2}} - \left(-e^{-\frac{u}{2}}\right)$ 1M

$$= e^{\frac{u}{2}} + e^{-\frac{u}{2}}$$

Let A square units be the area of the rectangle $PQSR$.

$$A = u\left(e^{\frac{u}{2}+e^{-\frac{u}{2}}}\right)$$ 1M

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \left[\left(e^{\frac{u}{2}} + e^{-\frac{u}{2}}\right) + u\left(\frac{1}{2}e^{\frac{u}{2}} - \frac{1}{2}e^{-\frac{u}{2}}\right)\right]\frac{\mathrm{d}u}{\mathrm{d}t}$$ 1M

When $u = 2$,

$$e^2 = \left[\left(e + e^{-1}\right) + \left(e - e^{-1}\right)\right]\frac{\mathrm{d}u}{\mathrm{d}t}$$

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{e}{2}$$ 1A

Required rate is $\frac{e}{2}$ units per minute.

(b) Let p units be the perimeter of the rectangle $PQSR$.

$$p = 2\left(u + e^{\frac{u}{2}} + e^{-\frac{u}{2}}\right)$$ 1M

$$\frac{\mathrm{d}p}{\mathrm{d}t} = 2\left(1 + \frac{1}{2}e^{\frac{u}{2}} - \frac{1}{2}e^{-\frac{u}{2}}\right)\frac{\mathrm{d}u}{\mathrm{d}t}$$ 1M

When $u = 2$,

$$\frac{\mathrm{d}p}{\mathrm{d}t} = 2\left(1 + \frac{1}{2}e - \frac{1}{2}e^{-1}\right)\left(\frac{e}{2}\right) \quad \text{1M}$$

$$= \frac{e^2}{2} + e - \frac{1}{2} \quad \text{1A}$$

Required rate is $\frac{e^2}{2} + e - \frac{1}{2}$ units per minute.

46. (a) Let r cm be the radius of the water surface in the vessel.

We have $r = h \tan \frac{\pi}{3} = \sqrt{3}h$. 1M

$$A = \pi(\sqrt{3}h)\sqrt{h^2 + (\sqrt{3}h)^2} \quad \text{1M}$$

$$= 2\sqrt{3}\pi h^2 \quad \text{1}$$

(b) When the volume of water in the vessel is $216\pi\ \text{cm}^3$.

$$\frac{1}{3}\pi(\sqrt{3}h)^2 h = 216\pi \quad \text{1M}$$

$$h = 6 \quad \text{1A}$$

From (a), we have $A = 2\sqrt{3}\pi h^2$.

$$\frac{\mathrm{d}A}{\mathrm{d}t} = 4\sqrt{3}\pi h\frac{\mathrm{d}h}{\mathrm{d}t} \quad \text{1M}$$

Put $\frac{\mathrm{d}A}{\mathrm{d}t} = -12\pi$ and $h = 6$.

$$-12\pi = 4\sqrt{3}\pi(6)\frac{\mathrm{d}h}{\mathrm{d}t}$$

$$\frac{\mathrm{d}h}{\mathrm{d}t} = -\frac{\sqrt{3}}{6} \quad \text{1A}$$

Required rate is $-\frac{\sqrt{3}}{6}$ cm/s.

47. (Cancelled)

48. (a) $y = 8e^{-x}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -8e^{-x} \quad \text{1M}$$

Let the coordinates of R be $(r,\ 0)$.

$$\frac{8e^{-u} - 0}{u - r} = -8e^{-u} \quad \text{1M}$$

$$r = u + 1$$

The coordinates of R are $(u + 1,\ 0)$. 1A

(b) When the y-coordinate of Q is -1,

$$-1 = 2\sin 2u$$
$$2u = -\frac{\pi}{6}$$
$$u = -\frac{\pi}{12} \qquad \text{1A}$$

Let A be the area of $\triangle PQR$.

$$A = \frac{[(u+1)-u][8e^{-u} - 2\sin 2u]}{2} \qquad \text{1M}$$
$$= 4e^{-u} - \sin 2u$$
$$\frac{dA}{dt} = (-4e^{-u} - 2\cos 2u)\frac{du}{dt} \qquad \text{1M}$$
$$-2 = \left[-4e^{\frac{\pi}{12}} - 2\cos\frac{-\pi}{6}\right]\frac{du}{dt} \qquad \text{1M}$$
$$\frac{du}{dt} = \frac{2}{4e^{\frac{\pi}{12}} + \sqrt{3}} \qquad \text{1A}$$

Required rate is $\frac{2}{4e^{\frac{\pi}{12}} + \sqrt{3}}$ per second.

49. (a) $PQ = \sqrt{u^2+64} + \sqrt{(u-9)^2+16}$ 1M

$$\frac{dPQ}{du} = \frac{u}{\sqrt{u^2+64}} + \frac{u-9}{\sqrt{(u-9)^2+16}} \qquad \text{1M}$$
$$= \frac{u\sqrt{(u-9)^2+16} + (u-9)\sqrt{u^2+64}}{\sqrt{u^2+64}\sqrt{(u-9)^2+16}}$$
$$= \frac{u^2[(u-9)^2+16] - (u-9)^2(u^2+64)}{\sqrt{u^2+64}\sqrt{(u-9)^2+16}(u\sqrt{(u-9)^2+16} - (u-9)\sqrt{u^2+64})}$$
$$= \frac{-48(u-6)(u-18)}{\sqrt{u^2+64}\sqrt{(u-9)^2+16}(u\sqrt{(u-9)^2+16} - (u-9)\sqrt{u^2+64})}$$

For $\frac{dPQ}{du} = 0$, we have $u = 6$. 1M

Thus, we have $a = 6$. 1A

(b) (i) Let A square units be the area of the rectangle $PQSR$.

$$A = uPQ \qquad \text{1M}$$
$$\frac{dA}{du} = PQ + u\frac{dPQ}{du} \qquad \text{1M}$$

When $u = 6$,

$$\frac{dA}{du} = \sqrt{6^2+64} + \sqrt{(6-9)^2+16} + 6(0)$$
$$= 15 \neq 0 \qquad \text{1M}$$

Hence, A does not attains its minimum value when $u = 6$.

The claim is disagreed. 1A

(ii) $OP = \sqrt{u^2 + (u^2 + 64)}$

$= \sqrt{2u^2 + 64}$

$\dfrac{dOP}{dt} = \dfrac{2u}{\sqrt{2u^2 + 64}} \dfrac{du}{dt}$ 1M

When $u = 6$,

$$12 = \frac{2(6)}{\sqrt{2(6)^2 + 64}} \left(\left. \frac{du}{dt} \right|_{u=6} \right)$$

$\left. \dfrac{du}{dt} \right|_{u=6} = 2\sqrt{34}$ 1M

Let w units be the perimeter of the rectangle $PQSR$.

$w = 2(u + PQ)$ 1M

$\dfrac{dw}{dt} = 2\dfrac{du}{dt} + 2\dfrac{dPQ}{du} \cdot \dfrac{du}{dt}$ 1M

When $u = 6$,

$\left. \dfrac{dw}{dt} \right|_{u=6} = 2(2\sqrt{34}) + 2(0)(2\sqrt{34})$

$= 4\sqrt{34}$ 1A

Required rate is $4\sqrt{34}$ units per second.

50. (a) $A(\theta) = \dfrac{[(1 - \sin\theta) + 1](2 - \cos\theta)}{2}$

$= \dfrac{(2 - \sin\theta)(2 - \cos\theta)}{2}$ 1A

$\dfrac{dA(\theta)}{d\theta} = \dfrac{1}{2}[(-\cos\theta)(2 - \cos\theta) + (2 - \sin\theta)(\sin\theta)]$ 1M

$= \dfrac{1}{2}[2(\sin\theta - \cos\theta) - (\sin^2\theta - \cos^2\theta)]$

$= \dfrac{1}{2}(\sin\theta - \cos\theta)(2 - \sin\theta - \cos\theta)$ 1

(b) (i) For $0 < \theta < \dfrac{\pi}{2}$, we have $\sin\theta < 1$ and $\cos\theta < 1$. 1M

$2 - \sin\theta - \cos\theta > 2 - 1 - 1$

$2 - \sin\theta - \cos\theta > 0$ 1

(ii) When $A(\theta) = 0$,

$\sin\theta - \cos\theta = 0$ 1M

$\tan\theta = 1$

$\theta = \dfrac{\pi}{4}$

θ	$0 < \theta < \frac{\pi}{4}$	$\frac{\pi}{4} < \theta < \frac{\pi}{2}$
$\frac{dA(\theta)}{d\theta}$	$-$	$+$

1M

$A(\theta)$ attains its minimum value at $\theta = \frac{\pi}{4}$. 1A

Required value

$$= \frac{1}{2}\left(2 - \sin\frac{\pi}{4}\right)\left(2 - \cos\frac{\pi}{4}\right)$$

$$= \frac{9 - 4\sqrt{2}}{4}$$ 1A

(c) Let t s be the time.

$$\frac{\mathrm{d}A(\theta)}{\mathrm{d}t} = \frac{\mathrm{d}A(\theta)}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t}$$ 1M

When $\theta = \frac{\pi}{3}$ and $\frac{\mathrm{d}\theta}{\mathrm{d}t} = -\frac{1}{10}$,

$$\frac{\mathrm{d}A(\theta)}{\mathrm{d}t} = \frac{1}{2}\left(\sin\frac{\pi}{3} - \cos\frac{\pi}{3}\right)\left(2 - \sin\frac{\pi}{3} - \cos\frac{\pi}{3}\right)\left(-\frac{1}{10}\right)$$ 1M

$$= \frac{1}{2} \cdot \frac{\sqrt{3} - 1}{2} \cdot \frac{4 - \sqrt{3} - 1}{2} \cdot \left(-\frac{1}{10}\right)$$

$$= \frac{3 - 2\sqrt{3}}{40}$$

Required rate is $\frac{3 - 2\sqrt{3}}{40}$ m²/s. 1A

51. (a) The coordinates of A and B are $\left(h,\ (h-18)^2 + 100\right)$ and $\left(h,\ -h^4 + 12\right)$ respectively.

$$AB = (h - 18)^2 + 100 - (-h^4 + 12)$$ 1M

$$= h^4 + h^2 - 36h + 412$$

$$\frac{\mathrm{d}AB}{\mathrm{d}h} = 4h^3 + 2h - 36$$ 1M

$$= 2(h - 2)(2h^2 + 4h + 9)$$

When AB attains its minimum value, $\frac{\mathrm{d}AB}{\mathrm{d}h} = 0$.

$$(h - 2)(2h^2 + 4h + 9) = 0$$ 1M

$$h = 2 \quad \text{or} \quad 2h^2 + 4h + 9 = 0$$

Consider the equation $2h^2 + 4h + 9 = 0$, $\Delta = 4^2 - 4(2)(9) = -56 < 0$.

The equation has no real roots.

Thus, we have $a = 2$. 1A

(b) (i) Let p units be the perimeter of the rectangle $ABDC$.

$$p = 2(AB + h)$$ 1M

$$\frac{\mathrm{d}p}{\mathrm{d}h} = 2\frac{\mathrm{d}AB}{\mathrm{d}h} + 2$$ 1M

$$\left.\frac{\mathrm{d}p}{\mathrm{d}h}\right|_{h=2} = 2(0) + 2$$ 1M

$$= 2 \neq 0$$

p does not attains its minimum value when $h = 2$.

The claim is disagreed. 1A

(ii) $$OB = \sqrt{h^2 + (-h^4 + 12)^2}$$

$$\frac{\mathrm{d}OB}{\mathrm{d}h} = \frac{2h + 2(-h^4 + 12)(-4h^3)}{2\sqrt{h^2 + (-h^4 + 12)^2}}$$ 1M

$$\left.\frac{\mathrm{d}OB}{\mathrm{d}h}\right|_{h=2} = \frac{4 + 2(-16 + 12)(-32)}{2\sqrt{4 + (-16 + 12)^2}}$$

$$= \frac{65}{\sqrt{5}}$$

$$\frac{\mathrm{d}OB}{\mathrm{d}t} = \frac{\mathrm{d}OB}{\mathrm{d}h} \cdot \frac{\mathrm{d}h}{\mathrm{d}t}$$

$$-12 = \left(\frac{65}{\sqrt{5}}\right) \left.\frac{\mathrm{d}h}{\mathrm{d}t}\right|_{h=2}$$

$$\left.\frac{\mathrm{d}h}{\mathrm{d}t}\right|_{h=2} = -\frac{12\sqrt{5}}{65}$$ 1M

Let S square units be the area of the rectangle $ABDC$.

$$S = (AB)h$$ 1M

$$\frac{\mathrm{d}S}{\mathrm{d}t} = h\frac{\mathrm{d}AB}{\mathrm{d}t} + AB\frac{\mathrm{d}h}{\mathrm{d}t}$$ 1M

$$\left.\frac{\mathrm{d}S}{\mathrm{d}t}\right|_{h=2} = 2(0) + [2^4 + 2^2 - 36(2) + 412]\left(-\frac{12\sqrt{5}}{65}\right)$$

$$= -\frac{864\sqrt{5}}{13}$$ 1A

Required rate is $-\frac{864\sqrt{5}}{13}$ square units per second.

52. (a) (i) (2, 16) is a stationary point of C_1.

$$y = ax + bx^3$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = a + 3bx^2$$

$$0 = a + 3b(2)^2$$ 1M

(2, 16) is a point on C_1.

$$16 = 2a + b(2)^3$$ 1M

Solving, we have $a = 12$ and $b = -1$. 1A

(ii) $$\frac{\mathrm{d}y}{\mathrm{d}x} = 12 - 3x^2$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -6x$$ 1M

$$\left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|_{x=2} = -12 < 0$$

Thus, (2, 16) is a maximum point of C_1. 1A

(iii) When $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$, $x = 0$.

x	$x < 0$	$x > 0$
$\frac{d^2y}{dx^2}$	+	−

1M

Point of inflexion is $(0,\ 0)$. 1A

(b) (i) $PQ = (12r - r^3) - \ln\left(12r - r^3\right)$

$= 12r - r^3 - \ln\left(12r - r^3\right)$ 1A

(ii) Let A square units be the area of $\triangle OPQ$.

$$A = \frac{1}{2}r[12r - r^3 - \ln\left(12r - r^3\right)]$$ 1M

$$= \frac{1}{2}[12r^2 - r^4 - r\ln\left(12r - r^3\right)]$$

$$\frac{dA}{dt} = \frac{1}{2}\left[24r - 4r^3 - \ln\left(12r - r^3\right) - \frac{r(12 - 3r^2)}{12r - r^3}\right]\frac{dr}{dt}$$ 1M

$$\left.\frac{dA}{dt}\right|_{r=2} = \frac{1}{2}\left[24(2) - 4(2)^3 - \ln\left(12(2) - 2^3\right) - \frac{2(12 - 3(2)^2)}{12(2) - 2^3}\right](4)$$ 1M

$$= 32 - 2\ln 16$$ 1A

Required rate is $32 - 2\ln 16$ square units per minute.

53. (a) $\angle POQ = \frac{\pi}{2}$

$\angle OQP = \pi - \frac{\pi}{2} - \theta = \frac{\pi}{2} - \theta$

$\angle QCR = \pi - 2\left(\frac{\pi}{2} - \theta\right) = 2\theta$ 1M

$A = \pi(1)^2 \times \frac{2\theta}{2\pi} - \frac{1}{2}(1)^2\sin 2\theta$ 1M+1M

$= \theta - \frac{1}{2}\sin 2\theta$ 1

(b) $x = \frac{2}{\tan\theta}$ 1A

(c) When $x = 2\sqrt{3}$,

$$2\sqrt{3} = \frac{2}{\tan\theta}$$

$$\theta = \frac{\pi}{6}$$ 1A

Consider the equation $x = \frac{2}{\tan\theta}$.

$$\tan\theta = \frac{2}{x}$$

$$\sec^2\theta\frac{d\theta}{dt} = -\frac{2}{x^2}\frac{dx}{dt}$$ 1M

$$\sec^2\frac{\pi}{6}\frac{d\theta}{dt} = -\frac{2}{(2\sqrt{3})^2}(-2)$$ 1A

$$\frac{d\theta}{dt} = \frac{1}{4}$$

Consider the equation $A = \theta - \frac{1}{2}\sin 2\theta$.

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}\theta}{\mathrm{d}t} - \cos 2\theta \frac{\mathrm{d}\theta}{\mathrm{d}t} \qquad \text{1M}$$

$$= \frac{1}{4} - \cos\frac{\pi}{3} \cdot \frac{1}{4} \qquad \text{1M}$$

$$= \frac{1}{8} \qquad \text{1A}$$

Required rate is $\frac{1}{8}$ m/s^2.

54. (a) The coordinates of S are $\left(a,\ e^{a^2+a}\right)$.

$$y = e^{x^2+x}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = e^{x^2+x}(2x+1) \qquad \text{1M}$$

Let $(r,\ 0)$ be the coordinates of R.

$$\frac{0 - e^{a^2+a}}{r-a} = e^{a^2+a}(2a+1) \qquad \text{1M}$$

$$r = a - \frac{1}{2a+1} \qquad \text{1A}$$

(b) $$A = \frac{1}{2}\left[a - \left(a - \frac{1}{2a+1}\right)\right] e^{a^2+a} \qquad \text{1M+1M}$$

$$= \frac{e^{a^2+a}}{2(2a+1)}$$

$$\frac{\mathrm{d}A}{\mathrm{d}a} = \frac{e^{a^2+a}(2a+1)(2a+1) - e^{a^2+a}(2)}{2(2a+1)^2} \qquad \text{1M}$$

$$= e^{a^2+a}\left[\frac{1}{2} - \frac{1}{(2a+1)^2}\right] \qquad \text{1A}$$

(c) Note that $ST = e^{a^2+a}$.

$$\frac{\mathrm{d}}{\mathrm{d}t}(ST) \le 2$$

$$e^{a^2+a}(2a+1)\frac{\mathrm{d}a}{\mathrm{d}t} \le 2 \qquad \text{1M}$$

$$\frac{\mathrm{d}a}{\mathrm{d}t} \le \frac{2}{(2a+1)e^{a^2+a}}$$

Note that for $a > \frac{1}{2}$, $\frac{1}{2} - \frac{1}{(2a+1)^2} > 0$.

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}a} \cdot \frac{\mathrm{d}a}{\mathrm{d}t} \qquad \text{1M}$$

$$= e^{a^2+a}\left[\frac{1}{2} - \frac{1}{(2a+1)^2}\right] \cdot \frac{\mathrm{d}a}{\mathrm{d}t}$$

$$\le e^{a^2+a}\left[\frac{1}{2} - \frac{1}{(2a+1)^2}\right] \cdot \frac{2}{(2a+1)e^{a^2+a}} \qquad \text{1M}$$

$$\frac{\mathrm{d}A}{\mathrm{d}t} \le \frac{1}{2a+1} - \frac{2}{(2a+1)^3}$$

Let $f(a) = \dfrac{1}{2a+1} - \dfrac{2}{(2a+1)^3}$.

$$f'(a) = -\frac{2}{(2a+1)^2} + \frac{12}{(2a+1)^4} \qquad \text{1M}$$
$$= \frac{-2(4a^2+4a-5)}{(2a+1)^4}$$

When $f'(a) = 0$, $a = \dfrac{-1+\sqrt{6}}{2}$ or $\dfrac{-1-\sqrt{6}}{2}$ (rejected). 1M

a	$\frac{1}{2} < a < \frac{-1+\sqrt{6}}{2}$	$a > \frac{-1+\sqrt{6}}{2}$
$f'(a)$	+	−

1M

Maximum value of $f(a)$ is $f\left(\dfrac{-1-\sqrt{6}}{2}\right) = \dfrac{\sqrt{6}}{9} < 0.3$. 1M

We have $\dfrac{dA}{dt} \le f(a) < 0.3$.

The claim is agreed. 1A

55. (a) (i) $T = \dfrac{\sqrt{3^2+x^2}}{10} + \dfrac{20-4-x}{24} + \dfrac{\sqrt{3^2+4^2}}{\left(\frac{600}{16-x}\right)}$ 1M

$$= \frac{\sqrt{9+x^2}}{10} - \frac{x}{20} + \frac{4}{5} \qquad \text{1}$$

(ii) $\dfrac{dT}{dx} = \dfrac{1}{10} \cdot \dfrac{2x}{2\sqrt{9+x^2}} - \dfrac{1}{20}$ 1M

$$= \frac{x}{10\sqrt{9+x^2}} - \frac{1}{20}$$

When $\dfrac{dT}{dx} = 0$,

$$\frac{x}{10\sqrt{9+x^2}} = \frac{1}{20}$$
$$(2x)^2 = (\sqrt{9+x^2})^2$$
$$x^2 = 3$$
$$x = \sqrt{3} \quad \text{or} \quad -\sqrt{3} \text{ (rejected)}$$

x	$0 < x < \sqrt{3}$	$\sqrt{3} < x < 16$
$\frac{dT}{dx}$	−	+

1M

T attains its minimum at $x = \sqrt{3}$.

Minimum value of $T = \dfrac{\sqrt{9+3}}{10} - \dfrac{\sqrt{3}}{20} + \dfrac{4}{5}$

$$= \frac{3\sqrt{3}+16}{20} \qquad \text{1A}$$

$SM = \sqrt{3^2+3}$

$= \sqrt{12}$ km 1A

(b) (i) $SC = (\sqrt{12})(\sqrt{3}) = 6\,\text{km}$ 1M

$CN = \sqrt{3^2 + (16-6)^2} = \sqrt{109}\,\text{km}$

$\sin\beta = \dfrac{3}{\sqrt{109}}$ and $\cos\beta = \dfrac{10}{\sqrt{109}}$. 1A

$$\frac{RN}{\sin\alpha} = \frac{CN}{\sin\angle CRN} \quad \text{1M}$$

$$\begin{aligned} RN &= \frac{\sqrt{109}\sin\alpha}{\sin(\pi-\alpha-\beta)} \\ &= \frac{\sqrt{109}\sin\alpha}{\sin(\alpha+\beta)} \\ &= \frac{\sqrt{109}\sin\alpha}{\sin\alpha\cos\beta + \sin\beta\cos\alpha} && \text{1M} \\ &= \frac{\sqrt{109}\sin\alpha}{\sin\alpha\left(\frac{10}{\sqrt{109}}\right) + \cos\alpha\left(\frac{3}{\sqrt{109}}\right)} \\ &= \frac{109\tan\alpha}{10\tan\alpha + 3} && \text{1} \end{aligned}$$

(ii) $$\frac{dRN}{dt} = 109\frac{\sec^2\alpha(10\tan\alpha+3) - 10\sec^2\alpha\tan\alpha}{(10\tan\alpha+3)^2}\cdot\frac{d\alpha}{dt} \quad \text{1M}$$

$$= \frac{327\sec^2\alpha}{(10\tan\alpha+3)^2}\cdot\frac{d\alpha}{dt}$$

$$\frac{dRN}{dt} = \frac{-24}{3600} = -\frac{1}{150}\,\text{km/s}$$

$$-\frac{1}{150} = \frac{327\sec^2 1}{(10\tan 1 + 3)^2}\cdot\frac{d\alpha}{dt}$$

$$\frac{d\alpha}{dt} \approx -0.0021$$

Required rate is −0.0021 rad/s. 1A

56. (a) Volume of the pyramid $= \dfrac{1}{3}(12)^2(6) = 288\,\text{cm}^3$.

Volume of pyramid below water surface

$$= 288\left[1 - \left(\frac{6-h}{6}\right)^3\right] \quad \text{1M}$$

$$= \frac{4}{3}[6-(6-h)][6^2 + 6(6-h) + (6-h)^2]$$

$$= \frac{4h^3}{3} - 24h^2 + 144h$$

$$V = 12h^2 - \left(\frac{4h^3}{3} - 24h^2 + 144h\right) \quad \text{1M}$$

$$= 24h^2 - \frac{4}{3}h^3 \quad \text{1}$$

(b) $$\frac{dV}{dt} = 48h\frac{dh}{dt} - 4h^2\frac{dh}{dt} \quad \text{1M}$$

Put $\frac{dh}{dt} = 1$,

$$\frac{dV}{dt} = 48h - 4h^2 > 140$$

$$-4h^2 + 48h - 140 > 0 \qquad \text{1M}$$

$$5 < h < 7 \qquad \text{1M}$$

$h \leq 5$ by considering the height of the container. It is not possible to have $\frac{dV}{dt} > 140$.

The claim is disagreed. 1A

(c) Put $\frac{dV}{dt} = -(12 - h)(\ln h + 1)^2$,

$$-(12 - h)(\ln h + 1)^2 = (48h - 4h^2)\frac{dh}{dt} \qquad \text{1M}$$

$$\frac{dh}{dt} = -\frac{(\ln h + 1)^2}{4h}$$

$$\frac{d}{dh}\left(\frac{dh}{dt}\right) = \frac{2(\ln h + 1)\left(\frac{1}{h}\right)(h) - (\ln h + 1)^2}{-4h^2} \qquad \text{1M}$$

$$= \frac{(\ln h + 1)(\ln h - 1)}{4h^2}$$

When $\frac{d}{dh}\left(\frac{dh}{dt}\right) = 0$,

$$\ln h = \pm 1$$

$$h = e \quad \text{or} \quad \frac{1}{e}$$

h	$0 < h < \frac{1}{e}$	$\frac{1}{e} < h < e$	$e < h < 5$
$\frac{d}{dh}\left(\frac{dh}{dt}\right)$	+	−	+

1M

$\left.\frac{dh}{dt}\right|_{h=\frac{1}{e}} = 0$ and $\left.\frac{dh}{dt}\right|_{h=e} = -\frac{1}{e}$ and $\left.\frac{dh}{dt}\right|_{h=5} = -\frac{(\ln 5 + 1)^2}{20}$.

When depth of water decreases from 5 cm to e cm, the rate of change of the depth of water decreases from $-\frac{(\ln 5 + 1)^2}{20}$ cm/s to a local minimum $-\frac{1}{e}$ cm/s. 1M

When the depth of water decreases from e cm to $\frac{1}{e}$ cm, the rate of change of the depth of water increases from $-\frac{1}{e}$ cm/s to a local maximum 0 cm/s.

When the depth of water decreases from $\frac{1}{e}$ cm to 0 cm, the rate of change of the depth of water decreases indefinitely. 1A

57. (a) $AD = 2\cos\theta + 2 + 2\cos\theta$ 1M

$$= (4\cos\theta + 2) \text{ m}$$

$V = \left[\frac{1}{2}(AD + BC)(BX)\right](5)$

$= \frac{1}{2}(4\cos\theta + 2 + 2)(2\sin\theta)(5)$ 1M

$= 20(\sin\theta\cos\theta + \sin\theta)\ \text{m}^3$ 1

(b) $\frac{dV}{d\theta} = 20[\sin\theta(-\sin\theta) + (\cos\theta)\cos\theta + \cos\theta]$ 1M

$= 20(-(1 - \cos^2\theta) + \cos^2\theta + \cos\theta)$

$= 20(2\cos^2\theta + \cos\theta - 1)$

$= 20(2\cos\theta - 1)(\cos\theta + 1)$

When $\frac{dV}{d\theta} = 0$, 1M

$\cos\theta = \frac{1}{2}$ or -1 (rejected)

$\theta = \frac{\pi}{3}$

θ	$0 < \theta < \frac{\pi}{3}$	$\frac{\pi}{3} < \theta < \frac{\pi}{2}$
$\frac{dV}{d\theta}$	+	−

1M

V is maximum at $\theta = \frac{\pi}{3}$.

Required value of θ is $\frac{\pi}{3}$. 1A

(c) (i) Let h m and Y m^3 be the depth of water and the volume of water.

Consider the cross section of the water in the plane $ABCD$.

Length of water surface

$= \frac{h}{\tan\frac{\pi}{3}} + 2 + \frac{h}{\tan\frac{\pi}{3}}$ 1M

$= \left(\frac{2h}{\sqrt{3}} + 2\right)$ m

Consider the volume of water.

$W = \left[\frac{1}{2}\left(\frac{2h}{\sqrt{3}} + 2 + 2\right)(h)\right](5)$ 1M

$= \frac{5h^2}{\sqrt{3}} + 10h$

$\frac{dW}{dt} = \left(\frac{10h}{\sqrt{3}} + 10\right)\frac{dh}{dt}$ 1M

$5 = \left[\frac{10}{\sqrt{3}}\left(\frac{\sqrt{3}}{2}\right) + 10\right]\frac{dh}{dt}$

$\frac{dh}{dt} = \frac{1}{3}$ 1A

Required rate is $\frac{1}{3}$ m/h.

(ii) Area of the water surface

$$= \left(\frac{2h}{\sqrt{3}} + 2\right) \times 5 \qquad \text{1M}$$

$$= \left(\frac{10h}{\sqrt{3}} + 10\right) \text{ m}^2$$

Note that $\frac{\text{d}h}{\text{d}t}$ is a constant.

$$\frac{\text{d}W}{\text{d}t} = \left(\frac{10h}{\sqrt{3}} + 10\right)\frac{\text{d}h}{\text{d}t}$$

$$= (\text{area of water surface})\frac{\text{d}h}{\text{d}t}$$

The claim is agreed. 1A

58. (a) The coordinates of P are $(r,\ \ln(2r+1))$.

$$y = \ln(2x+1)$$

$$\frac{\text{d}y}{\text{d}x} = \frac{2}{2x+1} \qquad \text{1M}$$

Let $(q,\ 0)$ be the coordinates of Q.

$$\frac{0 - \ln(2r+1)}{q-r} = -1 \div \frac{2}{2r+1} \qquad \text{1M}$$

$$\frac{2\ln(2r+1)}{2r+1} = q - r$$

$$q = \frac{2\ln(2r+1)}{2r+1} + r \qquad \text{1}$$

(b) Let A be the area of $\triangle PQR$.

$$A = \frac{1}{2}\left[\left(\frac{2\ln(2r+1)}{2r+1} + r\right) - r\right]\ln(2r+1) \qquad \text{1M}$$

$$= \frac{[\ln(2r+1)]^2}{2r+1} \qquad \text{1A}$$

$$\frac{\text{d}A}{\text{d}r} = \frac{(2r+1)(2)\left[\frac{2\ln(2r+1)}{2r+1}\right] - [\ln(2r+1)]^2(2)}{(2r+1)^2} \qquad \text{1M}$$

$$= \frac{2\ln(2r+1)[2 - \ln(2r+1)]}{(2r+1)^2}$$

When $\frac{\text{d}A}{\text{d}r} = 0$,

$$\ln(2r+1) = 0 \quad \text{or} \quad 2$$

$$r = 0 \text{ (rejected)} \quad \text{or} \quad \frac{e^2-1}{2}$$

r	$0 < r < \frac{e^2-1}{2}$	$r > \frac{e^2-1}{2}$
$\frac{\text{d}A}{\text{d}r}$	+	−

1M

A is maximum when $r = \dfrac{e^2 - 1}{2}$.

Required area

$$= \frac{\left[\ln\left(2\left(\frac{e^2-1}{2}\right) + 1\right)\right]^2}{2\left(\frac{e^2-1}{2}\right) + 1}$$

$$= \frac{4}{e^2}$$ 1A

(c) $OP = \sqrt{r^2 + [\ln(2r+1)]^2}$ 1M

$$\frac{\mathrm{d}OP}{\mathrm{d}t} = \frac{1}{2\sqrt{r^2 + [\ln(2r+1)]^2}}\left[2r + \frac{4\ln(2r+1)}{2r+1}\right]\frac{\mathrm{d}r}{\mathrm{d}t}$$ 1M

When $r = \dfrac{e-1}{2}$,

$$\frac{\mathrm{d}OP}{\mathrm{d}t} = \frac{1}{2\sqrt{\left(\frac{e-1}{2}\right)^2 + (\ln e)^2}}\left[(e-1) + \frac{4\ln e}{e}\right]\frac{\mathrm{d}r}{\mathrm{d}t}$$

$$= \frac{1}{\sqrt{(e-1)^2 + 4}}\left(\frac{e^2 - e + 4}{e}\right)\frac{\mathrm{d}r}{\mathrm{d}t}$$

Consider the rate of change the area of $\triangle PQR$ when $r = \dfrac{e-1}{2}$.

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}r}\cdot\frac{\mathrm{d}r}{\mathrm{d}t}$$

$$= \frac{2\ln e[2 - \ln e]}{e^2}\cdot\frac{e\sqrt{(e-1)^2+4}}{e^2 - e + 4}\cdot\frac{\mathrm{d}OP}{\mathrm{d}t}$$ 1M

$$= \frac{2\sqrt{(e-1)^2+4}}{e(e^2 - e + 4)}\cdot\frac{\mathrm{d}OP}{\mathrm{d}t}$$

Note that $0 \le \dfrac{\mathrm{d}OP}{\mathrm{d}t} \le \dfrac{e^2 - e + 4}{e}$.

$$0 \le \frac{\mathrm{d}A}{\mathrm{d}t} \le \frac{2\sqrt{(e-1)^2+4}}{e(e^2 - e + 4)}\cdot\frac{e^2 - e + 4}{e}$$

$$0 \le \frac{\mathrm{d}A}{\mathrm{d}t} \le \frac{2\sqrt{(e-1)^2+4}}{e^2}$$

$$0 \le \frac{\mathrm{d}A}{\mathrm{d}t} < \frac{6}{e^2}$$

The claim is correct. 1A

END OF PAPER