1. (a) $\frac{1}{4}(\cos x + \cos 3x + \cos 7x + \cos 9x) = \frac{1}{4}(\cos x + \cos 9x) + \frac{1}{4}(\cos 3x + \cos 7x)$

$= \frac{1}{2}\cos 5x \cos 4x + \frac{1}{2}\cos 5x \cos 2x$ 1M

$= \frac{\cos 5x}{2}(\cos 4x + \cos 2x)$

$= \cos 5x(\cos 3x \cos x)$ 1M

$= \cos x \cos 3x \cos 5x$ 1

(b) $\int_0^{\frac{\pi}{4}} 8\cos x \cos 3x \cos 5x \, \mathrm{d}x = 2\int_0^{\frac{\pi}{4}} (\cos x + \cos 3x + \cos 7x + \cos 9x)\, \mathrm{d}x$ 1M

$= 2\left[\sin x + \frac{\sin 3x}{3} + \frac{\sin 7x}{7} + \frac{\sin 9x}{9}\right]_0^{\frac{\pi}{4}}$ 1A

$= 2\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{6} - \frac{\sqrt{2}}{14} + \frac{\sqrt{2}}{18}\right) - 0$

$= \frac{82\sqrt{2}}{63}$ 1A

2. (a) $\frac{1}{\sqrt{x+2} - \sqrt{x-1}} = \frac{\sqrt{x+2} + \sqrt{x-1}}{(x+2) - (x-1)}$

$= \frac{\sqrt{x+2} + \sqrt{x-1}}{3}$ 1A

(b) $\int_2^7 \frac{\mathrm{d}x}{\sqrt{x+2} - \sqrt{x-1}} = \frac{1}{3}\int_2^7 (\sqrt{x+2} + \sqrt{x-1})\, \mathrm{d}x$ 1M

$= \frac{1}{3}\left[\frac{2(x+2)^{\frac{3}{2}}}{3} + \frac{2(x-1)^{\frac{3}{2}}}{3}\right]_2^7$ 1A

$= \frac{1}{3}\left[(18 + 4\sqrt{6}) - \left(\frac{16}{3} + \frac{2}{3}\right)\right]$

$= 4 + \frac{4\sqrt{6}}{3}$ 1A

3. $\int \sin 3x \cos 8x \, \mathrm{d}x = \frac{1}{2}\int (\sin 11x - \sin 5x)\, \mathrm{d}x$ 1M

$= -\frac{\cos 11x}{22} + \frac{\cos 5x}{10} + \text{constant}$ 1A

4. $\displaystyle\int_1^4 \frac{3x}{1+e^x}\,\mathrm{d}x - 3\int_4^1 \frac{xe^x}{1+e^x}\,\mathrm{d}x$

$$= \int_1^4 \frac{3x}{1+e^x} + 3\int_1^4 \frac{xe^x}{1+e^x}\,\mathrm{d}x \qquad \text{1M}$$

$$= \int_1^4 \frac{3x(1+e^x)}{1+e^x}\,\mathrm{d}x$$

$$= \int_1^4 3x\,\mathrm{d}x$$

$$= \left[\frac{3x^2}{2}\right]_1^4$$

$$= \frac{45}{2} \qquad \text{1A}$$

5. $\displaystyle\int_0^{\frac{\pi}{4}} \sin^4 x\,\mathrm{d}x = \frac{1}{4}\int_0^{\frac{\pi}{4}} (1-\cos 2x)^2\,\mathrm{d}x$ 1M

$$= \frac{1}{4}\int_0^{\frac{\pi}{4}} (1 - 2\cos 2x + \cos^2 2x)\,\mathrm{d}x$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{4}} \left[1 - 2\cos 2x + \frac{1}{2}(1+\cos 4x)\right]\mathrm{d}x \qquad \text{1M}$$

$$= \frac{1}{4}\left[\frac{3x}{2} - \sin 2x + \frac{\sin 4x}{8}\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{4}\left(\frac{3\pi}{8} - 1 + 0\right) - \frac{1}{4}(0)$$

$$= \frac{3\pi}{32} - \frac{1}{4} \qquad \text{1A}$$

6. (a) $\displaystyle\frac{\cos x + \cos 9x}{\cos 5x} = \frac{2\cos 5x\cos 4x}{\cos 5x}$ 1M

$$= 2\cos 4x \qquad \text{1}$$

(b) $\displaystyle\int_0^{\frac{\pi}{16}} \frac{(\cos x + \cos 9x)\sin 4x}{\cos 5x}\,\mathrm{d}x = \int_0^{\frac{\pi}{16}} 2\cos 4x\sin 4x\,\mathrm{d}x$ 1M

$$= \int_0^{\frac{\pi}{16}} \sin 8x\,\mathrm{d}x \qquad \text{1M}$$

$$= -\frac{1}{8}\Big[\cos 8x\Big]_0^{\frac{\pi}{16}} \qquad \text{1A}$$

$$= -\frac{1}{8}(0-1)$$

$$= \frac{1}{8} \qquad \text{1A}$$

7. (a) $$\frac{d}{dx}\left[\ln\left(x+\sqrt{x^2+9}\right)\right]=\frac{1+\frac{1}{2}(x^2+9)^{-\frac{1}{2}}(2x)}{x+\sqrt{x^2+9}}$$ 1M

$$=\frac{\sqrt{x^2+9}+x}{(x+\sqrt{x^2+9})\sqrt{x^2+9}}$$

$$=\frac{1}{\sqrt{x^2+9}}$$ 1A

(b) $$\int_0^1 \frac{dx}{\sqrt{x^2+9}}=\left[\ln\left(x+\sqrt{x^2+9}\right)\right]_0^1$$ 1M

$$=\ln\left(1+\sqrt{10}\right)-\ln 3$$ 1A

8. $$\int_{-\pi}^{\frac{\pi}{2}}\frac{e^{2x}-\pi e^x-2\pi^2}{e^x+\pi}\,dx=\int_{-\pi}^{\frac{\pi}{2}}\frac{e^x(e^x+\pi)-2\pi(e^x+\pi)}{e^x+\pi}\,dx$$

$$=\int_{-\pi}^{\frac{\pi}{2}}(e^x-2\pi)\,dx$$ 1M

$$=\left[e^x-2\pi x\right]_{-\pi}^{\frac{\pi}{2}}$$

$$=e^{\frac{\pi}{2}}-e^{-\pi}-3\pi^2$$ 1A

9. $$\int_0^2\frac{1}{\sqrt{2x+1}-\sqrt{2x}}\,dx=\int_0^2\left(\frac{1}{\sqrt{2x+1}-\sqrt{2x}}\times\frac{\sqrt{2x+1}+\sqrt{2x}}{\sqrt{2x+1}+\sqrt{2x}}\right)dx$$ 1M

$$=\int_0^2(\sqrt{2x+1}+\sqrt{2x})\,dx$$

$$=\left[\frac{(2x+1)^{\frac{3}{2}}}{3}+\frac{(2x)^{\frac{3}{2}}}{3}\right]_0^2$$ 1A

$$=\frac{5\sqrt{5}+7}{3}$$ 1A

10. $$\int_{-2}^{2}(3x+2)(7x+5)\,dx=\int_{-2}^{2}(21x^2+29x+10)\,dx$$ 1M

$$=\left[7x^3+\frac{29x^2}{2}+10x\right]_{-2}^{2}$$

$$=152$$ 1A

11. $$\int_0^{\frac{\pi}{4}}\tan^2 x\,dx-\int_{\frac{\pi}{4}}^{0}dx=\int_0^{\frac{\pi}{4}}\tan^2 x\,dx+\int_0^{\frac{\pi}{4}}dx$$

$$=\int_0^{\frac{\pi}{4}}\sec^2 x\,dx$$ 1A

$$=\left[\tan x\right]_0^{\frac{\pi}{4}}$$

$$=1$$ 1A

12. (a) $x + h + \dfrac{k}{x+1} = \dfrac{(x+h)(x+1)+k}{x+1}$

$= \dfrac{x^2 + (h+1)x + (h+k)}{x+1}$

So, $h + 1 = 0$ and $h + k = 1$. 1M

Solving, we have $h = -1$ and $k = 2$. 1A

(b) $\displaystyle\int_0^1 \frac{x^2+1}{x+1}\,dx = \int_0^1 \left(x - 1 + \frac{2}{x+1}\right) dx$ 1M

$= \left[\dfrac{x^2}{2} - x + 2\ln|x+1|\right]_0^1$ 1A

$= \left(\dfrac{1}{2} - 1 + 2\ln 2\right) - (0 - 0 + 2\ln 1)$

$= 2\ln 2 - \dfrac{1}{2}$ 1A

13. (a) $\displaystyle\int_0^1 (x+1)^4\,dx = \left[\frac{(x+1)^5}{5}\right]_0^1$ 1A

$= \dfrac{2^5}{5} - \dfrac{1}{5}$

$= \dfrac{31}{5}$ 1A

$\displaystyle\int_0^1 (x+1)^5\,dx = \left[\frac{(x+1)^6}{6}\right]_0^1$

$= \dfrac{2^6}{6} - \dfrac{1}{6}$

$= \dfrac{21}{2}$ 1A

(b) $\displaystyle\int_0^1 30x(x+1)^4\,dx = 30\int_0^1 [(x+1)(x+1)^4 - (x+1)^4]\,dx$ 1M

$\displaystyle= 30\int_0^1 (x+1)^5\,dx - 30\int_0^1 (x+1)^4\,dx$

$= 30 \times \dfrac{21}{2} - 30 \times \dfrac{31}{5}$

$= 129$ 1A

14. $\displaystyle\int_0^{n\pi} (\cos\theta + 1)\,d\theta = \left[\sin\theta + \theta\right]_0^{n\pi}$ 1M

$= \sin n\pi + n\pi$

$= n\pi$ 1

15. (a) $\displaystyle\int_0^4 \sqrt{2x+1}\,dx = \left[\frac{(2x+1)^{\frac{3}{2}}}{3}\right]_0^4$ 1A

$= 9 - \dfrac{1}{3}$

$= \dfrac{26}{3}$ 1A

$$\int_0^4 \frac{dx}{\sqrt{2x+1}} = \left[\sqrt{2x+1}\right]_0^4$$

$$= 3 - 1$$

$$= 2$$ 1A

(b) $$\int_0^4 \frac{6x\,dx}{\sqrt{2x+1}} = 3\int_0^4 \frac{2x+1-1}{\sqrt{2x+1}}\,dx$$

$$= 3\int_0^4 \sqrt{2x+1}\,dx - 3\int_0^4 \frac{dx}{\sqrt{2x+1}}$$ 1M

$$= 3 \times \frac{26}{3} - 3 \times 2$$

$$= 20$$ 1A

16. $$\int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1}{2(1+\sin x)}\,dx + \int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos 2x}{2(1+\sin x)}\,dx$$

$$= \int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1+(1-2\sin^2 x)}{2(1+\sin x)}\,dx$$ 1M

$$= \int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{(1+\sin x)(1-\sin x)}{1+\sin x}\,dx$$

$$= \int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} (1-\sin x)\,dx$$ 1A

$$= \left[x + \cos x\right]_{-\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$$ 1A

17. (a) $$\frac{d}{dx}[\ln(1+e^x)] = \frac{e^x}{1+e^x}$$ 1A

(b) (i) $$\int_0^1 \frac{e^x}{1+e^x}\,dx = \left[\ln(1+e^x)\right]_0^1$$ 1M

$$= \ln(1+e) - \ln 2$$ 1A

(ii) $$\int_0^1 \frac{dx}{1+e^{-x}} = \int_0^1 \frac{e^x}{e^x+1}\,dx$$ 1M

$$= \ln(1+e) - \ln 2$$ 1A

(iii) $$\int_0^1 \frac{dx}{1+e^x} = \int_0^1 \left(1 - \frac{e^x}{1+e^x}\right)dx$$ 1M

$$= \left[x - \ln(1+e^x)\right]_0^1$$

$$= [1 - \ln(1+e)] + \ln 2$$

$$= 1 - \ln(1+e) + \ln 2$$ 1A

18. $\displaystyle\int_0^{2n\pi} \sin\theta \, d\theta = \Big[-\cos\theta \Big]_0^{2n\pi}$ 1M

$= -\cos(2n\pi) + 1$

$= 0$ 1A

19. (a) $\sin(2x + x) = \sin 2x \cos x + \cos 2x \sin x$ 1M

$= 2\sin x \cos^2 x + (1 - 2\sin^2 x)\sin x$ 1M

$= 2\sin x(1 - \sin^2 x) + \sin x - 2\sin^3 x$

$= 3\sin x - 4\sin^3 x$ 1

(b) $\displaystyle\int_0^{\frac{\pi}{6}} 8\sin^3 x \, dx = \int_0^{\frac{\pi}{6}} (6\sin x - 2\sin 3x) \, dx$ 1M

$\displaystyle = \left[-6\cos x + \frac{2\cos 3x}{3} \right]_0^{\frac{\pi}{6}}$ 1A

$\displaystyle = \left(-3\sqrt{3} + 0\right) - \left(-6 + \frac{2}{3}\right)$

$\displaystyle = \frac{16}{3} - 3\sqrt{3}$ 1A

20. $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos 2x}{\cos x + \sin x} \, dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos^2 x - \sin^2 x}{\cos x + \sin x} \, dx$

$\displaystyle = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\cos x - \sin x) \, dx$ 1M

$\displaystyle = \Big[\sin x + \cos x \Big]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$ 1A

$\displaystyle = (1 + 0) - \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \right)$

$= 1 - \sqrt{2}$ 1A

21. $\displaystyle\int_1^2 \frac{1 + 2x}{2x - 1} \, dx = \int_1^2 \left(\frac{2x - 1}{2x - 1} + \frac{2}{2x - 1} \right) dx$ 1M

$= \Big[x + \ln|2x - 1| \Big]_1^2$ 1A

$= (2 + \ln 3) - (1 + \ln 1)$

$= 1 + \ln 3$ 1A

22. $\displaystyle\int_h^k (x-h)(x-k)\,\mathrm{d}x = \int_h^k (x^2-(h+k)x+hk)\,\mathrm{d}x$ 1M

$$=\left[\frac{x^3}{3}-\frac{(h+k)x^2}{2}+hkx\right]_h^k$$

$$=\frac{2(k^3-h^3)-3(h+k)(k^2-h^2)+6hk(k-h)}{6}$$

$$=\frac{h^3-3h^2k+3hk^2-k^3}{6}$$

$$=\frac{(h-k)^3}{6}$$ 1

23. $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{1+\cos 2x}\,\mathrm{d}x = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{1+(2\cos^2 x-1)}\,\mathrm{d}x$ 1M

$$=\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec^2 x\,\mathrm{d}x$$ 1A

$$=\frac{1}{2}\Big[\tan x\Big]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$=\frac{\sqrt{3}}{3}$$ 1A

24. $\displaystyle\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\mathrm{d}x}{\tan\frac{x}{2}+\cot\frac{x}{2}} = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\mathrm{d}x}{\tan\frac{x}{2}+\cot\frac{x}{2}} \times \frac{\sin\frac{x}{2}\cos\frac{x}{2}}{\sin\frac{x}{2}\cos\frac{x}{2}}\,\mathrm{d}x$

$$=\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\sin\frac{x}{2}\cos\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}}\,\mathrm{d}x$$

$$=\frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin x\,\mathrm{d}x$$ 1A

$$=-\frac{1}{2}\Big[\cos x\Big]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$$

$$=\frac{1}{4}$$ 1A

25. $\displaystyle\int_{-\pi}^{-\frac{\pi}{3}} (\sin 2x\cos x-\cos 2x\sin x)\,\mathrm{d}x = \int_{-\pi}^{-\frac{\pi}{3}} \sin(2x-x)\,\mathrm{d}x$ 1M

$$=\Big[-\cos x\Big]_{-\pi}^{-\frac{\pi}{3}}$$

$$=-\cos\left(-\frac{\pi}{3}\right)+\cos(-\pi)$$

$$=-\frac{3}{2}$$ 1A

26. $\int_0^4 f(x)\,dx + \int_5^2 f(x)\,dx = \int_0^2 f(x)\,dx + \int_2^4 f(x)\,dx - \int_2^5 f(x)\,dx$ 1M

$$= \int_0^2 f(x)\,dx - \left(\int_4^2 f(x)\,dx + \int_2^5 f(x)\,dx\right)$$

$$= \int_0^2 f(x)\,dx - \int_4^5 f(x)\,dx$$ 1M

$$= k$$ 1A

27. $\int_1^3 (x-2)\left(4+\frac{3}{x^2}\right) dx = \int_1^3 (4x - 8 + 3x^{-1} - 6x^{-2})\,dx$ 1M

$$= \left[2x^2 - 8x + 3\ln|x| + 6x^{-1}\right]_1^3$$

$$= 3\ln 3 - 4$$ 1A

28. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (e^x + e^{-x})^2 \cos x\,dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (e^{2x} + e^{-2x} + 2)\cos x\,dx$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (e^{2x} + e^{-2x})\cos x\,dx + 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\,dx$$ 1M

$$= k + 2\left[\sin x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$ 1A

$$= k + 2[1 - (-1)]$$

$$= k + 4$$ 1A

29. (a) $\frac{A}{x+1} + \frac{B}{x+3} = \frac{A(x+3) + B(x+1)}{(x+1)(x+3)}$

$$= \frac{(A+B)x + (3A+B)}{x^2+4x+3}$$

So, $A + B = 0$ and $3A + B = 1$ 1M

Solving, we have $A = \frac{1}{2}$ and $B = -\frac{1}{2}$. 1A

(b) $\int_1^2 \frac{dx}{x^2+4x+3} = \frac{1}{2}\int_1^2 \left(\frac{1}{x+1} - \frac{1}{x+3}\right) dx$ 1M

$$= \frac{1}{2}\left[\ln|x+1| - \ln|x+3|\right]_1^2$$ 1A

$$= \frac{1}{2}[(\ln 3 - \ln 5) - (\ln 2 - \ln 4)]$$

$$= \frac{1}{2}\ln\frac{6}{5}$$ 1A

30. (a) Let $u = 1 - e^{-x}$. Then $\mathrm{d}u = e^{-x}\,\mathrm{d}x$. 1M

$$\int_{\ln 2}^{\ln 3} \frac{e^{-x}}{1-e^{-x}}\,\mathrm{d}x = \int_{\frac{1}{2}}^{\frac{2}{3}} \frac{\mathrm{d}u}{u} \quad \text{1M}$$

$$= \Big[\ln|u|\Big]_{\frac{1}{2}}^{\frac{2}{3}} \quad \text{1A}$$

$$= \ln\frac{2}{3} - \ln\frac{1}{2}$$

$$= \ln\frac{4}{3} \quad \text{1A}$$

(b) $$\frac{A}{u} + \frac{B}{u-1} = \frac{A(u-1)+Bu}{u(u-1)}$$

$$= \frac{(A+B)u - A}{u(u-1)}$$

So, $A + B = 0$ and $-A = 1$. 1M

Solving, we have $A = -1$ and $B = 1$. 1A

(c) $$\int_{\ln 2}^{\ln 3} \frac{1}{e^x(e^x-1)}\,\mathrm{d}x = \int_{\ln 2}^{\ln 3}\left(\frac{1}{e^x-1} - \frac{1}{e^x}\right)\mathrm{d}x \quad \text{1M}$$

$$= \int_{\ln 2}^{\ln 3}\left(\frac{e^{-x}}{1-e^{-x}} - e^{-x}\right)\mathrm{d}x$$

$$= \ln\frac{4}{3} + \Big[e^{-x}\Big]_{\ln 2}^{\ln 3} \quad \text{1M+1A}$$

$$= \ln\frac{4}{3} + \left(\frac{1}{3} - \frac{1}{2}\right)$$

$$= \ln\frac{4}{3} - \frac{1}{6} \quad \text{1A}$$

31. Let $x = \sqrt{3}\tan\theta$. Then $\mathrm{d}x = \sqrt{3}\sec^2\theta\,\mathrm{d}\theta$. 1M

$$\int_{-3}^{3} \frac{1}{(x^2+3)^{\frac{3}{2}}}\,\mathrm{d}x = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{\sqrt{3}\sec^2\theta}{(3\tan^2\theta+3)^{\frac{3}{2}}}\,\mathrm{d}\theta \quad \text{1A}$$

$$= \frac{1}{3}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{\sec^2\theta}{\sec^3\theta}\,\mathrm{d}\theta$$

$$= \frac{1}{3}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \cos\theta\,\mathrm{d}\theta$$

$$= \frac{1}{3}\Big[\sin\theta\Big]_{-\frac{\pi}{3}}^{\frac{\pi}{3}}$$

$$= \frac{\sqrt{3}}{3} \quad \text{1A}$$

32. (a) $\dfrac{A(x-1)}{x^2-2x+2}+\dfrac{B(x+1)}{x^2+2x+2}$

$$= \frac{A(x-1)(x^2+2x+2)+B(x+1)(x^2-2x+2)}{(x^2-2x+2)(x^2+2x+2)}$$

$$= \frac{(A+B)x^2+(A-B)x+(-2A+2B)}{x^4+4}$$

We have $A+B=0$, $A-B=1$ and $-2A+2B=-2$. 1M

Solving, we have $A=\dfrac{1}{2}$ and $B=-\dfrac{1}{2}$. 1A

(b) Let $u=x^2-2x+2$. Then $\mathrm{d}u=(2x-2)\,\mathrm{d}x$. 1M

Let $w=x^2+2x+2$. Then $\mathrm{d}w=(2x+2)\,\mathrm{d}x$.

$$\int_0^1 \frac{x^2-2}{x^4+4}\,\mathrm{d}x = \int_0^1 \left(\frac{x-1}{2(x^2-2x+2)}-\frac{x+1}{2(x^2+2x+2)}\right)\mathrm{d}x \qquad \text{1M}$$

$$= \frac{1}{4}\int_2^1 \frac{\mathrm{d}u}{u}-\frac{1}{4}\int_2^5 \frac{\mathrm{d}w}{w}$$

$$= \frac{1}{4}\Big[\ln|u|\Big]_2^1-\frac{1}{4}\Big[\ln|w|\Big]_2^5 \qquad \text{1M}$$

$$= -\frac{1}{4}\ln 5 \qquad \text{1A}$$

33. Let $u=\sqrt{x}+1$. Then $\mathrm{d}u=\dfrac{1}{2\sqrt{x}}\,\mathrm{d}x$. 1M

$$\int_1^9 \frac{\mathrm{d}x}{x+\sqrt{x}} = 2\int_2^4 \frac{1}{u}\,\mathrm{d}u \qquad \text{1M}$$

$$= 2\Big[\ln|u|\Big]_2^4 \qquad \text{1A}$$

$$= 2(\ln 4-\ln 2)$$

$$= 2\ln 2 \qquad \text{1A}$$

34. (a) $\dfrac{A}{x}+\dfrac{B}{x-1}=\dfrac{A(x-1)+Bx}{x(x-1)}$

$$= \frac{(A+B)x-A}{x(x-1)}$$

We have $-A=1$ and $A+B=0$. 1M

Solving, we have $A=-1$ and $B=1$. 1A

(b) Let $u=1+e^x$. Then $\mathrm{d}u=e^x\,\mathrm{d}x$. 1M

$$\int_0^1 \frac{\mathrm{d}x}{1+e^x} = \int_2^{1+e} \frac{\mathrm{d}u}{u(u-1)} \qquad \text{1M}$$

$$= \int_2^{1+e}\left(-\frac{1}{u}+\frac{1}{u-1}\right)\mathrm{d}u$$

$$= \Big[-\ln|u|+\ln|u-1|\Big]_2^{1+e} \qquad \text{1M}$$

$$= 1+\ln 2-\ln(1+e) \qquad \text{1A}$$

35. (a) $\displaystyle\int \frac{1}{e^{4u}}\,\mathrm{d}u = \int e^{-4u}\,\mathrm{d}u$ 1M

$\displaystyle = -\frac{1}{4e^{4u}} + \text{constant}$ 1A

(b) Let $u = \sqrt{x}$. Then $\mathrm{d}u = \dfrac{1}{2\sqrt{x}}\,\mathrm{d}x$. 1M

$$\int_1^{16} \frac{1}{\sqrt{x}e^{4\sqrt{x}}}\,\mathrm{d}x = \int_1^4 \frac{2}{e^{4u}}\,\mathrm{d}u$$ 1A

$$= 2\left[-\frac{1}{4e^{4u}}\right]_1^4$$ 1M

$$= \frac{1}{2e^4} - \frac{1}{2e^{16}}$$ 1A

36. (a) $\sin^4 x + \cos^4 x = (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\cos^2 x$ 1M

$\displaystyle = 1 - \frac{1}{2}(2\sin x\cos x)^2$

$\displaystyle = 1 - \frac{1}{2}\sin^2 2x$ 1A

(b) $t = \tan 2x$

$\displaystyle 1 = 2\sec^2 2x\frac{\mathrm{d}x}{\mathrm{d}t}$ 1M

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{2(1+\tan^2 2x)}$$

$$= \frac{1}{2(1+t^2)}$$ 1

$$\int_0^{\frac{\pi}{8}} \frac{1}{\sin^4 x + \cos^4 x}\,\mathrm{d}x = \int_0^{\frac{\pi}{8}} \frac{1}{1-\frac{1}{2}\sin^2 2x}\,\mathrm{d}x$$ 1M

$$= \int_0^1 \frac{1}{1-\frac{1}{2}\left(\frac{t}{\sqrt{1+t^2}}\right)^2}\cdot\frac{1}{2(1+t^2)}\,\mathrm{d}t$$

$$= \int_0^1 \frac{1}{2(1+t^2)-t^2}\,\mathrm{d}t$$

$$= \int_0^1 \frac{1}{2+t^2}\,\mathrm{d}t$$

Let $t = \sqrt{2}\tan\theta$. Then $\mathrm{d}t = \sqrt{2}\sec^2\theta\,\mathrm{d}\theta$. 1M

$$\int_0^{\frac{\pi}{8}} \frac{1}{\sin^4 x + \cos^4 x}\,\mathrm{d}x = \int_0^{\tan^{-1}\frac{\sqrt{2}}{2}} \frac{\sqrt{2}\sec^2\theta}{2+2\tan^2\theta}\,\mathrm{d}\theta$$

$$= \frac{\sqrt{2}}{2}\int_0^{\tan^{-1}\frac{\sqrt{2}}{2}} \mathrm{d}\theta$$

$$= \frac{\sqrt{2}}{2}\Big[\theta\Big]_0^{\tan^{-1}\frac{\sqrt{2}}{2}}$$

$$= \frac{\sqrt{2}}{2}\tan^{-1}\frac{\sqrt{2}}{2}$$ 1A

37. Let $u = \sqrt{x}$. Then $\mathrm{d}u = \dfrac{1}{2\sqrt{x}}\,\mathrm{d}x$. 1M

$$\int_1^4 \frac{f(\sqrt{x})}{\sqrt{x}}\,\mathrm{d}x = 2\int_1^2 f(u)\,\mathrm{d}u \quad \text{1M}$$

$$= 2(9)$$

$$= 18 \quad \text{1A}$$

38. Let $u = x^4$. Then $\mathrm{d}u = 4x^3\,\mathrm{d}x$. 1M

$$\int_{-1}^0 x^3 f(x^4)\,\mathrm{d}x = \frac{1}{4}\int_1^0 f(u)\,\mathrm{d}u \quad \text{1M}$$

$$= \frac{1}{4}(-4)$$

$$= -1 \quad \text{1A}$$

39. (a) Let $u = 1 + x^3$. Then $\mathrm{d}u = 3x^2\,\mathrm{d}x$. 1M

$$\int_{\frac{1}{2}}^2 \frac{x^2}{1+x^3}\,\mathrm{d}x = \frac{1}{3}\int_{\frac{9}{8}}^9 \frac{\mathrm{d}u}{u} \quad \text{1M}$$

$$= \frac{1}{3}\Big[\ln|u|\Big]_{\frac{9}{8}}^9 \quad \text{1A}$$

$$= \frac{1}{3}\left(\ln 9 - \ln\frac{9}{8}\right)$$

$$= \ln 2 \quad \text{1A}$$

(b) $$\frac{x^2}{1+x^3} - \frac{1}{1+x} = \frac{x^2 - (x^2 - x + 1)}{1+x^3} \quad \text{1M}$$

$$= \frac{x-1}{1+x^3} \quad \text{1A}$$

(c) $$\int_{\frac{1}{2}}^2 \frac{x-1}{1+x^3}\,\mathrm{d}x = \int_{\frac{1}{2}}^2 \left(\frac{x^2}{1+x^3} - \frac{1}{1+x}\right)\mathrm{d}x \quad \text{1M}$$

$$= \ln 2 - \Big[\ln|1+x|\Big]_{\frac{1}{2}}^2 \quad \text{1A}$$

$$= \ln 2 - \left(\ln 3 - \ln\frac{3}{2}\right)$$

$$= 0 \quad \text{1A}$$

40. (a) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(x^{n-1}\sqrt{1+x^2}\right)$

$= (n-1)x^{n-2}\sqrt{1+x^2} + x^{n-1}\cdot\dfrac{1}{2}\cdot\dfrac{2x}{\sqrt{1+x^2}}$ 1M

$= \dfrac{(n-1)x^{n-2}(1+x^2)}{\sqrt{1+x^2}} + \dfrac{x^n}{\sqrt{1+x^2}}$

$= \dfrac{nx^n}{\sqrt{1+x^2}} + \dfrac{(n-1)x^{n-2}}{\sqrt{1+x^2}}$

Integrate both sides with respect to x.

$$\left[x^{n-1}\sqrt{1+x^2}\right]_0^1 = n\int_0^1 \frac{x^n}{\sqrt{1+x^2}}\,\mathrm{d}x + (n-1)\int_0^1 \frac{x^{n-2}}{\sqrt{1+x^2}}\,\mathrm{d}x$$ 1M

$$\sqrt{2} = n\int_0^1 \frac{x^n}{\sqrt{1+x^2}}\,\mathrm{d}x + (n-1)\int_0^1 \frac{x^{n-2}}{\sqrt{1+x^2}}\,\mathrm{d}x$$

$$\int_0^1 \frac{x^n}{\sqrt{1+x^2}}\,\mathrm{d}x = \frac{\sqrt{2}}{n} - \frac{n-1}{n}\int_0^1 \frac{x^{n-2}}{\sqrt{1+x^2}}\,\mathrm{d}x$$ 1

(b) (i) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\ln\left(\sqrt{1+x^2}+x\right)\right]$

$= \dfrac{\frac{x}{\sqrt{1+x^2}}+1}{\sqrt{1+x^2}+x}$

$= \dfrac{x+\sqrt{1+x^2}}{\sqrt{1+x^2}(\sqrt{1+x^2}+x)}$

$= \dfrac{1}{\sqrt{1+x^2}}$ 1A

(ii) $\displaystyle\int_0^1 \frac{x^2}{\sqrt{1+x^2}}\,\mathrm{d}x$

$= \dfrac{\sqrt{2}}{2} - \dfrac{1}{2}\displaystyle\int_0^1 \frac{1}{\sqrt{1+x^2}}\,\mathrm{d}x$ 1M

$= \dfrac{\sqrt{2}}{2} - \dfrac{1}{2}\left[\ln\left(\sqrt{1+x^2}+x\right)\right]_0^1$ 1M

$= \dfrac{\sqrt{2}-\ln\left(\sqrt{2}+1\right)}{2}$ 1A

(c) Let $u = \ln[(e-1)\sin x + 1]$. Then $\mathrm{d}u = \dfrac{(e-1)\cos x}{(e-1)\sin x + 1}\,\mathrm{d}x$. 1M

$\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin\left(\frac{\pi}{2}-x\right)\{\ln[(e-1)\sin x+1]\}^2}{[(e-1)\sin x+1]\sqrt{1+\{\ln[(e-1)\sin x+1]\}^2}}\,\mathrm{d}x$

$= \displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos x\{\ln[(e-1)\sin x+1]\}^2}{[(e-1)\sin x+1]\sqrt{1+\{\ln[(e-1)\sin x+1]\}^2}}\,\mathrm{d}x$

$= \dfrac{1}{e-1}\displaystyle\int_0^1 \frac{u^2}{\sqrt{1+u^2}}\,\mathrm{d}u$ 1M

$= \dfrac{\sqrt{2}-\ln\left(\sqrt{2}+1\right)}{2(e-1)}$ 1A

(d) Let $u = 1-x$. Then $\mathrm{d}u = -\,\mathrm{d}x$. 1M

$$\int_0^1 \sqrt{x^2-2x+2}\,dx$$

$$= \int_0^1 \sqrt{(x-1)^2+1}\,dx$$

$$= -\int_1^0 \sqrt{(-u)^2+1}\,du$$

$$= \int_0^1 \frac{1+u^2}{\sqrt{1+u^2}}\,du$$

$$= \int_0^1 \frac{1}{1+u^2}\,du + \int_0^1 \frac{u^2}{\sqrt{1+u^2}}\,du \qquad \text{1M}$$

$$= \ln\left(\sqrt{2}+1\right) + \frac{\sqrt{2}-\ln\left(\sqrt{2}+1\right)}{2}$$

$$= \frac{\sqrt{2}+\ln\left(\sqrt{2}+1\right)}{2} \qquad \text{1A}$$

41. (a) $\frac{d}{d\theta}\ln(\sec\theta+\tan\theta)$

$$= \frac{1}{\sec\theta+\tan\theta}\left[-(\cos\theta)^{-2}(-\sin\theta)+\sec^2\theta\right]$$

$$= \frac{\sec\theta\tan\theta+\sec^2\theta}{\sec\theta+\tan\theta}$$

$$= \frac{\sec\theta(\tan\theta+\sec\theta)}{\sec\theta+\tan\theta}$$

$$= \sec\theta \qquad \text{1A}$$

(b) (i) Let $u = \tan\theta$. Then $du = \sec^2\theta\,d\theta$. 1M

$$\int \frac{1}{\sqrt{u^2+1}}\,du = \int \frac{1}{\sqrt{\tan^2\theta+1}}\sec^2\theta\,d\theta$$

$$= \int \sec\theta\,d\theta$$

$$= \ln|\sec\theta+\tan\theta| + \text{constant} \qquad \text{1M}$$

$$= \ln|\sqrt{u^2+1}+u| + \text{constant} \qquad \text{1A}$$

(ii) Let $u = x^2+2$. Then $du = 2x\,dx$. 1M

$$\int_0^1 \frac{2x}{\sqrt{x^4+4x^2+5}}\,\mathrm{d}x$$

$$= \int_0^1 \frac{2x}{\sqrt{(x^2+2)^2+1}}\,\mathrm{d}x$$

$$= \int_2^3 \frac{1}{\sqrt{u^2+1}}\,\mathrm{d}u$$

$$= \Big[\ln|\sqrt{u^2+1}+u|\Big]_2^3$$ 1M

$$= \ln\frac{\sqrt{10}+3}{\sqrt{5}+2}\times\frac{\sqrt{5}-2}{\sqrt{5}-2}$$

$$= \ln\frac{5\sqrt{2}-2\sqrt{10}+3\sqrt{5}-6}{5-4}$$

$$= \ln\left(5\sqrt{2}+3\sqrt{5}-2\sqrt{10}-6\right)$$ 1A

42. (a) (i) $\sin 2x = 2\sin x\cos x$

$$= \frac{2\sin x\sec x}{\sec^2 x}$$

$$= \frac{2\tan x}{1+\tan^2 x}$$

$$= \frac{2t}{1+t^2}$$ 1

$\cos 2x = \cos^2 x - \sin^2 x$

$$= \frac{1-\tan^2 x}{\sec^2 x}$$

$$= \frac{1-t^2}{1+t^2}$$ 1

(ii) $3+2\cos 2x+\sin 2x = 3+2\left(\frac{1-t^2}{1+t^2}\right)+\frac{2t}{1+t^2}$

$$= \frac{3(1+t^2)+2(1-t^2)+2t}{1+t^2}$$

$$= \frac{5+2t+t^2}{1+t^2}$$ 1

(b) Let $t=\tan x$. Then $\mathrm{d}t = \sec^2 x\,\mathrm{d}x = (1+t^2)\,\mathrm{d}x$. 1M

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\mathrm{d}x}{3+2\cos 2x+\sin 2x}\,\mathrm{d}x$$

$$= \int_{-1}^{1} \frac{1}{\left(\frac{5+2t+t^2}{1+t^2}\right)}\cdot\frac{1}{1+t^2}\,\mathrm{d}t$$ 1M

$$= \int_{-1}^{1} \frac{\mathrm{d}t}{(t+1)^2+4}$$

Let $t+1 = 2\tan\theta$. Then $\mathrm{d}t = 2\sec^2\theta\,\mathrm{d}\theta$. 1M

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\mathrm{d}x}{3 + 2\cos 2x + \sin 2x}\,\mathrm{d}x$$

$$= \int_0^{\frac{\pi}{4}} \frac{2\sec^2\theta}{4\tan^2\theta + 4}\,\mathrm{d}\theta \qquad \text{1M}$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} \mathrm{d}\theta$$

$$= \frac{1}{2}\Big[\theta\Big]_0^{\frac{\pi}{4}}$$

$$= \frac{\pi}{8} \qquad \text{1A}$$

(c) $$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos^2 x + \sin x\cos x - \sin^2 x}{3 + 2\cos 2x + \sin 2x}\,\mathrm{d}x$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos 2x - \frac{1}{2}\sin 2x}{3 + 2\cos 2x + \sin 2x}\,\mathrm{d}x$$

$$= \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{2\cos 2x + \sin 2x}{3 + 2\cos 2x + \sin 2x}\,\mathrm{d}x$$

$$= \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left[1 - \frac{3}{3 + 2\cos 2x + \sin 2x}\right]\mathrm{d}x \qquad \text{1M}$$

$$= \frac{1}{2}\Big[x\Big]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} - \frac{3}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{3 + 2\cos 2x + \sin 2x}\,\mathrm{d}x$$

$$= \frac{\pi}{4} - \frac{3}{2}\cdot\frac{\pi}{8} \qquad \text{1M}$$

$$= \frac{\pi}{16} \qquad \text{1A}$$

43. Let $u = e^{3x} + 3x + 1$. Then $\mathrm{d}u = (3e^{3x} + 3)\,\mathrm{d}x$. 1M

$$\int_0^3 \frac{x}{e^{3x} + 3x + 1}\,\mathrm{d}x = \frac{1}{3}\int_0^3 \left(1 - \frac{e^{3x} + 1}{e^{3x} + 3x + 1}\right)\mathrm{d}x \qquad \text{1A}$$

$$= \frac{1}{3}\int_0^3 \mathrm{d}x - \frac{1}{9}\int_2^{e^9+10} \frac{\mathrm{d}u}{u} \qquad \text{1A}$$

$$= \frac{1}{3}\Big[x\Big]x_0^3 - \frac{1}{9}\Big[\ln|u|\Big]_2^{e^9+10}$$

$$= 1 - \frac{1}{9}\ln\frac{e^9 + 10}{2} \qquad \text{1A}$$

44. (a) Let $x = \sin\theta$. Then $\mathrm{d}x = \cos\theta\,\mathrm{d}\theta$.

$$\int_0^1 \sqrt{1-x^2}\,\mathrm{d}x = \int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2\theta}\cos\theta\,\mathrm{d}\theta \qquad \text{1M}$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} (1+\cos 2\theta)\,\mathrm{d}\theta \qquad \text{1M}$$

$$= \frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}} \qquad \text{1A}$$

$$= \frac{1}{2}\left(\frac{\pi}{2} + 0 - 0 - 0\right)$$

$$= \frac{\pi}{4} \qquad \text{1A}$$

(b) $$\frac{\mathrm{d}}{\mathrm{d}x}[(1-x^2)^{\frac{3}{2}}x] = (1-x^2)^{\frac{3}{2}} + \frac{3}{2}(1-x^2)^{\frac{1}{2}}(-2x)(x) \qquad \text{1M}$$

$$= \sqrt{1-x^2}[(1-x^2) - 3x^2]$$

$$= \sqrt{1-x^2} - 4x^2\sqrt{1-x^2} \qquad \text{1}$$

(c) By (b), $4x^2\sqrt{1-x^2} = \sqrt{1-x^2} - \frac{\mathrm{d}}{\mathrm{d}x}[(1-x^2)^{\frac{3}{2}}x]$

$$\int_0^1 x^2\sqrt{1-x^2}\,\mathrm{d}x = \frac{1}{4}\int_0^1 \sqrt{1-x^2}\,\mathrm{d}x - \frac{1}{4}\left[(1-x^2)^{\frac{3}{2}}x\right]_0^1 \qquad \text{1M}$$

$$= \frac{1}{4}\times\frac{\pi}{4} - \frac{1}{4}(0-0)$$

$$= \frac{\pi}{16} \qquad \text{1A}$$

45. Let $x = \sin^2\theta$, where $0 \le \theta \le \frac{\pi}{2}$. Then $\mathrm{d}x = 2\sin\theta\cos\theta\,\mathrm{d}x$. 1M

$$\int_0^{\frac{3}{4}} \sqrt{\frac{x}{1-x}}\,\mathrm{d}x = \int_0^{\frac{\pi}{3}} \sqrt{\frac{\sin^2\theta}{1-\sin^2\theta}}\cdot 2\sin\theta\cos\theta\,\mathrm{d}\theta \qquad \text{1M}$$

$$= \int_0^{\frac{\pi}{3}} 2\sin^2\theta\,\mathrm{d}\theta$$

$$= \int_0^{\frac{\pi}{3}} (1-\cos 2\theta)\,\mathrm{d}\theta \qquad \text{1M}$$

$$= \left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{3}} \qquad \text{1A}$$

$$= \frac{\pi}{3} - \frac{\sqrt{3}}{4} \qquad \text{1A}$$

46. (a) $a(\sin x + 2\cos x)b(\cos x - 2\sin x) = (2a+b)\cos x + (a-2b)\sin x$

We have $2a + b = 13$ and $a - 2b = -6$. 1M

Solving, we have $a = 4$ and $b = 5$. 1A

(b) Let $u = \sin x + 2\cos x$. Then $\mathrm{d}u = (\cos x - 2\sin x)\,\mathrm{d}x$. 1M

$$\int_0^{\frac{\pi}{4}} \frac{13 - 6\tan x}{\tan x + 2}\,\mathrm{d}x$$

$$= \int_0^{\frac{\pi}{4}} \frac{13\cos x - 6\sin x}{\sin x + 2\cos x}\,\mathrm{d}x$$

$$= \int_0^{\frac{\pi}{4}} \frac{4(\sin x + 2\cos x) + 5(\cos x - 2\sin x)}{\sin x + 2\cos x}\,\mathrm{d}x$$

$$= 4\int_0^{\frac{\pi}{4}} \mathrm{d}x + 5\int_0^{\frac{\pi}{4}} \frac{\cos x - 2\sin x}{\sin x + 2\cos x}\,\mathrm{d}x \qquad \text{1M}$$

$$= 4\Big[x\Big]_0^{\frac{\pi}{4}} + 5\int_2^{\frac{3\sqrt{2}}{2}} \frac{\mathrm{d}u}{u}$$

$$= \pi + 5\Big[\ln|u|\Big]_2^{\frac{3\sqrt{2}}{2}} \qquad \text{1M}$$

$$= \pi + 5\ln\frac{3\sqrt{2}}{4} \qquad \text{1A}$$

47\. (a) $\frac{A}{x} + \frac{Bx + C}{1 + x^2} = \frac{A(1 + x^2) + (Bx + C)(x)}{x(1 + x^2)}$

$$= \frac{(A + B)x^2 + Cx + A}{x(1 + x^2)}$$

We have $A + B = 0$, $C = 0$ and $A = 2$. 1M

Solving, we have $A = 2$, $B = -2$ and $C = 0$. 1A

(b) $\int_1^2 \frac{2}{x(1 + x^2)}\,\mathrm{d}x = \int_1^2 \left(\frac{2}{x} + \frac{-2x}{1 + x^2}\right)\mathrm{d}x$ 1M

$$= 2\Big[\ln|x|\Big]_1^2 - 2\int_1^2 \frac{x\,\mathrm{d}x}{1 + x^2} \qquad \text{1A}$$

Let $u = 1 + x^2$. Then $\mathrm{d}u = 2x\,\mathrm{d}x$. 1M

$$\int_1^2 \frac{2}{x(1 + x^2)}\,\mathrm{d}x = 2(\ln 2 - \ln 1) - \int_2^5 \frac{\mathrm{d}u}{u}$$

$$= 2\ln 2 - \Big[\ln|u|\Big]_2^5$$

$$= 2\ln 2 - (\ln 5 - \ln 2)$$

$$= 3\ln 2 - \ln 5 \qquad \text{1A}$$

48\. (a) $\sin^2 x \tan^2 x$

$$= (1 - \cos^2 x)(\sec^2 x - 1)$$

$$= \sec^2 x - 2 + \cos^2 x$$

$$= \sec^2 x - 2 + \frac{1}{2}(1 + \cos 2x) \qquad \text{1M}$$

$$= \sec^2 x + \frac{1}{2}\cos 2x - \frac{3}{2} \qquad \text{1}$$

(b) $\int_0^{\frac{\pi}{4}} \sin^4 x \sec^2 x \, \mathrm{d}x$

$= \int_0^{\frac{\pi}{4}} \sin^2 x \tan^2 x \, \mathrm{d}x$

$= \int_0^{\frac{\pi}{4}} \left(\sec^2 x + \frac{1}{2}\cos 2x - \frac{3}{2}\right) \mathrm{d}x$ 1M

$= \left[\tan x + \frac{\sin 2x}{4} - \frac{3x}{2}\right]_0^{\frac{\pi}{4}}$ 1A

$= \frac{5}{4} - \frac{3\pi}{8}$ 1A

49. $\int_0^3 \left(\frac{x}{e^{2x}}\right)^2 \mathrm{d}x = \int_0^3 x^2 e^{-4x} \, \mathrm{d}x$

$= \left[x^2 \times \frac{e^{-4x}}{-4}\right]_0^3 - \int_0^3 (2x)\left(\frac{e^{-4x}}{-4}\right) \mathrm{d}x$ 1M+1M

$= -\frac{9e^{-12}}{4} + \frac{1}{2}\left[\left[x\left(\frac{e^{-4x}}{-4}\right)\right]_0^3 - \int_0^3 \frac{e^{-4x}}{-4} \, \mathrm{d}x\right]$ 1M

$= -\frac{9e^{-12}}{4} - \frac{3e^{-12}}{8} + \frac{1}{8}\left[\frac{e^{-4x}}{-4}\right]_0^3$ 1A

$= -\frac{21e^{-12}}{8} - \frac{1}{32}(e^{-12} - 1)$

$= -\frac{85}{32e^{12}} + \frac{1}{32}$ 1A

50. Let $u = x + 1$. Then $\mathrm{d}u = \mathrm{d}x$. 1M

$\int_0^1 [\ln(x+1)]^2 \, \mathrm{d}x = \int_1^2 (\ln u)^2 \, \mathrm{d}u$

$= \left[u(\ln u)^2\right]_1^2 - \int_1^2 u(2)\ln u \times \frac{1}{u} \, \mathrm{d}u$ 1M+1M

$= 2(\ln 2)^2 - 2\int_1^2 \ln u \, \mathrm{d}u$

$= 2(\ln 2)^2 - 2\left[\left[u \ln u\right]_1^2 - \int_1^2 u \times \frac{1}{u} \, \mathrm{d}u\right]$ 1M

$= 2(\ln 2)^2 - 4\ln 2 + 2\left[u\right]_1^2$

$= 2(\ln 2)^2 - 4\ln 2 + 2$ 1A

51. (a) $\displaystyle\int_0^{\frac{\pi}{2}} e^{\frac{x}{2}} \sin x \,\mathrm{d}x = \left[2e^{\frac{x}{2}} \sin x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (2e^{\frac{x}{2}})(\cos x)\,\mathrm{d}x$ 1M+1M

$\displaystyle = 2e^{\frac{\pi}{4}} - 2\left[\left[2e^{\frac{x}{2}} \cos x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (2e^{\frac{x}{2}})(-\sin x)\,\mathrm{d}x\right]$ 1M

$\displaystyle = 2e^{\frac{\pi}{4}} - 4(0-1) - 4\int_0^{\frac{\pi}{2}} e^{\frac{x}{2}} \sin x\,\mathrm{d}x$

$\displaystyle 5\int_0^{\frac{\pi}{2}} e^{\frac{x}{2}} \sin x\,\mathrm{d}x = 2e^{\frac{\pi}{4}} + 4$ 1M

$\displaystyle \int_0^{\frac{\pi}{2}} e^{\frac{x}{2}} \sin x\,\mathrm{d}x = \frac{2}{5}(e^{\frac{\pi}{4}} + 2)$ 1A

(b) $\displaystyle\int_0^{\frac{\pi}{4}} e^x(\sin x + \cos x)^2\,\mathrm{d}x = \int_0^{\frac{\pi}{4}} e^x(1 + \sin 2x)\,\mathrm{d}x$

Let $u = 2x$. Then $\mathrm{d}u = 2\,\mathrm{d}x$.

When $x = 0$, $u = 0$; when $x = \frac{\pi}{4}$, $u = \frac{\pi}{2}$.

$\displaystyle\int_0^{\frac{\pi}{4}} e^x(\sin x + \cos x)^2\,\mathrm{d}x = \frac{1}{2}\int_0^{\frac{\pi}{2}} e^{\frac{u}{2}}(1 + \sin u)\,\mathrm{d}u$ 1M

$\displaystyle = \left[e^{\frac{u}{2}}\right]_0^{\frac{\pi}{2}} + \frac{1}{5}(e^{\frac{\pi}{4}} + 2)$ 1M

$\displaystyle = e^{\frac{\pi}{4}} - 1 + \frac{1}{5}(e^{\frac{\pi}{4}} + 2)$

$\displaystyle = \frac{1}{5}(6e^{\frac{\pi}{4}} - 3)$ 1A

52. (a) $\displaystyle\int \sin(\ln x)\,\mathrm{d}x = x\sin(\ln x) - \int \cos(\ln x)\,\mathrm{d}x$ 1M

$\displaystyle = x\sin(\ln x) - x\cos(\ln x) - \int \sin(\ln x)\,\mathrm{d}x$ 1M

$\displaystyle 2\int \sin(\ln x)\,\mathrm{d}x = x\sin(\ln x) - x\cos(\ln x) + \text{constant}$ 1M

$\displaystyle \int \sin(\ln x)\,\mathrm{d}x = \frac{x(\sin(\ln x) - \cos(\ln x))}{2} + \text{constant}$ 1A

(b) Let $u = x + 1$. Then $\mathrm{d}u = \mathrm{d}x$. 1M

$\displaystyle\int_0^{e^{\pi}-1} \sin[\ln(x+1)]\,\mathrm{d}x = \int_1^{e^{\pi}} \sin(\ln u)\,\mathrm{d}u$

$\displaystyle = \left[\frac{u(\sin(\ln u) - \cos(\ln u))}{2}\right]_1^{e^{\pi}}$ 1M

$\displaystyle = \frac{e^{\pi} + 1}{2}$ 1A

53. $\int_0^3 \left(\frac{x}{e^{2x}}\right)^2 \mathrm{d}x = \int_0^3 x^2 e^{-4x}\,\mathrm{d}x$

$= -\frac{1}{4}\left[x^2 e^{-4x}\right]_0^3 + \frac{1}{2}\int_0^3 xe^{-4x}\,\mathrm{d}x$ 1M+1A

$= -\frac{9e^{-12}}{4} - \frac{1}{8}\left[xe^{-4x}\right]_0^3 + \frac{1}{8}\int_0^3 e^{-4x}\,\mathrm{d}x$

$= -\frac{9e^{-12}}{4} - \frac{3e^{-12}}{8} - \frac{1}{32}\left[e^{-4x}\right]_0^3$

$= -\frac{85}{32e^{12}} + \frac{1}{32}$ 1A

54. (a) $\int x\cos kx\,\mathrm{d}x = x\left(\frac{\sin kx}{k}\right) - \int \frac{\sin kx}{k}\,\mathrm{d}x$ 1M+1M

$= \frac{x\sin kx}{k} + \frac{\cos kx}{k^2} + C$ 1A

(b) $\int_0^{\frac{\pi}{2}} x\cos x\cos 2x\,\mathrm{d}x = \frac{1}{2}\int_0^{\frac{\pi}{2}} x(\cos 3x + \cos x)\,\mathrm{d}x$ 1M

$= \frac{1}{2}\left[\frac{x\sin 3x}{3} + \frac{\cos 3x}{9}\right]_0^{\frac{\pi}{2}} + \frac{1}{2}\left[x\sin x + \cos x\right]_0^{\frac{\pi}{2}}$ 1M

$= \frac{1}{2}\left[\left(-\frac{\pi}{6} + 0\right) - \left(0 + \frac{1}{9}\right)\right] + \frac{1}{2}\left[\left(\frac{\pi}{2} + 0\right) - (0+1)\right]$

$= \frac{\pi}{6} - \frac{5}{9}$ 1A

55. (a) $\int_1^6 x\sqrt{x-1}\,\mathrm{d}x = \left[x \times \frac{2(x-1)^{\frac{3}{2}}}{3}\right]_1^6 - \int_1^6 \frac{2(x-1)^{\frac{3}{2}}}{3}\,\mathrm{d}x$ 1M+1M

$= (20\sqrt{5} - 0) - \frac{4}{15}\left[(x-1)^{\frac{5}{2}}\right]_1^6$ 1A

$= 20\sqrt{5} - \frac{20\sqrt{5}}{3}$

$= \frac{40\sqrt{5}}{3}$ 1A

(b) Let $u = x - 1$. Then $\mathrm{d}u = \mathrm{d}x$.

$\int_1^6 x\sqrt{x-1}\,\mathrm{d}x = \int_0^5 (u+1)\sqrt{u}\,\mathrm{d}u$ 1M

$= \int_0^5 (u^{\frac{3}{2}} + u^{\frac{1}{2}})\,\mathrm{d}u$ 1M

$= \left[\frac{2u^{\frac{5}{2}}}{5} + \frac{2u^{\frac{3}{2}}}{3}\right]_0^5$ 1A

$= \frac{40\sqrt{5}}{3}$ 1A

56. (a) Let $u = \ln x$. Then $\mathrm{d}u = \dfrac{\mathrm{d}x}{x}$. 1M

$$\int_1^e \frac{\ln x}{x}\,\mathrm{d}x = \int_0^1 u\,\mathrm{d}u \qquad \text{1M}$$
$$= \left[\frac{u^2}{2}\right]_0^1 \qquad \text{1A}$$
$$= \frac{1}{2} \qquad \text{1A}$$

(b)
$$\int_1^e \frac{(1+x^2)\ln x}{x}\,\mathrm{d}x = \int_1^e \frac{\ln x}{x}\,\mathrm{d}x + \int_1^e x\ln x\,\mathrm{d}x$$
$$= \frac{1}{2}+\left[\frac{x^2}{2}\ln x\right]_1^e - \int_1^e \left(\frac{x^2}{2}\right)\left(\frac{1}{x}\right)\mathrm{d}x \qquad \text{1M+1M}$$
$$= \frac{1}{2} + \frac{e^2}{2} - 0 - \frac{1}{2}\int_1^e x\,\mathrm{d}x$$
$$= \frac{1+e^2}{2} - \frac{1}{2}\left[\frac{x^2}{2}\right]_1^e$$
$$= \frac{1+e^2}{2} - \frac{e^2-1}{4}$$
$$= \frac{3+e^2}{4} \qquad \text{1A}$$

57. (a)
$$\int_0^{\frac{\pi}{2}} e^{-2x}\cos 2x\,\mathrm{d}x = \left[\left(\frac{e^{-2x}}{-2}\right)\cos 2x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \left(\frac{e^{-2x}}{-2}\right)(-2\sin 2x)\,\mathrm{d}x \qquad \text{1M+1M}$$
$$= -\frac{1}{2}(-e^{-\pi}-1) - \int_0^{\frac{\pi}{2}} e^{-2x}\sin 2x\,\mathrm{d}x$$
$$= \frac{e^{-\pi}+1}{2} - \left[\left[\left(\frac{e^{-2x}}{-2}\right)\sin 2x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}}\left(\frac{e^{-2x}}{-2}\right)(2\cos 2x)\,\mathrm{d}x\right] \qquad \text{1M}$$
$$= \frac{e^{-\pi}+1}{2} + 0 - \int_0^{\frac{\pi}{2}} e^{-2x}\cos 2x\,\mathrm{d}x$$
$$2\int_0^{\frac{\pi}{2}} e^{-2x}\cos 2x\,\mathrm{d}x = \frac{e^{-\pi}+1}{2} \qquad \text{1M}$$
$$\int_0^{\frac{\pi}{2}} e^{-2x}\cos 2x\,\mathrm{d}x = \frac{e^{-\pi}+1}{4} \qquad \text{1A}$$

(b)
$$\int_0^{\frac{\pi}{2}} e^{-2x}\cos^2 x\,\mathrm{d}x = \frac{1}{2}\int_0^{\frac{\pi}{2}} e^{-2x}(1+\cos 2x)\,\mathrm{d}x \qquad \text{1M}$$
$$= \frac{1}{2}\left[\frac{e^{-2x}}{-2}\right]_0^{\frac{\pi}{2}} + \frac{1}{2}\times\frac{e^{-\pi}+1}{4} \qquad \text{1M}$$
$$= -\frac{1}{4}(e^{-\pi}-1) + \frac{e^{-\pi}+1}{8}$$
$$= \frac{3-e^{-\pi}}{8} \qquad \text{1A}$$

58. (a) $\displaystyle\int x(x+2)e^x\,\mathrm{d}x = x(x+2)e^x - \int (2x+2)e^x\,\mathrm{d}x$ 1M+1M

$\displaystyle= (x^2+2x)e^x - 2\left[(x+1)e^x - \int e^x\,\mathrm{d}x\right]$ 1M

$= (x^2+2x)e^x - 2(x+1)e^x + 2e^x + C$

$= x^2e^x + C$ 1A

(b) $\displaystyle\int_1^e x(x+2)e^x \ln x\,\mathrm{d}x = \left[x^2e^x\ln x\right]_1^e - \int_1^e (x^2e^x)\left(\frac{1}{x}\right)\mathrm{d}x$ 1M+1M

$\displaystyle= (e^{2+e}-0) - \int_1^e xe^x\,\mathrm{d}x$

$\displaystyle= e^{2+e} - \left[\left[xe^x\right]_1^e - \int_1^e e^x\,\mathrm{d}x\right]$ 1M

$= e^{2+e} - (e^{1+e}-e) + \left[e^x\right]_1^e$ 1A

$= e^{2+e} - e^{1+e} + e + (e^e - e)$

$= e^{2+e} - e^{1+e} + e^e$ 1A

59. $\displaystyle\int_0^{\frac{\pi}{4}} e^x\sin 2x\,\mathrm{d}x = \left[e^x\sin 2x\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} e^x(2\cos 2x)\,\mathrm{d}x$ 1M+1M

$\displaystyle= (e^{\frac{\pi}{4}}-0) - 2\left[\left[e^x\cos 2x\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} e^x(-2\sin 2x)\,\mathrm{d}x\right]$ 1M

$\displaystyle 5\int_0^{\frac{\pi}{4}} e^x\sin 2x\,\mathrm{d}x = e^{\frac{\pi}{4}} - 2(0-1)$ 1M

$\displaystyle\int_0^{\frac{\pi}{4}} e^x\sin 2x\,\mathrm{d}x = \frac{1}{5}(e^{\frac{\pi}{4}}+2)$ 1A

60. (a) $\displaystyle\int e^x\cos(\pi x)\,\mathrm{d}x = e^x\cos(\pi x) + \pi\int e^x\sin(\pi x)\,\mathrm{d}x$ 1M

$\displaystyle= e^x\cos(\pi x) + \pi e^x\sin(\pi x) - \pi^2\int e^x\cos(\pi x)\,\mathrm{d}x$ 1M

$\displaystyle(1+\pi^2)\int e^x\cos(\pi x)\,\mathrm{d}x = e^x\cos(\pi x) + \pi e^x\sin(\pi x) + \text{constant}$ 1M

$\displaystyle\int e^x\cos(\pi x)\,\mathrm{d}x = \frac{1}{1+\pi^2}[e^x\cos(\pi x) + \pi e^x\sin(\pi x)] + \text{constant}$ 1A

(b) Let $u = 2 - x$. Then $\mathrm{d}u = -\,\mathrm{d}x$. 1M

$$\int_0^2 e^{2-x}\cos(\pi x)\,\mathrm{d}x$$
$$= -\int_2^0 e^u \cos(2\pi - u)\,\mathrm{d}u$$
$$= \int_0^2 e^u \cos u\,\mathrm{d}u$$
$$= \frac{1}{1+\pi^2}\left[e^u\cos(\pi u) + \pi e^u \sin(\pi u)\right]_0^2 \qquad \text{1M}$$
$$= \frac{e^2 - 1}{1+\pi^2} \qquad \text{1A}$$

61. (a) Let $x = a\tan\theta$. Then $\mathrm{d}x = a\sec^2\theta\,\mathrm{d}\theta$. 1M

$$\int \frac{\mathrm{d}x}{x^2+a^2} = \int \frac{a\sec^2\theta\,\mathrm{d}\theta}{a^2\tan^2\theta + a^2} \qquad \text{1M}$$
$$= \frac{1}{a}\int \mathrm{d}\theta$$
$$= \frac{1}{a}\tan^{-1}\frac{x}{a} + C \qquad \text{1A}$$

(b) $$\int_0^{3\sqrt{3}} \tan^{-1}\frac{x}{3}\,\mathrm{d}x = \left[x\tan^{-1}\frac{x}{3}\right]_0^{3\sqrt{3}} - \int_0^{3\sqrt{3}} x \times 3 \times \frac{1}{x^2+3^2}\,\mathrm{d}x \qquad \text{1M+1M}$$
$$= 3\sqrt{3}\times\frac{\pi}{3} - 0 - 3\int_0^{3\sqrt{3}} \frac{x}{x^2+9}\,\mathrm{d}x$$

Let $u = x^2 + 9$. Then $\mathrm{d}u = 2x\,\mathrm{d}x$. 1M

$$\int_0^{3\sqrt{3}} \tan^{-1}\frac{x}{3}\,\mathrm{d}x = \sqrt{3}\pi - \frac{3}{2}\int_9^{36}\frac{\mathrm{d}u}{u}$$
$$= \sqrt{3}\pi - \frac{3}{2}\Big[\ln|u|\Big]_9^{36} \qquad \text{1A}$$
$$= \sqrt{3}\pi - 3\ln 2 \qquad \text{1A}$$

62. (a) $$\int \ln x\,\mathrm{d}x = x\ln x - \int x\left(\frac{1}{x}\right)\mathrm{d}x \qquad \text{1M}$$
$$= x\ln x - x + \text{constant} \qquad \text{1A}$$

(b) Let $u = \sqrt{x} + 1$. Then $\mathrm{d}u = \frac{1}{2\sqrt{x}}\,\mathrm{d}x$. 1M

$$\int_0^{e^2} \frac{\ln\left(\sqrt{x}+1\right)}{\sqrt{x}}\,\mathrm{d}x = 2\int_1^{e+1}\ln u\,\mathrm{d}u \qquad \text{1A}$$
$$= 2\Big[u\ln u - u\Big]_1^{e+1}$$
$$= 2(e+1)\ln(e+1) - 2e \qquad \text{1A}$$

63. (a) $\displaystyle\int_1^e \ln x\,\mathrm{d}x = \Big[x\ln x\Big]_1^e - \int_1^e x \times \frac{1}{x}\,\mathrm{d}x$ 1M+1M

$= (e-0) - \Big[x\Big]_1^e$

$= e - (e-1)$

$= 1$ 1A

(b) $\displaystyle\int_1^e (\ln x)^2\,\mathrm{d}x = \Big[x(\ln x)^2\Big]_1^e - \int_1^e x\cdot 2\ln x \times \frac{1}{x}\,\mathrm{d}x$ 1M

$\displaystyle= (e-0) - 2\int_1^e \ln x\,\mathrm{d}x$

$= e - 2$ 1A

64. (a) $\displaystyle\int_0^{\frac{\pi}{8}} x\cos 2x\,\mathrm{d}x = \left[x \times \frac{\sin 2x}{2}\right]_0^{\frac{\pi}{8}} - \int_0^{\frac{\pi}{8}} \frac{\sin 2x}{2}\,\mathrm{d}x$ 1M

$\displaystyle= \left(\frac{\pi}{8} \times \frac{\sqrt{2}}{4} - 0\right) + \left[\frac{\cos 2x}{4}\right]_0^{\frac{\pi}{8}}$ 1A

$\displaystyle= \frac{\pi\sqrt{2}}{32} + \frac{\sqrt{2}}{8} - \frac{1}{4}$ 1A

(b) $\displaystyle\int_0^{\frac{\pi}{8}} x^2\sin 2x\,\mathrm{d}x = \left[x^2\left(-\frac{\cos 2x}{2}\right)\right]_0^{\frac{\pi}{8}} - \int_0^{\frac{\pi}{8}} (2x)\left(-\frac{\cos 2x}{2}\right)\mathrm{d}x$ 1M+1M

$\displaystyle= \left[\frac{\pi^2}{64} \times \left(-\frac{\sqrt{2}}{4}\right) - 0\right] + \int_0^{\frac{\pi}{8}} x\cos 2x\,\mathrm{d}x$

$\displaystyle= -\frac{\pi^2\sqrt{2}}{256} + \frac{\pi\sqrt{2}}{32} + \frac{\sqrt{2}}{8} - \frac{1}{4}$ 1A

65. $\displaystyle\int_0^{\frac{\pi}{6}} x^2\cos 3x\,\mathrm{d}x = \left[x^2\left(\frac{\sin 3x}{3}\right)\right]_0^{\frac{\pi}{6}} - \int_0^{\frac{\pi}{6}} (2x)\left(\frac{\sin 3x}{3}\right)\mathrm{d}x$ 1M+1M

$\displaystyle= \frac{\pi^2}{108} - 0 - \frac{2}{3}\left[\left[x\left(-\frac{\cos 3x}{3}\right)\right]_0^{\frac{\pi}{6}} - \int_0^{\frac{\pi}{6}}\left(-\frac{\cos 3x}{3}\right)\mathrm{d}x\right]$ 1M

$\displaystyle= \frac{\pi^2}{108} + \frac{2}{9}(0-0) - \frac{2}{9}\left[\frac{\sin 3x}{3}\right]_0^{\frac{\pi}{6}}$ 1A

$\displaystyle= \frac{\pi^2}{108} - \frac{2}{27}$ 1A

66. (a) $\displaystyle\int \sin(k\ln x)\,\mathrm{d}x = x\sin(k\ln x) - k\int \cos(k\ln x)\,\mathrm{d}x$ 1M

$\displaystyle= x\sin(k\ln x) - kx\cos(k\ln x) - k^2\int \sin(k\ln x)\,\mathrm{d}x$ 1M

$\displaystyle(1+k^2)\int \sin(k\ln x)\,\mathrm{d}x = x[\sin(k\ln x) - k\cos(k\ln x)] + \text{constant}$

$\displaystyle\int \sin(k\ln x)\,\mathrm{d}x = \frac{x[\sin(k\ln x) - k\cos(k\ln x)]}{1+k^2} + \text{constant}$ 1

(b) $\displaystyle\int_1^e \sin(\pi\ln x)\cos(\pi\ln x)\,dx = \frac{1}{2}\int_1^e \sin(2\pi\ln x)\,dx$ 1M

$\displaystyle= \frac{1}{2}\left[\frac{x[\sin(2\pi\ln x) - 2\pi\cos(2\pi\ln x)]}{1+4\pi^2}\right]_1^e$ 1M

$\displaystyle= \frac{(1-e)\pi}{1+4\pi^2}$ 1A

67. (a) $\displaystyle\int x^2e^x\,dx = x^2e^x - \int (2x)e^x\,dx$ 1M+1M

$\displaystyle= x^2e^x - 2\left[xe^x - \int e^x\,dx\right]$ 1M

$= x^2e^x - 2xe^x + 2e^x + C$ 1A

(b) Let $u = \sqrt[3]{x}$. Then $du = \frac{1}{3}x^{-\frac{2}{3}}\,dx$. 1M

$$\int_1^8 e^{\sqrt[3]{x}}\,dx = 3\int_1^8 x^{\frac{2}{3}}e^{\sqrt[3]{x}}\cdot\frac{1}{3}x^{-\frac{2}{3}}\,dx$$

$\displaystyle= 3\int_1^2 u^2e^u\,du$ 1M

$\displaystyle= 3\left[u^2e^u - 2ue^u + 2e^u\right]_1^2$ 1M

$= 3[(4e^2 - 4e^2 + 2e^2) - (e - 2e + 2e)]$

$= 6e^2 - 3e$ 1A

(c) $\displaystyle\int_1^8 2^{\sqrt[3]{x}}\,dx = \int_1^8 e^{\sqrt[3]{x}\ln 2}\,dx$ 1M

Let $u = \sqrt[3]{x}\ln 2$. Then $du = \frac{\ln 2}{3}x^{-\frac{2}{3}}\,dx$. 1M

$$\int_1^8 2^{\sqrt[3]{x}}\,dx = \frac{3}{\ln 2}\int_1^8 x^{\frac{2}{3}}e^{\sqrt[3]{x}}\cdot\frac{\ln 2}{3}x^{-\frac{2}{3}}\,dx$$

$\displaystyle= \frac{3}{\ln 2}\int_{\ln 2}^{2\ln 2}\left(\frac{u}{\ln 2}\right)^2 e^u\,du$ 1M

$\displaystyle= \frac{3}{(\ln 2)^3}\left[u^2e^u - 2ue^u + 2e^u\right]_{\ln 2}^{2\ln 2}$ 1M

$\displaystyle= \frac{42}{\ln 2} - \frac{36}{(\ln 2)^2} + \frac{12}{(\ln 2)^3}$ 1A

68. $\displaystyle\int_0^{\frac{\pi}{6}} x^2\cos 3x\,dx = \left[\frac{x^2\sin 3x}{3}\right]_0^{\frac{\pi}{6}} - \frac{2}{3}\int_0^{\frac{\pi}{6}} x\sin 3x\,dx$ 1M+1A

$$= \frac{\pi^2}{108} + \frac{2}{9}\left[x\cos 3x\right]_0^{\frac{\pi}{6}} - \frac{2}{9}\int_0^{\frac{\pi}{6}}\cos 3x\,dx$$

$$= \frac{\pi^2}{108} + 0 - \frac{2}{27}\left[\sin 3x\right]_0^{\frac{\pi}{6}}$$

$\displaystyle= \frac{\pi^2}{108} - \frac{2}{27}$ 1A

69. (a) Let $u = \frac{\pi}{4} - x$. Then $\mathrm{d}u = -\,\mathrm{d}x$. 1M

$$\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln\left[\cos\left(\frac{\pi}{4} - x\right)\right] \mathrm{d}x = -\int_{\frac{\pi}{6}}^{\frac{\pi}{12}} \ln(\cos u)\,\mathrm{d}u$$

$$= \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln(\cos u)\,\mathrm{d}u \qquad \text{1M}$$

$$= \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln(\cos x)\,\mathrm{d}x \qquad \text{1}$$

(b) $$\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln\left[\cos\left(\frac{\pi}{4} - x\right)\right] \mathrm{d}x - \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln(\cos x)\,\mathrm{d}x = 0$$

$$\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln \frac{\cos\frac{\pi}{4}\cos x + \sin\frac{\pi}{4}\sin x}{\cos x}\,\mathrm{d}x = 0 \qquad \text{1M}$$

$$\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \left[-\frac{1}{2}\ln 2 + \ln(1 + \tan x)\right] \mathrm{d}x = 0 \qquad \text{1M}$$

$$-\frac{\ln 2}{2}\Big[x\Big]_{\frac{\pi}{12}}^{\frac{\pi}{6}} + \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln(1 + \tan x)\,\mathrm{d}x = 0$$

$$\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln(1 + \tan x)\,\mathrm{d}x = \frac{\pi \ln 2}{24} \qquad \text{1}$$

(c) $$\tan\frac{\pi}{12} = \tan\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$$

$$= \frac{\tan\frac{\pi}{4} - \tan\frac{\pi}{6}}{1 + \tan\frac{\pi}{4}\tan\frac{\pi}{6}} \qquad \text{1M}$$

$$= \frac{1 - \frac{\sqrt{3}}{3}}{1 + \frac{\sqrt{3}}{3}} \times \frac{\sqrt{3} - 1}{\sqrt{3} - 1}$$

$$= \frac{4 - 2\sqrt{3}}{2}$$

$$= 2 - \sqrt{3} \qquad \text{1A}$$

(d) Note that $\frac{\mathrm{d}}{\mathrm{d}x}\ln(1 + \tan x) = \frac{\sec^2 x}{1 + \tan x}$.

$$\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \frac{(x+\pi)\sec^2 x}{1+\tan x}\,dx$$

$$= \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \frac{x\sec^2 x}{1+\tan x}\,dx + \pi\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \frac{\sec^2 x}{1+\tan x}\,dx$$

$$= \Big[x\ln(1+\tan x)\Big]_{\frac{\pi}{12}}^{\frac{\pi}{6}} - \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln(1+\tan x)\,dx + \pi\Big[\ln(1+\tan x)\Big]_{\frac{\pi}{12}}^{\frac{\pi}{6}}$$ 1M

$$= \frac{\pi}{6}\ln\left(1+\tan\frac{\pi}{6}\right) - \frac{\pi}{12}\ln\left(1+\tan\frac{\pi}{12}\right) - \frac{\pi\ln 2}{24} + \pi\ln\left(1+\tan\frac{\pi}{6}\right) - \pi\ln\left(1+\tan\frac{\pi}{12}\right)$$ 1M

$$= \frac{7\pi}{6}\ln\left(1+\frac{1}{\sqrt{3}}\right) - \frac{13\pi}{12}\ln\left(3-\sqrt{3}\right) - \frac{\pi\ln 2}{24}$$ 1M

$$= \frac{7\pi}{6}\left[\ln 2 - \ln\left(3-\sqrt{3}\right)\right] - \frac{13\pi}{12}\ln\left(3-\sqrt{3}\right) - \frac{\pi\ln 2}{24}$$ 1M

$$= \frac{9\pi\ln 2}{8} - \frac{9\pi}{4}\ln\left(3-\sqrt{3}\right)$$

$$= -\frac{9\pi}{8}\left[-\ln 2 + \ln\left(3-\sqrt{3}\right)^2\right]$$

$$= -\frac{9\pi}{8}\left[-\ln 2 + \ln\left(12-6\sqrt{3}\right)\right]$$

$$= -\frac{9\pi}{8}\ln\left(6-3\sqrt{3}\right)$$ 1

70. (a) $$\frac{A}{u^2+1} + \frac{B}{u^2+4} = \frac{A(u^2+4)+B(u^2+1)}{(u^2+1)(u^2+4)}$$

$$= \frac{(A+B)u^2+(4A+B)}{(u^2+1)(u^2+4)}$$

Comparing like terms, we have $A+B=0$ and $4A+B=1$. 1M

Solving, we have $A=\frac{1}{3}$ and $B=-\frac{1}{3}$.

Thus, $\frac{1}{(u^2+1)(u^2+4)} = \frac{1}{3(u^2+1)} - \frac{1}{3(u^2+4)}$. 1A

(b) Let $u=-x$. Then $du = -\,dx$. 1M

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

$$= -\int_{a}^{0} f(-u)\,du + \int_{0}^{a} f(x)\,dx$$ 1M

$$= \int_{0}^{a} f(u)\,du + \int_{0}^{a} f(x)\,dx$$

$$= 2\int_{0}^{a} f(x)\,dx$$ 1

(c) (i) Use the result of (b).

$$\int_{-\sqrt{3}}^{\sqrt{3}} \frac{du}{(u^2+1)(u^2+4)} = \int_{-\sqrt{3}}^{\sqrt{3}} \left(\frac{1}{3(u^2+1)} - \frac{1}{3(u^2+4)}\right)du$$ 1M

Let $u=\tan\theta$. Then $du = \sec^2\theta\,d\theta$. 1M

Let $u = 2\tan\phi$. Then $du = 2\sec^2\phi \, d\phi$.

$$\int_{-\sqrt{3}}^{\sqrt{3}} \frac{du}{(u^2+1)(u^2+4)} = \frac{1}{3}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{\sec^2\theta}{\tan^2\theta+1}\, d\theta - \frac{1}{3}\int_{-\tan^{-1}\frac{\sqrt{3}}{2}}^{\tan^{-1}\frac{\sqrt{3}}{2}} \frac{2\sec^2\phi}{4\tan^2\phi+4}\, d\phi \qquad \text{1A}$$

$$= \frac{1}{3}\Big[\theta\Big]_{-\frac{\pi}{3}}^{\frac{\pi}{3}} - \frac{1}{6}\Big[\phi\Big]_{-\tan^{-1}\frac{\sqrt{3}}{2}}^{\tan^{-1}\frac{\sqrt{3}}{2}}$$

$$= \frac{2\pi}{9} - \frac{1}{3}\tan^{-1}\frac{\sqrt{3}}{2} \qquad \text{1A}$$

(ii) Note that $\dfrac{1}{\tan^2(-x)+4} = \dfrac{1}{\tan^2 x+4}$.
Let $u = \tan x$. Then $du = \sec^2\theta \, dx = (u^2+1)\, dx$.

$$\int_0^{\frac{\pi}{3}} \frac{dx}{\tan^2 x+4} = \frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{dx}{\tan^2 x+4} \qquad \text{1M}$$

$$= \frac{1}{2}\int_{-\sqrt{3}}^{\sqrt{3}} \frac{du}{(u^2+1)(u^2+4)} \qquad \text{1M}$$

$$= \frac{\pi}{9} - \frac{1}{6}\tan^{-1}\frac{\sqrt{3}}{2} \qquad \text{1A}$$

71. (a) Let $x = \sqrt{3}\tan\theta$. Then $dx = \sqrt{3}\sec^2\theta \, d\theta$. 1M

$$\int \frac{1}{x^2+3}\, dx = \int \frac{\sqrt{3}\sec^2\theta}{3\tan^2\theta+3}\, d\theta$$

$$= \frac{\sqrt{3}}{3}\int d\theta \qquad \text{1M}$$

$$= \frac{\sqrt{3}}{3}\tan^{-1}\frac{\sqrt{3}x}{3} + \text{constant} \qquad \text{1A}$$

(b) Let $t = \tan x$. Then $dt = \sec^2 x \, dx$.

$$\int \frac{1}{1+2\cos^2 x}\, dx = \int \frac{\sec^2 x}{\sec^2 x+2}\, dx \qquad \text{1M}$$

$$= \int \frac{dt}{t^2+3} \qquad \text{1A}$$

$$= \frac{\sqrt{3}}{3}\tan^{-1}\frac{\sqrt{3}t}{3} + \text{constant}$$

$$= \frac{\sqrt{3}}{3}\tan^{-1}\frac{\sqrt{3}\tan x}{3} + \text{constant} \qquad \text{1A}$$

(c) Let $y = 2\pi - x$. Then $\mathrm{d}y = -\,\mathrm{d}x$.

$$\int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} f(x)\ln\left(1+e^{\sin x}\right)\mathrm{d}x = -\int_{\frac{5\pi}{4}}^{\frac{3\pi}{4}} f(2\pi - y)\ln\left(1+e^{\sin(2\pi-y)}\right)\mathrm{d}y \qquad \text{1M}$$

$$= \int_{\frac{5\pi}{4}}^{\frac{3\pi}{4}} f(y)\ln\left(1+e^{-\sin y}\right)\mathrm{d}y \qquad \text{1A}$$

$$= -\int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} f(y)\ln\left(\frac{1+e^{\sin y}}{e^{\sin y}}\right)\mathrm{d}y$$

$$= -\int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} f(y)\ln\left(1+e^{\sin y}\right)\mathrm{d}y + \int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} f(y)\sin y\,\mathrm{d}y$$

$$= -\int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} f(x)\ln\left(1+e^{\sin x}\right)\mathrm{d}x + \int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} f(x)\sin x\,\mathrm{d}x$$

$$\int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} f(x)\ln\left(1+e^{\sin x}\right)\mathrm{d}x = \frac{1}{2}\int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} f(x)\sin x\,\mathrm{d}x \qquad \text{1}$$

(d) Put $f(x) = \dfrac{\sin x}{1+2\cos^2 x}$.

$$f(2\pi - x) = \frac{\sin(2\pi - x)}{1+2\cos^2(2\pi - x)} = \frac{-\sin x}{1+2\cos^2 x} = -f(x) \qquad \text{1M}$$

$$\int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} \frac{\sin x\left(1+e^{\sin x}\right)}{1+2\cos^2 x}\,\mathrm{d}x$$

$$= \frac{1}{2}\int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} \frac{\sin^2 x}{1+2\cos^2 x}\,\mathrm{d}x$$

$$= \frac{1}{2}\int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} \frac{1-\cos^2 x}{1+2\cos^2 x}\,\mathrm{d}x$$

$$= \frac{1}{2}\int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} \frac{1-\frac{1}{2}(1+2\cos^2 x)+\frac{1}{2}}{1+2\cos^2 x}\,\mathrm{d}x \qquad \text{1M}$$

$$= \frac{3}{4}\int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} \frac{\mathrm{d}x}{1+2\cos^2 x} - \frac{1}{4}\int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} \mathrm{d}x$$

$$= \frac{3}{4}\left[\frac{\sqrt{3}}{3}\tan^{-1}\frac{\sqrt{3}\tan x}{3}\right]_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} - \frac{1}{4}\Big[x\Big]_{\frac{3\pi}{4}}^{\frac{5\pi}{4}}$$

$$= \frac{\sqrt{3}\pi}{12} - \frac{\pi}{8} \qquad \text{1A}$$

72. (a) $$\frac{1+\sin 2t}{1+\cos 2t} = \frac{1+2\sin t\cos t}{1+(2\cos^2 t - 1)} \qquad \text{1M}$$

$$= \frac{1+2\sin t\cos t}{2\cos^2 t}$$

$$= \frac{1}{2}\sec^2 t + \tan t \qquad \text{1}$$

(b) $$\int e^x f'(x)\,\mathrm{d}x = e^x f(x) - \int e^x f(x)\,\mathrm{d}x \qquad \text{1M}$$

$$\int e^x\left[f(x)+f'(x)\right]\mathrm{d}x = e^x f(x) + \text{constant} \qquad \text{1}$$

(c) Let $f(x) = \tan\frac{x}{2}$. Then $f'(x) = \frac{1}{2}\sec^2\frac{x}{2}$. 1M

$$\int_0^{\frac{\pi}{2}} \frac{e^x(1+\sin x)}{1+\cos x}\,\mathrm{d}x = \int_0^{\frac{\pi}{2}} e^x\left(\frac{1}{2}\sec^2\frac{x}{2} + \tan\frac{x}{2}\right)\mathrm{d}x \quad \text{1M}$$

$$= \left[e^x \tan\frac{x}{2}\right]_0^{\frac{\pi}{2}} \quad \text{1M}$$

$$= e^{\frac{\pi}{2}} \quad \text{1A}$$

(d) Let $u = \frac{\pi}{2} - x$. Then $\mathrm{d}u = -\mathrm{d}x$. 1M

$$\int_0^{\frac{\pi}{2}} \frac{1+\cos x}{e^x(1+\sin x)}\,\mathrm{d}x = -\int_{\frac{\pi}{2}}^{0} \frac{1+\cos\left(\frac{\pi}{2}-x\right)}{e^{\frac{\pi}{2}-u}\left(1+\sin\left(\frac{\pi}{2}-u\right)\right)}\,\mathrm{d}u$$

$$= e^{-\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{e^u(1+\sin u)}{1+\cos u}\,\mathrm{d}u$$

$$= e^{-\frac{\pi}{2}} \cdot e^{\frac{\pi}{2}} \quad \text{1M}$$

$$= 1 \quad \text{1A}$$

73. (a) (i) $$\frac{\mathrm{d}}{\mathrm{d}\theta}\ln(\csc\theta + \cot\theta) = \frac{-(\sin\theta)^{-2}\cos\theta - (\tan\theta)^{-2}\sec^2\theta}{\csc\theta + \cot\theta} \quad \text{1M}$$

$$= \frac{-\csc\theta\cot\theta - \csc^2\theta}{\csc\theta + \cot\theta}$$

$$= -\csc\theta$$

Thus, $\int \csc\theta\,\mathrm{d}\theta = -\ln(\csc\theta + \cot\theta) + \text{constant}$. 1

(ii) $$\int \frac{1}{\sin x\cos x}\,\mathrm{d}x = 2\int \csc 2x\,\mathrm{d}x \quad \text{1M}$$

$$= -\ln(\csc 2x + \cot 2x) + \text{constant} \quad \text{1A}$$

(b) Let $u = \frac{\pi}{2} - x$. Then $\mathrm{d}u = -\mathrm{d}x$. 1M

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} f(x)\ln\left(1 + e^{\sin x - \cos x}\right)\mathrm{d}x = -\int_{\frac{\pi}{3}}^{\frac{\pi}{6}} f\left(\frac{\pi}{2} - u\right)\ln\left[1 + e^{\sin\left(\frac{\pi}{2}-u\right) - \cos\left(\frac{\pi}{2}-u\right)}\right]\mathrm{d}u$$

$$= -\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} f(u)\ln\left(\frac{e^{\sin u - \cos u} + 1}{e^{\sin u - \cos u}}\right)\mathrm{d}u \quad \text{1M}$$

$$= -\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} f(u)\ln\left(1 + e^{\sin x - \cos x}\right)\mathrm{d}x$$

$$+ \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} f(x)(\sin x - \cos x)\,\mathrm{d}x \quad \text{1M}$$

$$2\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} f(x)\ln\left(1 + e^{\sin x - \cos x}\right)\mathrm{d}x = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} f(x)(\sin x - \cos x)\,\mathrm{d}x \quad \text{1M}$$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} f(x)\ln\left(1 + e^{\sin x - \cos x}\right)\mathrm{d}x = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} f(x)(\sin x - \cos x)\,\mathrm{d}x \quad \text{1}$$

(c) Put $f(x) = \frac{\sin x - \cos x}{\sin x\cos x}$. 1M

$$f\left(\frac{\pi}{2}-x\right)=\frac{\sin\left(\frac{\pi}{2}-x\right)-\cos\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)\cos\left(\frac{\pi}{2}-x\right)}$$

$$=\frac{\cos x-\sin x}{\cos x\sin x}$$

$$=-f(x)$$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{(\sin x-\cos x)\ln\left(1+e^{\sin x-\cos x}\right)}{\sin x\cos x}\,\mathrm{d}x$$

$$=\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{(\sin x-\cos x)^2}{\sin x\cos x}\,\mathrm{d}x$$ 1M

$$=\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{1-2\sin x\cos x}{\sin x\cos x}\,\mathrm{d}x$$

$$=\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{1}{\sin x\cos x}\,\mathrm{d}x-\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\mathrm{d}x$$

$$=\frac{1}{2}\Big[-\ln(\csc 2x+\cot 2x)\Big]_{\frac{\pi}{6}}^{\frac{\pi}{3}}-\Big[x\Big]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$ 1M

$$=\frac{1}{2}\left(\ln 3-\frac{\pi}{3}\right)$$ 1A

74. (a) (i) Let $u=x-\dfrac{\pi}{2}$. Then $\mathrm{d}u=\mathrm{d}x$.

$$\int_{\frac{\pi}{2}}^{\pi}\frac{\sin^4 x}{\sin^4 x+\cos^4 x}\,\mathrm{d}x=\int_0^{\frac{\pi}{2}}\frac{\sin^4\left(u+\frac{\pi}{2}\right)}{\sin^4\left(u+\frac{\pi}{2}\right)+\cos^4\left(u+\frac{\pi}{2}\right)}\,\mathrm{d}u$$ 1M

$$=\int_0^{\frac{\pi}{2}}\frac{\cos^4 x}{\sin^4 x+\cos^4 x}\,\mathrm{d}x$$ 1

(ii) $$\int_0^{\pi}\frac{\sin^4 x}{\sin^4 x+\cos^4 x}\,\mathrm{d}x$$

$$=\int_0^{\frac{\pi}{2}}\frac{\sin^4 x}{\sin^4 x+\cos^4 x}\,\mathrm{d}x+\int_{\frac{\pi}{2}}^{\pi}\frac{\sin^4 x}{\sin^4 x+\cos^4 x}\,\mathrm{d}x$$ 1M

$$=\int_0^{\frac{\pi}{2}}\frac{\sin^4 x}{\sin^4 x+\cos^4 x}\,\mathrm{d}x+\int_0^{\frac{\pi}{2}}\frac{\cos^4 x}{\sin^4 x+\cos^4 x}\,\mathrm{d}x$$ 1M

$$=\int_0^{\frac{\pi}{2}}\mathrm{d}x$$ 1M

$$=\Big[x\Big]_0^{\frac{\pi}{2}}$$

$$=\frac{\pi}{2}$$ 1A

(b) Let $u = \lambda - x$. Then $\mathrm{d}u = -\,\mathrm{d}x$. 1M

$$\int_0^{\lambda} x f(x)\,\mathrm{d}x = -\int_{\lambda}^{0} (\lambda - u) f(\lambda - u)\,\mathrm{d}u$$

$$= \int_0^{\lambda} (\lambda - u) f(u)\,\mathrm{d}u \qquad \text{1M}$$

$$= \lambda \int_0^{\lambda} f(x)\,\mathrm{d}x - \int_0^{\lambda} x f(x)\,\mathrm{d}x$$

$$= \frac{\lambda}{2} \int_0^{\lambda} f(x)\,\mathrm{d}x \qquad \text{1}$$

(c) Put $f(x) = \dfrac{4\sin^2 x - \sin^2 2x}{\sin^4 x + \cos^4 x}$.

We have $f(\pi - x) = \dfrac{4\sin^2(\pi - x) - \sin^2(2\pi - 2x)}{\sin^4(\pi - x) + \cos^4(\pi - x)} = \dfrac{4\sin^2 x - \sin^2 2x}{\sin^4 x + \cos^4 x} = f(x)$.

$$\int_0^{\pi} \frac{4x\sin^2 x - x\sin^2 2x}{\sin^4 x + \cos^4 x}\,\mathrm{d}x = \int_0^{\pi} x\left(\frac{4\sin^2 x - \sin^2 2x}{\sin^4 x + \cos^4 x}\right)\mathrm{d}x$$

$$= \frac{\pi}{2}\int_0^{\pi} \frac{4\sin^2 x - \sin^2 2x}{\sin^4 x + \cos^4 x}\,\mathrm{d}x \qquad \text{1M}$$

$$= \frac{\pi}{2}\int_0^{\pi} \frac{4\sin^2 x - (2\sin x\cos x)^2}{\sin^4 x + \cos^4 x}\,\mathrm{d}x \qquad \text{1M}$$

$$= \frac{\pi}{2}\int_0^{\pi} \frac{4\sin^2 x(1 - \cos^2 x)}{\sin^4 x + \cos^4 x}\,\mathrm{d}x$$

$$= 2\pi\int_0^{\pi} \frac{\sin^4 x}{\sin^4 x + \cos^4 x}\,\mathrm{d}x$$

$$= 2\pi\left(\frac{\pi}{2}\right) \qquad \text{1M}$$

$$= \pi^2 \qquad \text{1A}$$

75. (a) Let $u = \sqrt{3}k\tan\theta$. Then $\mathrm{d}u = \sqrt{3}k\sec^2\theta\,\mathrm{d}\theta$. 1M

$$\int_0^{k} \frac{k}{u^2 + 3u^2}\,\mathrm{d}u = \int_0^{\frac{\pi}{6}} \frac{k \cdot \sqrt{3}k\sec^2\theta}{3k^2\tan^2\theta + 3k^2}\,\mathrm{d}\theta \qquad \text{1M}$$

$$= \frac{\sqrt{3}}{3}\int_0^{\frac{\pi}{6}} \mathrm{d}\theta$$

$$= \frac{\sqrt{3}}{3}\Big[\theta\Big]_0^{\frac{\pi}{6}}$$

$$= \frac{\sqrt{3}\pi}{18} \qquad \text{1A}$$

(b) Let $u = a - x$. Then $\mathrm{d}u = -\,\mathrm{d}x$. 1M

$$\int_0^a xf(x)\,\mathrm{d}x = -\int_a^0 (a-u)f(a-u)\,\mathrm{d}u$$

$$= \int_0^a (a-u)f(u)\,\mathrm{d}u$$

$$= a\int_0^a f(x)\,\mathrm{d}x - \int_0^a xf(x)\,\mathrm{d}x \quad \text{1M}$$

$$= \frac{a}{2}\int_0^a f(x)\,\mathrm{d}x \quad \text{1}$$

(c) Put $f(x) = \dfrac{\sin x}{\cos 2x + 7}$.

Then $f(\pi - x) = \dfrac{\sin(\pi - x)}{\cos(2\pi - 2x) + 7} = \dfrac{\sin x}{\cos 2x + 7} = f(x)$. 1M

$$\int_0^\pi \frac{x\sin x}{\cos 2x + 7}\,\mathrm{d}x = \frac{\pi}{2}\int_0^\pi \frac{\sin x}{\cos 2x + 7}\,\mathrm{d}x \quad \text{1M}$$

$$= \frac{\pi}{2}\int_0^\pi \frac{\sin x}{2\cos^2 x - 1 + 7}$$

$$= \frac{\pi}{4}\int_0^\pi \frac{\sin x}{\cos^2 x + 3}\,\mathrm{d}x$$

Let $u = \cos x$. Then $\mathrm{d}u = -\sin x\,\mathrm{d}x$. 1M

$$\int_0^\pi \frac{x\sin x}{\cos 2x + 7}\,\mathrm{d}x = -\frac{\pi}{4}\int_1^{-1} \frac{\mathrm{d}u}{u^2 + 3}$$

$$= \frac{\pi}{4}\int_{-1}^{1} \frac{\mathrm{d}u}{u^2 + 3}$$

$$= \frac{\pi}{2}\int_0^{1} \frac{\mathrm{d}u}{u^2 + 3}$$

$$= \frac{\pi}{2}\left(\frac{\sqrt{3}\pi}{18}\right) \quad \text{1M}$$

$$= \frac{\sqrt{3}\pi^2}{36} \quad \text{1A}$$

76. (a) Let $t = \pi - x$. Then $\mathrm{d}t = -\,\mathrm{d}x$. 1M

$$\int_0^\pi xf(\sin x)\,\mathrm{d}x = -\int_\pi^0 (\pi - t)f[\sin(\pi - t)]\,\mathrm{d}t$$

$$= \int_0^\pi (\pi - t)f(\sin t)\,\mathrm{d}t \quad \text{1M}$$

$$= \pi\int_0^\pi f(\sin x)\,\mathrm{d}x - \int_0^\pi xf(\sin x)\,\mathrm{d}x$$

$$= \frac{\pi}{2}\int_0^\pi f(\sin x)\,\mathrm{d}x \quad \text{1}$$

(b) Let $u = \pi - x$. Then $\mathrm{d}u = -\,\mathrm{d}x$. 1M

$$\begin{aligned}\int_0^{\pi} x f(\sin x)\,\mathrm{d}x &= \frac{\pi}{2}\int_0^{\pi} f(\sin x)\,\mathrm{d}x \\ &= \frac{\pi}{2}\left[\int_0^{\frac{\pi}{2}} f(\sin x)\,\mathrm{d}x + \int_{\frac{\pi}{2}}^{\pi} f(\sin x)\,\mathrm{d}x\right] && \text{1M} \\ &= \frac{\pi}{2}\int_0^{\frac{\pi}{2}} f(\sin x)\,\mathrm{d}x - \frac{\pi}{2}\int_{\frac{\pi}{2}}^{0} f[\sin(\pi - u)]\,\mathrm{d}u \\ &= \frac{\pi}{2}\int_0^{\frac{\pi}{2}} f(\sin x)\,\mathrm{d}x + \frac{\pi}{2}\int_0^{\frac{\pi}{2}} f(\sin x)\,\mathrm{d}x && \text{1M} \\ &= \pi\int_0^{\pi} f(\sin x)\,\mathrm{d}x && \text{1}\end{aligned}$$

(c) Note that $\dfrac{\sin x}{1+\cos^2 x} = \dfrac{\sin x}{1+(1-\sin^2 x)} = \dfrac{\sin x}{2-\sin^2 x}$.

Put $f(\sin x) = \dfrac{\sin x}{2-\sin^2 x}$.

$$\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,\mathrm{d}x = \pi\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}\,\mathrm{d}x \qquad \text{1M}$$

Let $u = \cos x$. Then $\mathrm{d}u = -\sin x\,\mathrm{d}x$. 1M

$$\begin{aligned}\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,\mathrm{d}x &= -\pi\int_1^{0} \frac{\mathrm{d}u}{1+u^2} \\ &= \pi\int_0^{1} \frac{\mathrm{d}u}{1+u^2}\end{aligned}$$

Let $u = \tan\theta$. Then $\mathrm{d}u = \sec^2\theta\,\mathrm{d}\theta$. 1M

$$\begin{aligned}\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,\mathrm{d}x &= \pi\int_0^{\frac{\pi}{4}} \frac{\sec^2\theta}{1+\tan^2\theta}\,\mathrm{d}\theta \\ &= \pi\int_0^{\frac{\pi}{4}} \mathrm{d}\theta \\ &= \pi\Big[\theta\Big]_0^{\frac{\pi}{4}} \\ &= \frac{\pi^2}{4} && \text{1A}\end{aligned}$$

(d) Put $f(x) = \dfrac{x}{2-x^2}$. 1M

Since $f(-x) = \dfrac{-x}{2-(-x)^2} = \dfrac{-x}{2-x^2} = -f(x)$, $f(x)$ is an odd function.

Note that $(-x)f(\sin(-x)) = -x(-f(\sin x)) = xf(\sin x) = \dfrac{x\sin x}{1+\cos^2 x}$.

$$\begin{aligned}\int_{-\pi}^{\pi} xf(\sin x)\,\mathrm{d}x &= 2\int_0^{\pi} xf(\sin x)\,\mathrm{d}x \\ &= 2\left(\frac{\pi^2}{4}\right) \\ &= \frac{\pi^2}{2} \\ &\neq 0\end{aligned}$$

The claim is disagreed. 1A

77. (a) Let $u = \pi - x$. Then $\mathrm{d}u = -\,\mathrm{d}x$. 1M

$$\begin{aligned}\int_0^{\pi} xf(\sin x)\,\mathrm{d}x &= -\int_{\pi}^{0} (\pi - u)f(\sin(\pi - u))\,\mathrm{d}u \\ &= \int_0^{\pi} (\pi - u)f(\sin u)\,\mathrm{d}u \qquad \text{1M} \\ &= \pi\int_0^{\pi} f(\sin u)\,\mathrm{d}u - \int_0^{\pi} uf(\sin u)\,\mathrm{d}u \\ &= \pi\int_0^{\pi} f(\sin x)\,\mathrm{d}x - \int_0^{\pi} xf(\sin x)\,\mathrm{d}x \\ 2\int_0^{\pi} xf(\sin x)\,\mathrm{d}x &= \pi\int_0^{\pi} f(\sin x)\,\mathrm{d}x \\ \int_0^{\pi} xf(\sin x)\,\mathrm{d}x &= \frac{\pi}{2}\int_0^{\pi} f(\sin x)\,\mathrm{d}x \qquad \text{1}\end{aligned}$$

(b) (i) Let $v = -x$. Then $\mathrm{d}v = -\,\mathrm{d}x$. 1M

$$\begin{aligned}\int_{-\pi}^{0} \frac{g(x)}{1+e^x}\,\mathrm{d}x &= -\int_{\pi}^{0} \frac{g(-v)}{1+e^{-v}}\,\mathrm{d}v \\ &= \int_0^{\pi} \frac{e^v g(v)}{e^v(1+e^{-v})}\,\mathrm{d}v \\ &= \int_0^{\pi} \frac{e^v g(v)}{1+e^v}\,\mathrm{d}v \\ &= \int_0^{\pi} \frac{e^x g(x)}{1+e^x}\,\mathrm{d}x \qquad \text{1}\end{aligned}$$

(ii)
$$\begin{aligned}\int_{-\pi}^{\pi} \frac{g(x)}{1+e^x}\,\mathrm{d}x &= \int_{-\pi}^{0} \frac{g(x)}{1+e^x}\,\mathrm{d}x + \int_0^{\pi} \frac{g(x)}{1+e^x}\,\mathrm{d}x \\ &= \int_0^{\pi} \frac{e^x g(x)}{1+e^x}\,\mathrm{d}x + \int_0^{\pi} \frac{g(x)}{1+e^x}\,\mathrm{d}x \\ &= \int_0^{\pi} \frac{(1+e^x)g(x)}{1+e^x}\,\mathrm{d}x \\ &= \int_0^{\pi} g(x)\,\mathrm{d}x \qquad \text{1}\end{aligned}$$

(c) Put $g(x) = \dfrac{\sqrt{2+\cos x}}{\sqrt{2+\cos x}+\sqrt{2-\cos x}}$.

$$g(-x)=\frac{\sqrt{2+\cos(-x)}}{\sqrt{2+\cos(-x)}+\sqrt{2-\cos(-x)}}=\frac{\sqrt{2+\cos x}}{\sqrt{2+\cos x}+\sqrt{2-\cos x}}=g(x) \qquad \text{1M}$$

$g(x)$ is a continuous even function on $[-\pi,\ \pi]$.

$$\begin{aligned}
&\int_{-\pi}^{\pi}\frac{\sqrt{2+\cos x}}{(1+e^x)(\sqrt{2+\cos x}+\sqrt{2-\cos x})}\,\mathrm{d}x\\
&=\int_0^{\pi} g(x)\,\mathrm{d}x\\
&=\Big[xg(x)\Big]_0^{\pi}-\int_0^{\pi} xg'(x)\,\mathrm{d}x && \text{1M}
\end{aligned}$$

$$g'(x)=\frac{\frac{-\sin x}{2\sqrt{2+\cos x}}(\sqrt{2+\cos x}+\sqrt{2-\cos x})-(\sqrt{2+\cos x})\left(\frac{-\sin x}{2\sqrt{2+\cos x}}+\frac{\sin x}{2\sqrt{2-\cos x}}\right)}{(\sqrt{2+\cos x}+\sqrt{2-\cos x})^2} \qquad \text{1M}$$

$$=\frac{-\sin x}{2\sqrt{3+\sin^2 x}+(3+\sin^2 x)} \qquad \text{1A}$$

Take $f(x)=\dfrac{-x}{2\sqrt{3+x^2}+3+x^2}$ such that it is continuous on $[0,\ 1]$ and $g'(x)=f(\sin x)$.

$$\begin{aligned}
&\int_{-\pi}^{\pi}\frac{\sqrt{2+\cos x}}{(1+e^x)(\sqrt{2+\cos x}+\sqrt{2-\cos x})}\,\mathrm{d}x\\
&=\Big[xg(x)\Big]_0^{\pi}-\int_0^{\pi} xf(\sin x)\,\mathrm{d}x\\
&=\pi g(\pi)-\frac{\pi}{2}\int_0^{\pi} f(\sin x)\,\mathrm{d}x && \text{1M}\\
&=\pi g(\pi)-\frac{\pi}{2}\int_0^{\pi} g'(x)\,\mathrm{d}x\\
&=\pi g(\pi)-\frac{\pi}{2}\Big[g(x)\Big]_0^{\pi}\\
&=\frac{\pi}{2}\left(\frac{\sqrt{2-1}}{\sqrt{2-1}+\sqrt{2-1}}+\frac{\sqrt{2+1}}{\sqrt{2+1}+\sqrt{2-1}}\right)\\
&=\frac{\pi}{2} && \text{1A}
\end{aligned}$$

78. (a)
$$\begin{aligned}
\frac{2}{3\cos 2x+5}&=\frac{2}{3(\cos^2 x-1)+5}\\
&=\frac{2}{6\cos^2 x+2}\\
&=\frac{\sec^2 x}{3+\sec^2 x} && \text{1}
\end{aligned}$$

(b) Let $u=\tan x$. Then $\mathrm{d}u=\sec^2 x\,\mathrm{d}x$. 1M

$$\begin{aligned}
\int_0^{\frac{\pi}{4}}\frac{2}{3\cos 2x+5}\,\mathrm{d}x&=\int_0^{\frac{\pi}{4}}\frac{\sec^2 x}{3+\sec^2 x}\,\mathrm{d}x\\
&=\int_0^{1}\frac{1}{4+u^2}\,\mathrm{d}u
\end{aligned}$$

Let $u = 2\tan\theta$. Then $du = 2\sec^2\theta\, d\theta$. 1M

$$\int_0^{\frac{\pi}{4}} \frac{2}{3\cos 2x + 5}\, dx = \int_0^{\tan^{-1}\frac{1}{2}} \frac{2\sec^2\theta}{4 + 4\tan^2\theta}\, d\theta$$

$$= \frac{1}{2}\Big[\theta\Big]_0^{\tan^{-1}\frac{1}{2}}$$

$$= \frac{1}{2}\tan^{-1}\frac{1}{2} \qquad \text{1A}$$

(c) Let $u = -x$. Then $du = -dx$. 1M

$$\int_{-\alpha}^{\alpha} f(x)\ln(e^x + 1)\, dx$$

$$= \int_{-\alpha}^{0} f(x)\ln(e^x + 1)\, dx + \int_0^{\alpha} f(x)\ln(e^x + 1)\, dx \qquad \text{1M}$$

$$= -\int_{\alpha}^{0} f(-u)\ln(e^{-u} + 1)\, du + \int_0^{\alpha} f(x)\ln(e^x + 1)\, dx$$

$$= -\int_0^{\alpha} f(x)\ln\left(\frac{1 + e^x}{e^x}\right) dx + \int_0^{\alpha} f(x)\ln(e^x + 1)\, dx$$

$$= -\int_0^{\alpha} f(x)\ln(1 + e^x)\, dx + \int_0^{\alpha} f(x)\ln e^x\, dx + \int_0^{\alpha} f(x)\ln(e^x + 1)\, dx$$

$$= \int_0^{\alpha} x f(x)\, dx \qquad \text{1}$$

(d) Put $f(x) = \dfrac{6\sin 2x}{(3\cos 2x + 5)^2}$. 1M

Then $f(-x) = \dfrac{6\sin(-2x)}{(3\cos(-2x) + 5)^2} = \dfrac{-6\sin 2x}{(3\cos 2x + 5)^2} = -f(x)$.

Note that $\dfrac{d}{dx}\left(\dfrac{2}{3\cos 2x + 5}\right) = \dfrac{6\sin 2x}{(3\cos 2x + 5)^2}$.

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{6\sin 2x}{(3\cos 2x + 5)^2}\ln(e^x + 1)\, dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{6x\sin 2x}{(3\cos 2x + 5)^2}\, dx \qquad \text{1M}$$

$$= \left[\frac{x}{3\cos 2x + 5}\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \frac{1}{3\cos 2x + 5}\, dx \qquad \text{1M}$$

$$= \frac{\frac{\pi}{4}}{3(0) + 5} - \frac{1}{2}\left(\frac{1}{2}\tan^{-1}\frac{1}{2}\right) \qquad \text{1M}$$

$$= \frac{\pi}{20} - \frac{1}{4}\tan^{-1}\frac{1}{2} \qquad \text{1A}$$

79. (a) (i)

$$\frac{\sin 3x}{\sin x} - (2\cos 2x + 1) = \frac{1}{\sin x}\left[\sin 3x - 2\sin x\cos 2x - \sin x\right]$$

$$= \frac{1}{\sin x}\left[\sin 3x - (\sin 3x - \sin x) - \sin x\right] \qquad \text{1M}$$

$$= 0$$

Thus, $\dfrac{\sin 3x}{\sin x} = 2\cos 2x + 1$. 1

(ii) Put $x = \frac{\pi}{4} + y$.

$$\frac{\sin 3\left(\frac{\pi}{4} + y\right)}{\sin\left(\frac{\pi}{4} + y\right)} = 2\cos 2\left(\frac{\pi}{4} + y\right) + 1$$

$$\frac{\sin\frac{3\pi}{4}\cos 3y + \cos\frac{3\pi}{4}\sin 3y}{\sin\frac{\pi}{4}\cos y + \cos\frac{\pi}{4}\sin y} = -2\sin 2y + 1$$ 1A

$$\frac{\cos 3y - \sin 3y}{\cos y + \sin y} = -2\sin 2y + 1$$

$$\frac{\sin 3y - \cos 3y}{\sin x + \cos y} = 2\sin 2y - 1$$ 1A

(b) (i) Let $u = a - x$. Then $\mathrm{d}u = -\,\mathrm{d}x$.

$$\int_0^a f(x)\,\mathrm{d}x = -\int_a^0 f(a-u)\,\mathrm{d}u$$ 1M

$$= \int_0^a f(a-u)\,\mathrm{d}u$$

$$= \int_0^a f(a-x)\,\mathrm{d}x$$ 1A

(ii) Put $f(x) = \frac{\sin 3x}{\sin x + \cos x}$.

$$f\left(\frac{\pi}{2} - x\right) = \frac{\sin 3\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)} = \frac{-\cos 3x}{\cos x + \sin x}$$ 1

We have $\int_0^{\frac{\pi}{2}} \frac{\sin 3x}{\sin x + \cos x}\,\mathrm{d}x = \int_0^{\frac{\pi}{2}} \frac{-\cos 3x}{\sin x + \cos x}\,\mathrm{d}x$. 1A

$$2\int_0^{\frac{\pi}{2}} \frac{\sin 3x}{\sin x + \cos x}\,\mathrm{d}x = \int_0^{\frac{\pi}{2}} \frac{\sin 3x}{\sin x + \cos x}\,\mathrm{d}x + \int_0^{\frac{\pi}{2}} \frac{-\cos 3x}{\sin x + \cos x}\,\mathrm{d}x$$ 1M

$$\int_0^{\frac{\pi}{2}} \frac{\sin 3x}{\sin x + \cos x}\,\mathrm{d}x = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\sin 3x - \cos 3x}{\sin x + \cos x}\,\mathrm{d}x$$ 1A

(c) $$\int_0^{\frac{\pi}{2}} \frac{\sin 3x}{\sin x + \cos x}\,\mathrm{d}x = \frac{1}{2}\int_0^{\frac{\pi}{2}} (2\sin 2x - 1)\,\mathrm{d}x$$ 1M

$$= \frac{1}{2}\Big[-\cos 2x - x\Big]_0^{\frac{\pi}{2}}$$ 1A

$$= 1 - \frac{\pi}{4}$$ 1A

80. (a) $$\frac{1 - \cos 2x}{1 + \cos 2x} = \frac{1 - (1 - 2\sin^2 x)}{1 + (2\cos^2 x - 1)}$$ 1M

$$= \frac{2\sin^2 x}{2\cos^2 x}$$

$$= \tan^2 x$$ 1

(b) Let $u = a - x$. Then $\mathrm{d}u = -\,\mathrm{d}x$.

$$\int_0^a f(a-x)\,\mathrm{d}x = -\int_a^0 f(u)\,\mathrm{d}u$$
$$= \int_0^a f(u)\,\mathrm{d}u$$
$$= \int_0^a f(x)\,\mathrm{d}x \qquad 1$$

(c) Let $u = \cos x$. Then $\mathrm{d}u = -\sin x\,\mathrm{d}x$.

$$\int \tan x\,\mathrm{d}x = \int \frac{\sin x}{\cos x}\,\mathrm{d}x$$
$$= -\int \frac{\mathrm{d}u}{u} \qquad 1\mathrm{M}$$
$$= -\ln|u| + \text{constant}$$
$$= -\ln|\cos x| + \text{constant} \qquad 1$$

(d) $$\int x\sec^2 x\,\mathrm{d}x = x\tan x - \int \tan x\,\mathrm{d}x \qquad 1\mathrm{M}$$
$$= x\tan x - (-\ln|\cos x|) + \text{constant}$$
$$= x\tan x + \ln|\cos x| + \text{constant} \qquad 1\mathrm{A}$$

(e) $$\int_0^{\frac{\pi}{4}} \frac{(\cos x - \sin x + \sqrt{x})(\cos x - \sin x - \sqrt{x})}{1+\sin 2x}\,\mathrm{d}x$$
$$= \int_0^{\frac{\pi}{4}} \frac{(\cos x - \sin x)^2 - x}{1+\sin 2x}\,\mathrm{d}x$$
$$= \int_0^{\frac{\pi}{4}} \frac{\sin^2 x - 2\sin x\cos x + \cos^2 x - x}{1+\sin 2x}\,\mathrm{d}x$$
$$= \int_0^{\frac{\pi}{4}} \frac{1 - \sin 2x - x}{1+\sin 2x}\,\mathrm{d}x \qquad 1\mathrm{M}$$
$$= \int_0^{\frac{\pi}{4}} \frac{1 - \sin\left(\frac{\pi}{2} - 2x\right) - \left(\frac{\pi}{4} - x\right)}{1+\sin\left(\frac{\pi}{2} - 2x\right)}\,\mathrm{d}x \qquad 1\mathrm{M}$$
$$= \int_0^{\frac{\pi}{4}} \frac{1-\cos 2x}{1+\cos 2x}\,\mathrm{d}x + \int_0^{\frac{\pi}{4}} \frac{x - \frac{\pi}{4}}{1+\cos 2x}\,\mathrm{d}x$$
$$= \int_0^{\frac{\pi}{4}} \tan^2 x\,\mathrm{d}x + \int_0^{\frac{\pi}{4}} \frac{x - \frac{\pi}{4}}{2\cos^2 x}\,\mathrm{d}x \qquad 1\mathrm{M}$$
$$= \int_0^{\frac{\pi}{4}} (\sec^2 x - 1)\,\mathrm{d}x + \frac{1}{2}\int_0^{\frac{\pi}{4}} x\sec^2 x\,\mathrm{d}x - \frac{\pi}{8}\int_0^{\frac{\pi}{4}} \sec^2 x\,\mathrm{d}x \qquad 1\mathrm{M}$$
$$= \Big[\tan x - x\Big]_0^{\frac{\pi}{4}} + \frac{1}{2}\Big[x\tan x + \ln(\cos x)\Big]_0^{\frac{\pi}{4}} - \frac{\pi}{8}\Big[\tan x\Big]_0^{\frac{\pi}{4}} \qquad 1\mathrm{M}$$
$$= 1 - \frac{\pi}{4} - \frac{1}{4}\ln 2 12 \qquad 1\mathrm{A}$$

81. (a) $\int_0^{\pi} \cos^4 x \, dx$

$= \int_0^{\pi} \left(\frac{1+\cos 2x}{2}\right)^2 dx$ 1M

$= \int_0^{\pi} \left(\frac{1}{4} + \frac{\cos 2x}{2} + \frac{\cos^2 2x}{8}\right) dx$

$= \int_0^{\pi} \left(\frac{1}{4} + \frac{\cos 2x}{2} + \frac{1+\cos 4x}{16}\right) dx$ 1M

$= \left[\frac{3x}{8} + \frac{\sin 2x}{4} + \frac{\sin 4x}{32}\right]_0^{\pi}$

$= \frac{3\pi}{8}$ 1A

(b) Let $u = k - x$. Then $du = -dx$. 1M

$$\int_0^k f(x)g(x)\,dx = -\int_k^0 f(k-u)g(k-u)\,du$$
$$= \int_0^k f(u)g(k-u)\,du$$
$$= \int_0^k f(x)g(k-x)\,dx$$
$$= \frac{1}{2}\left[\int_0^k f(x)g(x)\,dx + \int_0^k f(x)g(k-x)\,dx\right] \quad \text{1M}$$
$$= \frac{1}{2}\int_0^k f(x)[g(x)+g(k-x)]\,dx$$
$$= \frac{a}{2}\int_0^k f(x)\,dx \quad \text{1}$$

(c) (i) Put $f(x) = \cos^4 x$ and $g(x) = \frac{1}{1+e^{\sin x}}$. 1M

$f(2\pi - x) = \cos^4(2\pi - x) = \cos^4 x = f(x)$

$$g(x) + g(2\pi - x) = \frac{1}{1+e^{\sin x}} + \frac{1}{1+e^{\sin(2\pi - x)}}$$
$$= \frac{1}{1+e^{\sin x}} + \frac{1}{1+e^{-\sin x}} \times \frac{e^{\sin x}}{e^{\sin x}}$$
$$= \frac{1}{1+e^{\sin x}} + \frac{e^{\sin x}}{1+e^{\sin x}}$$
$$= 1 \quad \text{1M}$$

Let $u = 2\pi - x$. Then $\mathrm{d}u = -\,\mathrm{d}x$.

$$\int_0^{2\pi} \frac{\cos^4 x}{1+e^{\sin x}}\,\mathrm{d}x = \frac{1}{2}\int_0^{2\pi} \cos^4 x\,\mathrm{d}x$$ 1M

$$= \frac{1}{2}\left(\int_0^{\pi} \cos^4 x\,\mathrm{d}x + \int_{\pi}^{2\pi} \cos^4 x\,\mathrm{d}x\right)$$

$$= \frac{1}{2}\int_0^{\pi} \cos^4 x\,\mathrm{d}x - \frac{1}{2}\int_{\pi}^{0} \cos^4(2\pi - u)\,\mathrm{d}u$$

$$= \frac{1}{2}\int_0^{\pi} \cos^4 x\,\mathrm{d}x + \frac{1}{2}\int_0^{\pi} \cos^4 u\,\mathrm{d}u$$

$$= \frac{1}{2}\left(\frac{3\pi}{8}\right) + \frac{1}{2}\left(\frac{3\pi}{8}\right)$$

$$= \frac{3\pi}{8}$$ 1A

(ii) $$\int_0^{2\pi} \frac{e^{\sin x}\cos^4 x}{1+e^{\sin x}}\,\mathrm{d}x = \int_0^{2\pi}\left[\frac{(1+e^{\sin x})\cos^4 x}{1+e^{\sin x}} - \frac{\cos^4 x}{1+e^{\sin x}}\right]\mathrm{d}x$$

$$= \int_0^{2\pi} \cos^4 x\,\mathrm{d}x - \int_0^{2\pi} \frac{\cos^4 x}{1+e^{\sin x}}\,\mathrm{d}x$$ 1M

$$= 2\left(\frac{3\pi}{8}\right) - \frac{3\pi}{8}$$

$$= \frac{3\pi}{8}$$ 1A

82. (a) Let $u = k - x$. Then $\mathrm{d}u = -\,\mathrm{d}x$. 1M

$$\int_0^{k} x f(x)\,\mathrm{d}x = -\int_k^{0} (k-u) f(\pi - u)\,\mathrm{d}u$$

$$= \int_0^{k} (k-u) f(u)\,\mathrm{d}u$$

$$= \int_0^{k} (k-x) f(x)\,\mathrm{d}x$$ 1

Put $k = \pi$.

$$\int_0^{\pi} x f(x)\,\mathrm{d}x = \int_0^{\pi} (\pi - x) f(x)\,\mathrm{d}x$$

$$= \frac{1}{2}\left[\int_0^{\pi} (\pi - x) f(x)\,\mathrm{d}x + \int_0^{\pi} x f(x)\,\mathrm{d}x\right]$$ 1M

$$= \frac{\pi}{2}\int_0^{\pi} f(x)\,\mathrm{d}x$$ 1

(b) Let $u = \cos x$. Then $\mathrm{d}u = -\sin x\,\mathrm{d}x$. 1M

$$\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\,\mathrm{d}x = -\int_1^{-1} \frac{\mathrm{d}u}{1+u^2}$$ 1A

$$= \int_{-1}^{1} \frac{\mathrm{d}u}{1+u^2}$$

Let $u = \tan\theta$. Then $\mathrm{d}u = \sec^2\theta\,\mathrm{d}\theta$. 1M

$$\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\,\mathrm{d}x = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sec^2\theta}{1+\tan^2\theta}\,\mathrm{d}\theta$$ 1A

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \mathrm{d}\theta$$

$$= \Big[\theta\Big]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$= \frac{\pi}{2}$$ 1A

(c) $$\frac{\cos\left(\frac{\pi}{6}-x\right)-\cos\left(\frac{\pi}{6}+x\right)}{3+\cos 2x}$$

$$= \frac{-2\sin\frac{\pi}{6}\sin(-x)}{3+(2\cos^2 x-1)}$$ 1M

$$= \frac{1}{2}\cdot\frac{\sin x}{1+\cos^2 x}$$

Put $f(x) = \dfrac{\sin x}{1+\cos^2 x}$. 1M

Then $f(\pi-x)\dfrac{\sin(\pi-x)}{1+\cos^2(\pi-x)} = \dfrac{\sin x}{1+\cos^2 x} = f(x)$.

$$\int_0^{\pi} \frac{x\left[\cos\left(\frac{\pi}{6}-x\right)-\cos\left(\frac{\pi}{6}+x\right)\right]}{3+\cos 2x}\,\mathrm{d}x$$

$$= \frac{1}{2}\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}\,\mathrm{d}x$$

$$= \frac{1}{2}\cdot\frac{\pi}{2}\int_0^{\pi}\frac{\sin x}{1+\cos^2 x}\,\mathrm{d}x$$

$$= \frac{\pi}{4}\cdot\frac{\pi}{2}$$

$$= \frac{\pi^2}{8}$$ 1A

83. (a) Let $u = \pi - x$. Then $\mathrm{d}u = -\,\mathrm{d}x$. 1M

$$\int_0^{\pi} f(\pi-x)\,\mathrm{d}x = -\int_{\pi}^{0} f(u)\,\mathrm{d}u$$

$$= \int_0^{\pi} f(u)\,\mathrm{d}u$$ 1M

$$= \int_0^{\pi} f(x)\,\mathrm{d}x$$ 1

(b) $$\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}\,\mathrm{d}x = \int_0^{\pi}\frac{(\pi-x)\sin(\pi-x)}{1+\cos^2(\pi-x)}\,\mathrm{d}x$$ 1M

$$= \int_0^{\pi}\frac{(\pi-x)\sin x}{1+\cos^2 x}\,\mathrm{d}x$$

$$= \frac{1}{2}\left[\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}\,\mathrm{d}x + \int_0^{\pi}\frac{(\pi-x)\sin x}{1+\cos^2 x}\,\mathrm{d}x\right]$$ 1M

$$= \frac{\pi}{2}\int_0^{\pi}\frac{\sin x}{1+\cos^2 x}\,\mathrm{d}x$$ 1A

(c) (i) Let $x = \tan\theta$. Then $\mathrm{d}x = \sec^2\theta\,\mathrm{d}\theta$. 1M

$$\int \frac{1}{1+x^2}\,\mathrm{d}x = \int \frac{\sec^2\theta}{1+\tan^2\theta}\,\mathrm{d}\theta$$
$$= \int \mathrm{d}\theta$$
$$= \theta + \text{constant}$$
$$= \tan^{-1}x + \text{constant}$$ 1A

(ii) $$\int_0^{\pi} \frac{(x+\cos^5 x)\sin x}{1+\cos^2 x}\,\mathrm{d}x$$
$$= \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,\mathrm{d}x + \int_0^{\pi} \frac{\cos^5 x}{1+\cos^2 x}\,\mathrm{d}x$$
$$= \frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\,\mathrm{d}x + \int_0^{\pi} \frac{\cos^5 x\sin x}{1+\cos^2 x}\,\mathrm{d}x$$ 1M

Since $\dfrac{\cos^5(-x)\sin(-x)}{1+\cos^2(-x)} = \dfrac{-\cos^5 x\sin x}{1+\cos^2 x}$, it is an odd function.

Let $u = \cos x$. Then $\mathrm{d}u = -\sin x\,\mathrm{d}x$. 1M

$$\int_0^{\pi} \frac{(x+\cos^5 x)\sin x}{1+\cos^2 x}\,\mathrm{d}x$$
$$= -\frac{\pi}{2}\int_1^{-1} \frac{\mathrm{d}u}{1+u^2} + 0$$ 1M
$$= \frac{\pi}{2}\int_{-1}^{1} \frac{\mathrm{d}u}{1+u^2}$$
$$= \frac{\pi}{2}\Big[\tan^{-1}x\Big]_{-1}^{1}$$ 1M
$$= \frac{\pi^2}{4}$$ 1A

84. (a) $$\int g(x)\,\mathrm{d}x = \sin^2 x\left(\frac{\sin 2x}{2}\right) - \frac{1}{2}\int (2\sin x\cos x)\sin 2x\,\mathrm{d}x$$ 1M
$$= \frac{1}{2}\sin^2 x\sin 2x - \frac{1}{2}\int \sin^2 2x\,\mathrm{d}x$$ 1

(b) $$\int_0^{\pi} g(x)\,\mathrm{d}x = \frac{1}{2}\Big[\sin^2 x\sin 2x\Big]_0^{\pi} - \frac{1}{2}\int_0^{\pi}\sin^2 2x\,\mathrm{d}x$$
$$= 0 - \frac{1}{4}\int_0^{\pi}(1-\cos 4x)\,\mathrm{d}x$$ 1M
$$= -\frac{1}{4}\left[x - \frac{\sin 4x}{4}\right]_0^{\pi}$$
$$= -\frac{\pi}{4}$$ 1A

(c) Let $u = \pi - x$. Then $\mathrm{d}u = -\,\mathrm{d}x$. 1M

$$\int_0^{\pi} xg(x)\,\mathrm{d}x = -\int_{\pi}^{0} (\pi - u)g(\pi - u)\,\mathrm{d}u$$

$$= \int_0^{\pi} (\pi - u)\sin^2(\pi - u)\cos(2\pi - 2u)\,\mathrm{d}u \quad \text{1M}$$

$$= \int_0^{\pi} (\pi - x)\sin^2 x\cos 2x\,\mathrm{d}x$$

$$= \pi\int_0^{\pi} g(x)\,\mathrm{d}x - \int_0^{\pi} xg(x)\,\mathrm{d}x$$

$$= \frac{\pi}{2}\int_0^{\pi} g(x)\,\mathrm{d}x \quad \text{1M}$$

$$= \frac{\pi}{2}\left(-\frac{\pi}{4}\right)$$

$$= -\frac{\pi^2}{8} \quad \text{1A}$$

(d) Note that $(-x)g(-x) = -x\sin^2(-x)\cos(-2x) = -x\sin^2\cos 2x = -xg(x)$. 1M

Thus, $xg(x)$ is an odd function.

Let $u = x - \pi$. Then $\mathrm{d}u = \mathrm{d}x$. 1M

$$\int_{-\pi}^{2\pi} xg(x)\,\mathrm{d}x = \int_{-\pi}^{\pi} xg(x)\,\mathrm{d}x + \int_{\pi}^{2\pi} xg(x)\,\mathrm{d}x$$

$$= 0 + \int_{\pi}^{2\pi} xg(x)\,\mathrm{d}x$$

$$= \int_0^{\pi} (u + \pi)\sin^2(u + \pi)\cos(2u + 2\pi)\,\mathrm{d}u$$

$$= \int_0^{\pi} (\pi + x)g(x)\,\mathrm{d}x \quad \text{1M}$$

$$= \pi\int_0^{\pi} g(x)\,\mathrm{d}x + \int_0^{\pi} xg(x)\,\mathrm{d}x$$

$$= \pi\left(-\frac{\pi}{4}\right) + \left(-\frac{\pi^2}{8}\right)$$

$$= -\frac{3\pi^2}{8} \quad \text{1A}$$

85. (a) $\sin x + \sin x\tan^2\dfrac{x}{2} = 2\sin\dfrac{x}{2}\cos\dfrac{x}{2}\left(1 + \dfrac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}\right)$

$$= 2\tan\frac{x}{2}\left(\cos^2\frac{x}{2} + \sin^2\frac{x}{2}\right)$$

$$= 2\tan\frac{x}{2} \quad \text{1}$$

(b) Note that $x^2 + ax + a^2 = \left(x + \dfrac{a}{2}\right)^2 + \dfrac{3a^2}{4}$.

Let $x + \dfrac{a}{2} = \dfrac{\sqrt{3}a}{2}\tan\theta$. Then $\mathrm{d}x = \dfrac{\sqrt{3}a}{2}\sec^2\theta\,\mathrm{d}\theta$. 1M

$$\int_0^a \frac{\mathrm{d}x}{x^2+ax+a^2} = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\frac{\sqrt{3}a}{2}\sec^2\theta}{\frac{3a^2}{4}\tan^2\theta + \frac{3a^2}{4}}\,\mathrm{d}\theta \quad \text{1M}$$

$$= \frac{2}{\sqrt{3}a}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \mathrm{d}\theta$$

$$= \frac{2}{\sqrt{3}a}\Big[\theta\Big]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= \frac{\pi}{3\sqrt{3}a} \quad \text{1A}$$

$$= \frac{\sqrt{3}\pi}{9a}$$

(c) Use the result of (a).

$$\sin x + \sin x\tan^2\frac{x}{2} = 2\tan\frac{x}{2}$$

$$\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$$

Let $t = \tan\dfrac{x}{2}$. Then $\mathrm{d}t = \dfrac{1}{2}\sec^2\dfrac{x}{2}\,\mathrm{d}x = \dfrac{1+t^2}{2}\,\mathrm{d}x$. 1M

$$\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}x}{\sin x + 2} = \int_0^1 \frac{1}{\frac{2t}{1+t^2}+2}\cdot\frac{2}{1+t^2}\,\mathrm{d}t \quad \text{1M}$$

$$= \int_0^1 \frac{\mathrm{d}t}{t^2+t+1}$$

$$= \frac{\sqrt{3}\pi}{9(1)}$$

$$= \frac{\sqrt{3}\pi}{9} \quad \text{1A}$$

(d) $$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\cos x}{\sin x+2}\right) = \frac{(-\sin x)(\sin x+2) - \cos x(\cos x)}{(\sin x+2)^2} \quad \text{1M}$$

$$= \frac{-2\sin x - 1}{(\sin x+2)^2}$$

$$= \frac{-2(\sin x+2)+3}{(\sin x+2)^2}$$

$$= -\frac{2}{\sin x+2} + \frac{3}{(\sin x+2)^2}$$

Integrate both sides with respect to x.

$$\left[\frac{\cos x}{\sin x+2}\right]_0^{\frac{\pi}{2}} = -2\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}x}{\sin x+2} + 3\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}x}{(\sin x+2)^2} \quad \text{1M}$$

$$\frac{1}{2} - 0 = -2\left(\frac{\sqrt{3}\pi}{9}\right) + 3\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}x}{(\sin x+2)^2}$$

$$\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}x}{(\sin x+2)^2} = \frac{2\sqrt{3}\pi}{27} - \frac{1}{6}$$

Note that $\dfrac{1}{[\cos(-x)+2]^2} = \dfrac{1}{(\cos x+2)^2}$.

Thus, $\dfrac{1}{(\cos x+2)^2}$ is an even function.

Let $u=\dfrac{\pi}{2}-x$. Then $\mathrm{d}u=-\,\mathrm{d}x$. 1M

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\mathrm{d}x}{(\cos+2)^2} = 2\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}x}{(\cos x+2)^2} \qquad \text{1M}$$

$$= -2\int_{-\frac{\pi}{2}}^{0} \frac{\mathrm{d}u}{\left[\cos\left(\frac{\pi}{2}-u\right)+2\right]^2}$$

$$= 2\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}u}{(\sin u+2)^2} \qquad \text{1M}$$

$$= 2\left(\frac{2\sqrt{3}\pi}{27}-\frac{1}{6}\right)$$

$$= \frac{4\sqrt{3}\pi}{27}-\frac{1}{3} \qquad \text{1A}$$

86. (a) $$\int_0^{\pi} x^2\cos^2 x\,\mathrm{d}x = \frac{1}{2}\int_0^{\pi} x^2(1+\cos 2x)\,\mathrm{d}x \qquad \text{1M}$$

$$= \left[\frac{x^3}{6}\right]_0^{\pi} + \frac{1}{4}\left[x^2\sin 2x\right]_0^{\pi} - \frac{1}{2}\int_0^{\pi} x\sin 2x\,\mathrm{d}x \qquad \text{1M}$$

$$= \frac{\pi^3}{6}+0+\frac{1}{4}\left[x\cos 2x\right]_0^{\pi} - \frac{1}{4}\int_0^{\pi}\cos 2x\,\mathrm{d}x \qquad \text{1M}$$

$$= \frac{\pi^3}{6}+\frac{\pi}{4}-\frac{1}{8}\left[\sin 2x\right]_0^{\pi}$$

$$= \frac{\pi^3}{6}+\frac{\pi}{4} \qquad \text{1A}$$

(b) (i) Let $u=-x$. Then $\mathrm{d}u=-\,\mathrm{d}x$. 1M

$$\int_{-a}^{a} f(x)\,\mathrm{d}x = \int_{-a}^{0} f(x)\,\mathrm{d}x + \int_0^a f(x)\,\mathrm{d}x$$

$$= -\int_a^0 f(-u)\,\mathrm{d}u + \int_0^a f(x)\,\mathrm{d}x$$

$$= \int_0^a f(-u)\,\mathrm{d}u + \int_0^a f(x)\,\mathrm{d}x$$

$$= \int_0^a [f(x)+f(-x)]\,\mathrm{d}x \qquad \text{1}$$

(ii) Let $u=a-x$. Then $\mathrm{d}u=-\,\mathrm{d}x$. 1M

$$\int_0^a xg(x)\,\mathrm{d}x = -\int_a^0 (a-u)g(a-u)\,\mathrm{d}u$$

$$= \int_0^a (a-u)g(u)\,\mathrm{d}u$$

$$= a\int_0^a g(x)\,\mathrm{d}x - \int_0^a xg(x)\,\mathrm{d}x$$

$$= \frac{a}{2}\int_0^a g(x)\,\mathrm{d}x \qquad \text{1}$$

(c) Put $f(x) = \ln(1+e^x)(\sin^3 x + x)\cos^2 x$.

$$\begin{aligned} f(x) + f(-x) &= \ln(1+e^x)(\sin^3 x + x)\cos^2 x + \ln(1+e^{-x})(\sin^3(-x) - x)\cos^2(-x) \\ &= \ln(1+e^x)(\sin^3 x + x)\cos^2 x - \ln\left(\frac{e^x+1}{e^x}\right)(\sin^3 x + x)\cos^2 x \\ &= \ln(e^x)(\sin^3 x + x)\cos^2 x \\ &= x(\sin^3 x + x)\cos^2 x \end{aligned}$$

Use the result of (b)(i).

$$\begin{aligned} \int_{-\pi}^{\pi} \ln(1+e^x)(\sin^3 x + x)\cos^2 x\,dx &= \int_0^{\pi} x(\sin^3 x + x)\cos^2 x\,dx && \text{1M} \\ &= \int_0^{\pi} x\sin^3 x\cos^2 x\,dx + \int_0^{\pi} x^2\cos^2 x\,dx \\ &= \int_0^{\pi} x\sin^3 x\cos^2 x\,dx + \frac{\pi^3}{6} + \frac{\pi}{4} \end{aligned}$$

Put $g(x) = \sin^3 x\cos^2 x$. Then $g(\pi - x) = \sin^3(\pi - x)\cos^2(\pi - x) = \sin^3 x\cos^2 x = g(x)$.
Let $u = \cos x$. Then $du = -\sin x\,dx$. 1M

$$\begin{aligned} \int_0^{\pi} x\sin^3 x\cos^2 x\,dx &= \frac{\pi}{2}\int_0^{\pi} \sin^3 x\cos^2 x\,dx && \text{1M} \\ &= -\frac{\pi}{2}\int_1^{-1}(1-u^2)u^2\,du \\ &= \frac{\pi}{2}\int_{-1}^{1}(u^2 - u^4)\,du \\ &= \frac{\pi}{2}\left[\frac{u^3}{3} - \frac{u^5}{5}\right]_{-1}^{1} \\ &= \frac{2\pi}{15} \end{aligned}$$

Combine the results.

$$\begin{aligned} \int_{-\pi}^{\pi} \ln(1+e^x)(\sin^3 x + x)\cos^2 x\,dx &= \frac{2\pi}{15} + \left(\frac{\pi^3}{6} + \frac{\pi}{4}\right) \\ &= \frac{\pi^3}{6} + \frac{23\pi}{60} && \text{1A} \end{aligned}$$

87. (a)
$$\begin{aligned} \frac{\sin 3\theta}{\sin\theta} &= \frac{\sin 2\theta\cos\theta + \cos 2\theta\sin\theta}{\sin\theta} && \text{1M} \\ &= \frac{2\sin\theta\cos^2\theta + \cos 2\theta\sin\theta}{\sin\theta} \\ &= (1+\cos 2\theta) + \cos 2\theta \\ &= 2\cos 2\theta + 1 && \text{1} \end{aligned}$$

(b) $$\frac{\sin\left[3\left(x+\frac{\pi}{4}\right)\right]}{\sin\left(x+\frac{\pi}{4}\right)} = \frac{\sin 3x\cos\frac{3\pi}{4}+\cos 3x\sin\frac{3\pi}{4}}{\sin x\cos\frac{\pi}{4}+\cos x\sin\frac{\pi}{4}}$$ 1M

$$= \frac{-\sin 3x\cdot\frac{\sqrt{2}}{2}+\cos 3x\cdot\frac{\sqrt{2}}{2}}{\sin x\cdot\frac{\sqrt{2}}{2}+\cos x\cdot\frac{\sqrt{2}}{2}}$$

$$= \frac{\cos 3x-\sin 3x}{\cos x+\sin x}$$ 1

(c) (i) Let $u=\dfrac{\pi}{2}-x$. Then $\mathrm{d}u=-\mathrm{d}x$.

$$\int_0^{\frac{\pi}{2}}\frac{\cos 3x}{\cos x+\sin x}\,\mathrm{d}x = -\int_{\frac{\pi}{2}}^{0}\frac{\cos\left(\frac{3\pi}{2}-3u\right)}{\cos\left(\frac{\pi}{2}-u\right)+\sin\left(\frac{\pi}{2}-u\right)}\,\mathrm{d}u$$

$$= \int_{\frac{\pi}{2}}^{0}\frac{\sin 3u}{\sin u+\cos u}\,\mathrm{d}u$$

$$= \int_{\frac{\pi}{2}}^{0}\frac{\sin 3x}{\cos x+\sin x}\,\mathrm{d}x$$ 1

(ii) $$\int_0^{\frac{\pi}{2}}\frac{\cos 3x}{\cos x+\sin x}\,\mathrm{d}x$$

$$= \frac{1}{2}\left[\int_0^{\frac{\pi}{2}}\frac{\cos 3x}{\cos x+\sin x}\,\mathrm{d}x+\int_{\frac{\pi}{2}}^{0}\frac{\sin 3x}{\cos x+\sin x}\,\mathrm{d}x\right]$$ 1M

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{\cos 3x}{\cos x+\sin x}\,\mathrm{d}x-\frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{\sin 3x}{\cos x+\sin x}\,\mathrm{d}x$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{\cos 3x-\sin 3x}{\cos x+\sin x}\,\mathrm{d}x$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{\sin\left(3x+\frac{3\pi}{4}\right)}{\sin\left(x+\frac{\pi}{4}\right)}\,\mathrm{d}x$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}}\left[2\cos\left(2x+\frac{\pi}{2}\right)+1\right]\mathrm{d}x$$ 1M

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}}(1-2\sin 2x)\,\mathrm{d}x$$

$$= \frac{1}{2}\Big[x+\cos 2x\Big]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{4}-1$$ 1A

88. (a) Let $u=a-x$. Then $\mathrm{d}u=-\mathrm{d}x$. 1M

$$\int_0^a f(a-x)\,\mathrm{d}x = -\int_a^0 f(u)\,\mathrm{d}u$$

$$= \int_0^a f(u)\,\mathrm{d}u$$ 1M

$$= \int_0^a f(x)\,\mathrm{d}x$$ 1

(b) (i) $I = \displaystyle\int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1 + \cos^2(\pi - x)}\,dx$ 1M

$\quad = \displaystyle\int_0^{\pi} \frac{(\pi - x)\sin x}{1 + \cos^2 x}\,dx$ 1

(ii) Use the result of (b)(i).

$$I = \frac{1}{2}\left[\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx + \int_0^{\pi} \frac{(\pi - x)\sin x}{1+\cos^2 x}\,dx\right]$$ 1M

$$= \frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx$$

Let $u = \pi - x$. Then $du = -\,dx$. 1M

$$I = \frac{\pi}{2}\left[\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}\,dx + \int_{\frac{\pi}{2}}^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx\right]$$

$$= \frac{\pi}{2}\left[\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}\,dx - \int_{\frac{\pi}{2}}^{0} \frac{\sin(\pi - u)}{1+\cos^2(\pi - u)}\,du\right]$$

$$= \frac{\pi}{2}\left[\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}\,dx + \int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}\,dx\right]$$

$$= \pi\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}\,dx$$ 1

(iii) Let $u = \cos x$. Then $du = -\sin x\,dx$. 1M

$$I = \pi\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}\,dx$$

$$= -\pi\int_1^{-1} \frac{du}{1+u^2}$$

$$= \pi\int_{-1}^{1} \frac{du}{1+u^2}$$

Let $u = \tan\theta$. Then $du = \sec^2\theta\,d\theta$. 1M

$$I = \pi\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sec^2\theta}{1+\tan^2\theta}\,d\theta$$

$$= \pi\Big[\theta\Big]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$= \frac{\pi^2}{4}$$ 1

89. (a) Let $u = \dfrac{\pi}{4}$. Then $du = -\,dx$. 1M

$$\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln\left[\cos\left(\frac{\pi}{4} - x\right)\right]dx = -\int_{\frac{\pi}{6}}^{\frac{\pi}{12}} \ln\cos(u)\,du$$

$$= \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln\cos(u)\,du$$ 1M

$$= \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln(\cos x)\,dx$$ 1

(b) $$\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln\left[\cos\left(\frac{\pi}{4}-x\right)\right] dx - \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln(\cos x)\, dx = 0$$

$$\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln \frac{\cos\frac{\pi}{4}\cos x + \sin\frac{\pi}{4}\sin x}{\cos x}\, dx = 0 \qquad \text{1M}$$

$$\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln \frac{1+\tan x}{\sqrt{2}}\, dx = 0$$

$$\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln(\tan x + 1)\, dx = \left(\frac{1}{2}\ln 2\right)\Big[x\Big]_{\frac{\pi}{12}}^{\frac{\pi}{6}} \qquad \text{1M}$$

$$\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln(\tan x + 1)\, dx = \frac{\pi}{24}\ln 2 \qquad \text{1A}$$

(c) (i) $$\tan\frac{\pi}{12} = \tan\left(\frac{\pi}{3}-\frac{\pi}{4}\right)$$

$$= \frac{\tan\frac{\pi}{3} - \tan\frac{\pi}{4}}{1+\tan\frac{\pi}{3}\tan\frac{\pi}{4}} \qquad \text{1M}$$

$$= \frac{\sqrt{3}-1}{1+\sqrt{3}} \times \frac{\sqrt{3}-1}{\sqrt{3}-1}$$

$$= 2-\sqrt{3} \qquad \text{1}$$

(ii) Note that $\frac{d}{dx}\ln(\tan x + 1) = \frac{\sec^2 x}{\tan x + 1}$. 1M

$$\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \frac{x\sec^2 x}{\tan x + 1}\, dx$$

$$= \Big[x\ln(\tan x + 1)\Big]_{\frac{\pi}{12}}^{\frac{\pi}{6}} - \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln(\tan x + 1)\, dx \qquad \text{1M}$$

$$= \frac{\pi}{6}\ln\left(\tan\frac{\pi}{6}+1\right) - \frac{\pi}{12}\ln\left(\tan\frac{\pi}{12}+1\right) - \frac{\pi}{24}\ln 2 \qquad \text{1M}$$

$$= \frac{\pi}{24}\ln\frac{(\sqrt{3}+3)^4}{3^4} - \frac{\pi}{24}\ln\left(3-\sqrt{3}\right)^2 - \frac{\pi}{24}\ln 2$$

$$= \frac{\pi}{24}\ln\left[\frac{(3+\sqrt{3})^4}{3^4(2)(3-\sqrt{3})^2} \times \frac{(3+\sqrt{3})^2}{(3+\sqrt{3})^2}\right] \qquad \text{1M}$$

$$= \frac{\pi}{24}\ln\left[\frac{(3+\sqrt{3})^6}{(3^6)(2^3)}\right]$$

$$= \frac{\pi}{4}\ln\frac{3+\sqrt{3}}{3\sqrt{2}}$$

$$= \frac{\pi}{4}\ln\frac{\sqrt{6}+3\sqrt{2}}{6} \qquad \text{1}$$

90. (a) Let $u = \cos \pi x$. Then $\mathrm{d}u = -\pi \sin \pi x \, \mathrm{d}x$. 1M

$$\int \cos^4 \pi x \sin^3 \pi x \, \mathrm{d}x = -\frac{1}{\pi} \int u^4 (1 - u^2) \, \mathrm{d}u \quad \text{1M}$$

$$= -\frac{1}{\pi} \int (u^4 - u^6) \, \mathrm{d}u$$

$$= -\frac{u^5}{5\pi} + \frac{u^7}{7\pi} + \text{constant}$$

$$= -\frac{\cos^5 \pi x}{5\pi} + \frac{\cos^7 \pi x}{7\pi} + \text{constant} \quad \text{1A}$$

(b) Let $u = 3 - x$. Then $\mathrm{d}u = -\,\mathrm{d}x$. 1M

$$\int_0^3 x \cos^4 \pi x \sin^3 \pi x \, \mathrm{d}x = -\int_3^0 (3 - u) \cos^4 (3\pi - \pi u) \sin^3 (3\pi - \pi u) \, \mathrm{d}u$$

$$= \int_0^3 (3 - x) \cos^4 \pi x \sin^3 \pi x \, \mathrm{d}x$$

$$2 \int_0^3 x \cos^4 x\pi x \sin^3 \pi x \, \mathrm{d}x = 3 \int_0^3 \cos^4 \pi x \sin^3 \pi x \, \mathrm{d}x \quad \text{1M}$$

$$\int_0^3 x \cos^4 \pi x \sin^3 \pi x \, \mathrm{d}x = \frac{3}{2} \left[-\frac{\cos^5 \pi x}{5\pi} + \frac{\cos^7 \pi x}{7\pi} \right]_0^3 \quad \text{1M}$$

$$= \frac{6}{35\pi} \quad \text{1A}$$

91. (a) Let $u = k - x$. Then $\mathrm{d}u = -\,\mathrm{d}x$.

$$\int_0^k \ln(1 + \tan k \tan x) \, \mathrm{d}x = -\int_k^0 \ln \left[1 + \tan k \tan(k - u)\right] \mathrm{d}u$$

$$= \int_0^k \ln \left[1 + \tan k \cdot \frac{\tan k - \tan u}{1 + \tan k \tan u} \right] \mathrm{d}u \quad \text{1M}$$

$$= \int_0^k \ln \left[\frac{1 + \tan k \tan u + \tan k (\tan k - \tan u)}{1 + \tan k \tan u} \right] \mathrm{d}u$$

$$= \int_0^k \ln \frac{\sec^2 k}{1 + \tan k \tan u} \, \mathrm{d}u \quad \text{1M}$$

$$= \int_0^k \ln\left(\sec^2 k\right) \mathrm{d}x - \int_0^k \ln(1 + \tan k \tan x) \, \mathrm{d}x$$

$$2 \int_0^k \ln(1 + \tan k \tan x) \, \mathrm{d}x = 2 \ln(\sec k) \Big[x \Big]_0^k \quad \text{1M}$$

$$\int_0^k \ln(1 + \tan k \tan x) \, \mathrm{d}x = k \ln(\sec k) \quad \text{1}$$

(b) Put $x = \frac{\pi}{4}$.

$$\int_0^{\frac{\pi}{4}} \ln \left(1 + \tan \frac{\pi}{4} \tan x \right) \mathrm{d}x = \frac{\pi}{4} \ln \left(\sec \frac{\pi}{4} \right) \quad \text{1M}$$

$$\int_0^{\frac{\pi}{4}} \ln(1 + \tan x) \, \mathrm{d}x = \frac{\pi}{4} \ln \sqrt{2}$$

$$= \frac{\pi \ln 2}{8} \quad \text{1A}$$

(c) $\dfrac{d}{dx}[\ln(1+\tan x)] = \dfrac{\sec^2 x}{1+\tan x}$ 1A

(d) Let $x = \tan\theta$. Then $dx = \sec^2\theta\, d\theta$. 1M

$$\int_0^1 \frac{\tan^{-1} x}{1+x}\, dx = \int_0^{\frac{\pi}{4}} \frac{\theta}{1+\tan\theta}\sec^2\theta\, d\theta \quad \text{1M}$$

$$= \Big[\theta\ln(1+\tan\theta)\Big]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \ln(1+\tan\theta)\, d\theta \quad \text{1M}$$

$$= \frac{\pi}{4}\ln 2 - \frac{\pi\ln 2}{8}$$

$$= \frac{\pi\ln 2}{8} \quad \text{1A}$$

92. (a) $\displaystyle\int xe^x\, dx = xe^x - \int e^x\, dx$ 1M

$= xe^x - e^x + \text{constant}$ 1A

(b) Let $u = x^2$. Then $du = 2x\, dx$. 1M

$$\int_0^1 x^3 e^{x^2}\, dx = \frac{1}{2}\int_0^1 ue^u\, du$$

$$= \frac{1}{2}\Big[ue^u - e^u\Big]_0^1 \quad \text{1M}$$

$$= \frac{1}{2} \quad \text{1A}$$

(c) Let $f(x) = x^5 e^{x^4}$.

$f(-x) = (-x)^5 e^{(-x)^4} = -x^5 e^{x^4} = -f(x)$ 1M

$f(x)$ is an odd function.

Thus, $\displaystyle\int_{-9}^{9} x^5 e^{x^4}\, dx = 0$. 1A

93. (a) Let $x = 3\tan\theta$. Then $dx = 3\sec^2\theta\, d\theta$. 1M

$$\int_0^1 \frac{1}{x^2+9}\, dx = \int_0^{\tan^{-1}\frac{1}{3}} \frac{3\sec^2\theta}{9\tan^2\theta + 9}\, d\theta$$

$$= \frac{1}{3}\Big[\theta\Big]_0^{\tan^{-1}\frac{1}{3}} \quad \text{1M}$$

$$= \frac{1}{3}\tan^{-1}\frac{1}{3} \quad \text{1A}$$

(b) (i) $\dfrac{1-\tan^2\theta}{1+\tan^2\theta} = \dfrac{1-\frac{\sin^2\theta}{\cos^2\theta}}{1+\frac{\sin^2\theta}{\cos^2\theta}} \times \dfrac{\cos^2\theta}{\cos^2\theta}$

$$= \frac{\cos^2\theta - \sin^2\theta}{\cos^2\theta + \sin^2\theta}$$

$= \cos 2\theta$ 1

(ii) Let $t = \tan\theta$. Then $\mathrm{d}t = \sec^2\theta\,\mathrm{d}\theta = (1+t^2)\,\mathrm{d}\theta$. 1M

$$\int_0^{\frac{\pi}{4}} \frac{1}{4\cos 2\theta + 5}\,\mathrm{d}\theta = \int_0^1 \frac{1}{4\left(\frac{1-t^2}{1+t^2}\right)+5}\cdot\frac{1}{1+t^2}\,\mathrm{d}t \qquad \text{1M}$$

$$= \int_0^1 \frac{1}{t^2+9}\,\mathrm{d}t$$

$$= \frac{1}{3}\tan^{-1}\frac{1}{3} \qquad \text{1A}$$

(c) Let $u = \dfrac{\pi}{4} - \theta$. Then $\mathrm{d}u = -\,\mathrm{d}\theta$. 1M

$$\int_0^{\frac{\pi}{4}} \frac{1}{4\cos 2\theta + 5}\,\mathrm{d}\theta = -\int_{\frac{\pi}{4}}^0 \frac{1}{4\cos\left(\frac{\pi}{4} - 2u\right)+5}\,\mathrm{d}u$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{4\sin 2u + 5}\,\mathrm{d}u$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{4\sin 2\theta + 5}\,\mathrm{d}\theta \qquad \text{1}$$

(d) $$\int_0^{\frac{\pi}{4}} \frac{4\sin 2\theta + 4\cos 2\theta + 10}{(4\sin 2\theta + 5)(4\cos 2\theta + 5)}\,\mathrm{d}\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{\cos 2\theta + 5}\,\mathrm{d}\theta + \int_0^{\frac{\pi}{4}} \frac{1}{4\sin 2\theta + 5}\,\mathrm{d}\theta \qquad \text{1M}$$

$$= 2\int_0^{\frac{\pi}{4}} \frac{1}{4\cos 2\theta + 5}\,\mathrm{d}\theta$$

$$= 2\left(\frac{1}{3}\tan^{-1}\frac{1}{3}\right) \qquad \text{1M}$$

$$= \frac{2}{3}\tan^{-1}\frac{1}{3} \qquad \text{1A}$$

94. (a) Let $u = a\tan\theta$. Then $\mathrm{d}u = a\sec^2\theta\,\mathrm{d}\theta$. 1M

$$\int \frac{u^2}{a^2+u^2}\,\mathrm{d}u = \int \frac{a^2\tan^2\theta}{a^2 + a^2\tan^2\theta}\cdot a\sec^2\theta\,\mathrm{d}\theta \qquad \text{1M}$$

$$= a\int \tan^2\theta\,\mathrm{d}\theta$$

$$= a\int (\sec^2\theta - 1)\,\mathrm{d}\theta$$

$$= a\tan\theta - a\theta + \text{constant}$$

$$= u - a\tan^{-1}\frac{u}{a} + \text{constant} \qquad \text{1A}$$

(b) Let $u = -x$. Then $\mathrm{d}u = -\,\mathrm{d}x$. 1M

$$\int_{-b}^{b} g(x)\log_{25}\left(5^{h(x)}+1\right)\mathrm{d}x$$

$$=\int_{-b}^{0} g(x)\log_{25}\left(5^{h(x)}+1\right)\mathrm{d}x+\int_{0}^{b} g(x)\log_{25}\left(5^{h(x)}+1\right)\mathrm{d}x \qquad \text{1M}$$

$$=-\int_{b}^{0} g(-u)\log_{25}\left(5^{h(-u)}+1\right)\mathrm{d}u+\int_{0}^{b} g(x)\log_{25}\left(5^{h(x)}+1\right)\mathrm{d}x$$

$$=-\int_{0}^{b} g(u)\log_{25}\left(5^{-h(u)}+1\right)\mathrm{d}u+\int_{0}^{b} g(x)\log_{25}\left(5^{h(x)}+1\right)\mathrm{d}x \qquad \text{1M}$$

$$=-\int_{0}^{b} g(x)\log_{25}\left(5^{-h(x)}+1\right)\mathrm{d}x+\int_{0}^{b} g(x)\log_{25}\left(5^{h(x)}+1\right)\mathrm{d}x$$

$$=\int_{0}^{b} g(x)\log_{25}\left[\left(5^{h(x)}+1\right)\div\left(\frac{1+5^{h(x)}}{5^{h(x)}}\right)\right]\mathrm{d}x$$

$$=\int_{0}^{b} g(x)\log_{25}\left(5^{h(x)}\right)\mathrm{d}x$$

$$=\frac{1}{2}\int_{0}^{b} g(x)h(x)\,\mathrm{d}x \qquad 1$$

(c) Put $g(x)=\dfrac{16^x-16^{-x}}{16^x+16^{-x}-1}$ and $h(x)=4^x-4^{-x}$.

$g(-x)=\dfrac{16^{-x}-16^x}{16^{-x}+16^x-1}=-g(x)$

$h(-x)=4^{-x}-4^x=-h(x)$

Both $g(x)$ and $h(x)$ are odd functions. 1M

Let $u=4^x-4^{-x}$. Then $\mathrm{d}u=(4^x+4^{-x})(\ln 4)\,\mathrm{d}x$.

$$\int_{-1}^{1}\frac{(16^x-16^{-x})\log_{25}\left(5^{4^x-4^{-x}}+1\right)}{16^x+16^{-x}-1}\mathrm{d}x$$

$$=\frac{1}{2}\int_{0}^{1}\frac{(16^x-16^{-x})(4^x-4^{-x})}{16^x+16^{-x}-1}\mathrm{d}x \qquad \text{1M}$$

$$=\frac{1}{2}\int_{0}^{1}\frac{(4^x+4^{-x})(4^x-4^{-x})^2}{(4^x+4^{-x})^2+1}\mathrm{d}x$$

$$=\frac{1}{2\ln 4}\int_{0}^{\frac{15}{4}}\frac{u^2}{1+u^2}\mathrm{d}u \qquad \text{1M}$$

$$=\frac{1}{4\ln 2}\left[x-\tan^{-1}x\right]_0^{\frac{15}{4}} \qquad \text{1M}$$

$$=\frac{15}{16\ln 2}-\frac{1}{4\ln 2}\tan^{-1}\frac{15}{4} \qquad \text{1A}$$

95. (a) Let $u=-x$. Then $\mathrm{d}u=-\,\mathrm{d}x$. 1M

$$\int_{-a}^{a} f(x)\,\mathrm{d}x=\int_{-a}^{0} f(x)\,\mathrm{d}x+\int_{0}^{a} f(x)\,\mathrm{d}x$$

$$=-\int_{a}^{0} f(-u)\,\mathrm{d}u+\int_{0}^{a} f(x)\,\mathrm{d}x$$

$$=\int_{0}^{a} f(-u)\,\mathrm{d}u+\int_{0}^{a} f(x)\,\mathrm{d}x \qquad \text{1M}$$

$$=\int_{0}^{a}[f(x)+f(-x)]\,\mathrm{d}x \qquad 1$$

(b) $\tan\left(\frac{\pi}{8}+\frac{\pi}{8}\right) = \dfrac{2\tan\frac{\pi}{8}}{1-\tan^2\frac{\pi}{8}}$ 1M

$$1 = \frac{2\tan\frac{\pi}{8}}{1-\tan^2\frac{\pi}{8}}$$

$$1-\tan^2\frac{\pi}{8} = 2\tan\frac{\pi}{8}$$

$$1-2\tan\frac{\pi}{8} = \tan^2\frac{\pi}{8}$$ 1A

(c) $\displaystyle\int_{-1}^{1}\frac{\tan\frac{\pi}{8}}{(1+e^x)\left(x^2+1-2\tan\frac{\pi}{8}\right)}\,\mathrm{d}x$

$$= \int_{-1}^{1}\frac{\tan\frac{\pi}{8}}{(1+e^x)\left(x^2+\tan^2\frac{\pi}{8}\right)}\,\mathrm{d}x$$ 1M

$$= \int_{0}^{1}\left[\frac{\tan\frac{\pi}{8}}{(1+e^x)\left(x^2+\tan^2\frac{\pi}{8}\right)}+\frac{\tan\frac{\pi}{8}}{(1+e^{-x})\left((-x)^2+\tan^2\frac{\pi}{8}\right)}\right]\mathrm{d}x$$ 1M+1M

$$= \int_{0}^{1}\left[\frac{\tan\frac{\pi}{8}}{(1+e^x)\left(x^2+\tan^2\frac{\pi}{8}\right)}+\frac{e^x\tan\frac{\pi}{8}}{(e^{x}+1)\left((-x)^2+\tan^2\frac{\pi}{8}\right)}\right]\mathrm{d}x$$

$$= \int_{0}^{1}\frac{\tan\frac{\pi}{8}}{x^2+\tan^2\frac{\pi}{8}}\,\mathrm{d}x$$ 1A

Let $x = \tan\frac{\pi}{8}\tan\theta$. Then $\mathrm{d}x = \tan\frac{\pi}{8}\sec^2\theta\,\mathrm{d}\theta$. 1M

$$\int_{-1}^{1}\frac{\tan\frac{\pi}{8}}{(1+e^x)\left(x^2+1-2\tan\frac{\pi}{8}\right)}\,\mathrm{d}x$$

$$= \int_{0}^{\frac{3\pi}{8}}\frac{\tan\frac{\pi}{8}\cdot\tan\frac{\pi}{8}\sec^2\theta}{\tan^2\frac{\pi}{8}\tan^2\theta+\tan^2\frac{\pi}{8}}\,\mathrm{d}\theta$$ 1A+1A

$$= \int_{0}^{\frac{3\pi}{8}}\mathrm{d}\theta$$

$$= \Big[\theta\Big]_{0}^{\frac{3\pi}{8}}$$

$$= \frac{3\pi}{8}$$ 1A

96. (a) (i) We have $p+q=0$ and $p(4+\sqrt{7})+q(4-\sqrt{7})=1$.

$$p(4+\sqrt{7})+(-p)(4-\sqrt{7}) = 1$$

$$p = \frac{1}{2\sqrt{7}}$$

Thus, $p = \dfrac{1}{2\sqrt{7}}$ and $q = -\dfrac{1}{2\sqrt{7}}$. 1A

(ii) $$ {}^{1}_{0}\frac{1}{t^2+8t+9}\,\mathrm{d}t = \int_0^1 \frac{1}{(t+4)^2-7}\,\mathrm{d}y $$

$$ = \int_0^1 \frac{p(t+4+\sqrt{7})+q(t+4-\sqrt{7})}{(t+4+\sqrt{7})(t+4-\sqrt{7})}\,\mathrm{d}t $$ 1M

$$ = \frac{1}{2\sqrt{7}}\int_0^1 \left(\frac{1}{t+4-\sqrt{7}} - \frac{1}{t+4+\sqrt{7}}\right)\mathrm{d}t $$

$$ = \frac{1}{2\sqrt{7}}\Big[\ln|t+4-\sqrt{7}| - \ln|t+4+\sqrt{7}|\Big]_0^1 $$ 1M

$$ = \frac{1}{2\sqrt{7}}\ln\frac{(4+\sqrt{7})(5-\sqrt{7})}{(4-\sqrt{7})(5+\sqrt{7})} $$

$$ = \frac{1}{2\sqrt{7}}\ln\frac{13+\sqrt{7}}{13-\sqrt{7}} $$ 1A

(b) (i) $$ \sin 2x = 2\sin x\cos x $$

$$ = \frac{2\sin x\cos x\sec^2 x}{\sec^2 x} $$

$$ = \frac{2\tan x}{1+\tan^2 x} $$ 1

$$ \cos 2x = \cos^2 x - \sin^2 x $$

$$ = \frac{(\cos^2 x - \sin^2 x)\sec^2 x}{\sec^2 x} $$

$$ = \frac{1-\tan^2 x}{1+\tan^2 x} $$ 1

(ii) Let $t = \tan x$. Then $\mathrm{d}t = \sec^2 x\,\mathrm{d}x$. 1M

$$ \int_0^{\frac{\pi}{4}} \frac{1}{4\sin 2x + 4\cos 2x + 5}\,\mathrm{d}x $$

$$ = \int_0^1 \frac{1}{4\left(\frac{2t}{1+t^2}\right) + 4\left(\frac{1-t^2}{1+t^2}\right) + 5}\cdot\frac{1}{t^2+1}\,\mathrm{d}t $$

$$ = \int_0^1 \frac{1}{t^2+8t+9}\,\mathrm{d}t $$

$$ = \frac{1}{2\sqrt{7}}\ln\frac{13+\sqrt{7}}{13-\sqrt{7}} $$ 1A

(c) Let $u = \dfrac{\pi}{4} - x$. Then $\mathrm{d}u = -\,\mathrm{d}x$.

$$ \int_0^{\frac{\pi}{4}} \frac{8\sin 2x + 5}{4\sin 2x + 4\cos 2x + 5}\,\mathrm{d}x $$

$$ = -\int_{\frac{\pi}{4}} \frac{8\sin\left(\frac{\pi}{2}-2u\right)+5}{4\sin\left(\frac{\pi}{2}-2u\right) + 4\cos\left(\frac{\pi}{2}-2u\right) + 5}\,\mathrm{d}u $$ 1M

$$ = \int_0^{\frac{\pi}{4}} \frac{8\cos 2u + 5}{4\cos 2u + 4\sin 2u + 5}\,\mathrm{d}u $$

$$ = \int_0^{\frac{\pi}{4}} \frac{8\cos 2x + 5}{4\sin 2x + 4\cos 2x + 5}\,\mathrm{d}x $$ 1

(d) $\displaystyle\int_0^{\frac{\pi}{4}} \frac{16\sin 2x + 11}{4\sin 2x + 4\cos 2x + 5}\,dx$

$= 2\displaystyle\int_0^{\frac{\pi}{4}} \frac{8\sin 2x + 5}{4\sin 2x + 4\cos 2x + 5}\,dx + \int_0^{\frac{\pi}{4}} \frac{1}{4\sin 2x + 4\cos 2x + 5}\,dx$ 1M

$= \displaystyle\int_0^{\frac{\pi}{4}} \frac{8\sin 2x + 5}{4\sin 2x + 4\cos 2x + 5}\,dx + \int_0^{\frac{\pi}{4}} \frac{8\cos 2x + 5}{4\sin 2x + 4\cos 2x + 5}\,dx + \int_0^{\frac{\pi}{4}} \frac{1}{4\sin 2x + 4\cos 2x + 5}\,dx$

$= \displaystyle\int_0^{\frac{\pi}{4}} \frac{8\sin 2x + 8\cos 2x + 10}{4\sin 2x + 4\cos 2x + 5}\,dx + \int_0^{\frac{\pi}{4}} \frac{1}{4\sin 2x + 4\cos 2x + 5}\,dx$

$= \displaystyle\int_0^{\frac{\pi}{4}} 2\,dx + \frac{1}{2\sqrt{7}} \ln \frac{13+\sqrt{7}}{13-\sqrt{7}}$ 1M

$= \Big[2x\Big]_0^{\frac{\pi}{4}} + \dfrac{1}{2\sqrt{7}} \ln \dfrac{13+\sqrt{7}}{13-\sqrt{7}}$

$= \dfrac{\pi}{2} + \dfrac{1}{2\sqrt{7}} \ln \dfrac{13+\sqrt{7}}{13-\sqrt{7}}$ 1A

97. (a) Let $u = T - x$. Then $du = -dx$. 1M

$$\int_0^T x f(x)\,dx = -\int_T^0 (T-u) f(T-u)\,du$$

$$= \int_0^T (T-u) f(u)\,du$$ 1M

$$= T\int_0^T f(x)\,dx - \int_0^T x f(x)\,dx$$

$$2\int_0^T x f(x)\,dx = T\int_0^T f(x)\,dx$$

$$\int_0^T x f(x)\,dx = \frac{T}{2}\int_0^T f(x)\,dx$$ 1

(b) (i) Let $u = \cos x$. Then $du = -\sin x\,dx$.

$$\int_0^{\pi} \frac{\sin^3 x}{2-\sin^2 x}\,dx = -\int_1^{-1} \frac{1-u^2}{2-(1-u^2)}\,du$$ 1M

$$= \int_{-1}^{1} \frac{1-u^2}{1+u^2}\,du$$

$$= \int_{-1}^{1} \left(\frac{2}{1+u^2} - 1\right) du$$ 1M

$$= 2\int_{-1}^{1} \frac{1}{1+u^2}\,du - \Big[u\Big]_{-1}^{1}$$

$$= 2\int_{-1}^{1} \frac{1}{1+u^2}\,du - 2$$ 1

(ii) Let $u = \tan\theta$. Then $\mathrm{d}u = \sec^2\theta\,\mathrm{d}\theta$. 1M

$$\int_0^{\pi} \frac{\sin^3 x}{2-\sin^2 x}\,\mathrm{d}x = 2\int_{-1}^{1} \frac{1}{1+u^2}\,\mathrm{d}u\,\mathrm{d}u - 2$$

$$= 2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sec^2\theta}{1+\tan^2\theta}\,\mathrm{d}\theta - 2 \qquad \text{1A}$$

$$= 2\Big[\theta\Big]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} - 2$$

$$= \pi - 2 \qquad \text{1A}$$

(c) Put $f(x) = \dfrac{\sin^3 x}{2-\sin^2 x}$.

$$f(\pi - x) = \frac{\sin^3(\pi - x)}{2-\sin^2(\pi - x)} = \frac{\sin^3 x}{2-\sin^2 x} = f(x) \qquad \text{1M}$$

$$\int_0^{\pi} \frac{x\sin^3 x}{2-\sin^2 x}\,\mathrm{d}x = \frac{\pi}{2}\int_0^{\pi} \frac{\sin^3 x}{2-\sin^2 x}\,\mathrm{d}x \qquad \text{1M}$$

$$= \frac{\pi}{2}(\pi - 2) \qquad \text{1A}$$

98. (a) (i) $$\frac{1}{1+2\sin^2 x} = \frac{1}{1+2(1-\cos^2 x)} \div \frac{\cos^2 x}{\cos^2 x}$$

$$= \frac{\sec^2 x}{3\sec^2 x - 2} \qquad \text{1}$$

(ii) Let $u = \tan x$. Then $\mathrm{d}u = \sec^2 x\,\mathrm{d}x$. 1M

$$\int_0^{\frac{\pi}{4}} \frac{1}{1+2\sin^2 x}\,\mathrm{d}x = \int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{3\sec^2 x - 2}\,\mathrm{d}x$$

$$= \int_0^{1} \frac{\mathrm{d}u}{3(u^2+1)-2}$$

$$= \int_0^{1} \frac{\mathrm{d}u}{1+3u^2}$$

Let $\sqrt{3}u = \tan\theta$. Then $\sqrt{3}\,\mathrm{d}u = \sec^2\theta\,\mathrm{d}\theta$. 1M

$$\int_0^{\frac{\pi}{4}} \frac{1}{1+2\sin^2 x}\,\mathrm{d}x = \frac{\sqrt{3}}{3}\int_0^{\frac{\pi}{3}} \frac{\sec^2\theta}{1+\tan^2\theta}\,\mathrm{d}\theta$$

$$= \frac{\sqrt{3}}{3}\Big[\theta\Big]_0^{\frac{\pi}{3}}$$

$$= \frac{\sqrt{3}\pi}{9} \qquad \text{1A}$$

(b) Let $u = -x$. Then $\mathrm{d}u = -\,\mathrm{d}x$. 1M

$$\int_{-m}^{m} g(x)h(x)\,\mathrm{d}x = \int_{-m}^{0} g(x)h(x)\,\mathrm{d}x + \int_{0}^{m} g(x)h(x)\,\mathrm{d}x \quad \text{1M}$$

$$= -\int_{m}^{0} g(-u)h(-u)\,\mathrm{d}u + \int_{0}^{m} g(x)h(x)\,\mathrm{d}x$$

$$= -\int_{0}^{m} g(-u)h(u)\,\mathrm{d}u + \int_{0}^{m} g(x)h(x)\,\mathrm{d}x$$

$$= -\int_{0}^{m} g(-x)h(x)\,\mathrm{d}x + \int_{0}^{m} g(x)h(x)\,\mathrm{d}x$$

$$= \int_{0}^{m} [g(x) - g(-x)]h(x)\,\mathrm{d}x \quad \text{1M}$$

$$= \int_{0}^{m} xh(x)\,\mathrm{d}x \quad \text{1}$$

(c) Put $g(x) = \ln(e^x + 1)$ and $h(x) = \dfrac{\sin 2x}{(1 + 2\sin^2 x)^2}$. 1M

$$h(-x) = \frac{\sin(-2x)}{(1 + 2\sin^2(-2x))^2} = \frac{-\sin 2x}{(1 + 2\sin^2 2x)^2} = -h(x).$$

$$g(x) - g(-x) = \ln(e^x + 1) - \ln(e^{-x} + 1)$$

$$= \ln(e^x + 1) - \ln\left(\frac{1 + e^x}{e^x}\right)$$

$$= \ln e^x$$

$$= x$$

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\ln(e^x + 1)\sin 2x}{(1 + 2\sin^2 x)^2}\,\mathrm{d}x$$

$$= \int_{0}^{\frac{\pi}{4}} \frac{x\sin 2x}{(1 + 2\sin^2 x)^2}\,\mathrm{d}x \quad \text{1M}$$

$$= \left[\frac{-x}{2(1 + \sin^2 x)}\right]_0^{\frac{\pi}{4}} + \int_{0}^{\frac{\pi}{4}} \frac{1}{2(1 + 2\sin^2 x)}\,\mathrm{d}x \quad \text{1M}$$

$$= \frac{-\frac{\pi}{4}}{2\left(1 + 2\sin^2\frac{\pi}{4}\right)} + \frac{1}{2}\left(\frac{\sqrt{3}\pi}{9}\right) \quad \text{1M}$$

$$= \frac{(8\sqrt{3} - 9)\pi}{144} \quad \text{1A}$$

END OF PAPER