1. (a) $\dfrac{1}{4} = \dfrac{2}{3k+2}$

$3k + 2 = 8$

$k = 2$ 1A

$\dfrac{1}{4} = e^{2h}$

$\ln\dfrac{1}{4} = 2h$

$h = \dfrac{1}{2}\ln\dfrac{1}{4}$

$= -\ln 2$ 1A

(b) Required area

$= \displaystyle\int_0^2 \left(e^{-x\ln 2} - \frac{2}{3x+2}\right) dx + \int_2^4 \left(\frac{2}{3x+2} - e^{-x\ln 2}\right) dx$ 1M

$= \left[\dfrac{e^{-x\ln 2}}{-\ln 2} - \dfrac{2}{3}\ln|3x+2|\right]_0^2 + \left[\dfrac{2}{3}\ln|3x+2| + \dfrac{e^{-x\ln 2}}{\ln 2}\right]_2^4$ 1M

≈ 0.260 1A

2. (a) $\displaystyle\int ye^{-y}\, dy = -ye^{-y} + \int e^{-y}\, dy$ 1M

$= -ye^{-y} - e^{-y} + \text{constant}$ 1A

(b) $0 = \dfrac{(1-y)(e^y - 1)}{e^y}$

$y = 1 \quad \text{or} \quad 0$ 1A

Required area

$= \displaystyle\int_0^1 \frac{(1-y)(e^y-1)}{e^y}\, dy$ 1M

$= \displaystyle\int_0^1 (1 - e^{-y} - y + ye^{-y})\, dy$

$= \left[y + e^{-y} - \dfrac{y^2}{2} - ye^{-y} - e^{-y}\right]_0^1$ 1M

$= \dfrac{1}{2} - \dfrac{1}{e}$ 1A

3. (a) $\displaystyle\int u(4^u)\, du = \frac{u(4^u)}{\ln 4} - \frac{1}{\ln 4}\int 4^u\, du$ 1M

$= \dfrac{u(4^u)}{\ln 4} - \dfrac{4^u}{(\ln 4)^2} + \text{constant}$ 1A

$= \dfrac{u(4^u)}{2\ln 2} - \dfrac{4^u}{4(\ln 2)^2} + \text{constant}$

(b) Required area

$$= \int_{-1}^{0} x^2(4^x)\,\mathrm{d}x \qquad \text{1A}$$

$$= \left[\frac{x^2(4^x)}{\ln 4}\right]_{-1}^{0} - \frac{2}{\ln 4}\int_{-1}^{0} x(4^x)\,\mathrm{d}x \qquad \text{1M}$$

$$= \frac{1}{8\ln 2} - \frac{1}{\ln 2}\left[\frac{x(4^x)}{\ln 4} - \frac{4^x}{(\ln 4)^2}\right]_{-1}^{0} \qquad \text{1M}$$

$$= \frac{3 - 2\ln 2 - 2(\ln 2)^2}{16(\ln 2)^3} \qquad \text{1A}$$

4. (a) $\int u(21^u)\,\mathrm{d}u = \frac{1}{\ln 21}\left[u(21^u) - \int 21^u\,\mathrm{d}u\right]$ 1M

$= \frac{u(21^u)}{\ln 21} - \frac{21^u}{(\ln 21)^2} + \text{constant}$ 1A

(b) Let $u = 2x$. Then $\mathrm{d}u = 2\,\mathrm{d}x$. 1M

Required area

$$= \int_{0}^{1} x(21^{2x})\,\mathrm{d}x \qquad \text{1M}$$

$$= \frac{1}{4}\int_{0}^{2} u(21^u)\,\mathrm{d}u$$

$$= \frac{1}{4}\left[\frac{u(21^u)}{\ln 21} - \frac{21^u}{(\ln 21)^2}\right]_{0}^{2} \qquad \text{1M}$$

$$= \frac{441}{2\ln 21} - \frac{110}{(\ln 21)^2} \qquad \text{1A}$$

5. (a) $y = e^{h-x}$

$\frac{\mathrm{d}y}{\mathrm{d}x} = -e^{h-x}$ 1A

Consider the slope of C at $x = h$.

$-h = -e^{h-h}$

$h = -1$ 1A

(b) Required area

$$= \int_{0}^{1} \left[e^{1-x} - (-x+2)\right]\mathrm{d}x \qquad \text{1M}$$

$$= \left[-e^{1-x} + \frac{x^2}{2} - 2x\right]_{0}^{1} \qquad \text{1A}$$

$$= e - \frac{5}{2} \qquad \text{1A}$$

6. (a) $\int x^2 \ln x\,\mathrm{d}x = \frac{x^3 \ln x}{3} - \frac{1}{3}\int x^2\,\mathrm{d}x$ 1M+1M

$= \frac{x^3 \ln x}{3} - \frac{x^3}{9} + \text{constant}$ 1A

(b) $x^2 = x^2 \ln x$

$x^2(1 - \ln x) = 0$

$x = e$ or 0 (rejected) 1A

Required area

$= \int_1^e (x^2 - x^2 \ln x)\, \mathrm{d}x$ 1M

$= \left[\frac{x^3}{3} - \frac{x^3 \ln x}{3} - \frac{x^3}{9} \right]_1^e$ 1M

$= \frac{e^3 - 4}{9}$ 1A

7. (a) $y = x - 3$ 1A

(b) Required area $= \int_2^3 \left[\left(x - 3 + \frac{16}{x^2 - 4x + 7} \right) - (x - 3) \right] \mathrm{d}x$ 1M

$= \int_2^3 \frac{16}{x^2 - 4x + 7}\, \mathrm{d}x$

$= \int_2^3 \frac{16}{(x-2)^2 + 3}\, \mathrm{d}x$ 1M

Let $x - 2 = \sqrt{3} \tan \theta$. Then $\mathrm{d}x = \sqrt{3} \sec^2 \theta \, \mathrm{d}\theta$. 1M

Required area $= \int_0^{\frac{\pi}{6}} \frac{16\sqrt{3} \sec^2 \theta}{(\sqrt{3} \tan \theta)^2 + 3}\, \mathrm{d}\theta$

$= \frac{16\sqrt{3}}{3} \int_0^{\frac{\pi}{6}} \frac{\sec^2 \theta}{\sec^2 \theta}\, \mathrm{d}\theta$ 1M

$= \frac{16\sqrt{3}}{3} \int_0^{\frac{\pi}{6}} \mathrm{d}\theta$

$= \frac{16\sqrt{3}}{3} \Big[\theta \Big]_0^{\frac{\pi}{6}}$

$= \frac{8\sqrt{3}\pi}{9}$ 1A

8. (a) $\int (x^2 + 1) \ln x \, \mathrm{d}x = \left(\frac{x^3}{3} + x \right) \ln x - \int \left(\frac{x^2}{3} + 1 \right) \mathrm{d}x$ 1M

$= \left(\frac{x^3}{3} + x \right) \ln x - \frac{x^3}{9} - x + \text{constant}$ 1A

(b) Required area

$= \int_{\frac{1}{3}}^{\frac{e}{3}} (9x^2 + 1) \ln 3x \, \mathrm{d}x$ 1M

Let $u = 3x$. Then $\mathrm{d}u = 3\, \mathrm{d}x$.

Required area

$= \dfrac{1}{3}\displaystyle\int_1^e (u^2 + 1) \ln u \, \mathrm{d}u$ 1M

$= \dfrac{1}{3}\left[\left(\dfrac{u^3}{3} + u \right) \ln u - \dfrac{u^3}{9} - u \right]_1^e$ 1M

$= \dfrac{2e^3}{27} + \dfrac{10}{27}$ 1A

9. (a) Let $y = x^2 + 3$. Then $\mathrm{d}u = 2x \, \mathrm{d}x$.

$y = \displaystyle\int \dfrac{-48x}{(x^2+3)^2} \mathrm{d}x$ 1M

$= -\displaystyle\int \dfrac{24}{u^2} \mathrm{d}u$

$= \dfrac{24}{x^2+3} + C$

Γ passes through $(0,\ 2)$.

$2 = \dfrac{24}{0+3} + C$ 1M

$C = -6$

The equation of Γ is $y = \dfrac{24}{x^2+3} - 6$. 1A

(b) $\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \dfrac{-48(x^2+3)^2 - (-48x)(2)(x^2+3)(2x)}{(x^2+3)^4}$ 1M

$= \dfrac{-48(x^2+3-4x^2)}{(x^2+3)^3}$

$= \dfrac{144(x+1)(x-1)}{(x^2+3)^3}$

When $\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 0$, $x = -1$ or 1.

x	$x < -1$	$-1 < x < 1$	$x > 1$
$\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2}$	$+$	$-$	$+$

1M

The points of inflexion of Γ are $(-1,\ 0)$ and $(1,\ 0)$. 1A

(c) Let $x = \sqrt{3} \tan \theta$. Then $\mathrm{d}x = \sqrt{3} \sec^2 \theta \, \mathrm{d}\theta$. 1M

Required area

$= \int_{-1}^{1} \left(\frac{24}{x^2+3} - 6 \right) \mathrm{d}x$ 1M

$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{24\sqrt{3}\sec^2\theta}{3\tan^2\theta + 3} \mathrm{d}\theta - 6\Big[x\Big]_{-1}^{1}$

$= 8\sqrt{3} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \mathrm{d}\theta - 12$

$= 8\sqrt{3}\Big[\theta\Big]_{\frac{\pi}{6}}^{\frac{\pi}{3}} - 12$

$= \frac{8\sqrt{3}\pi}{3} - 12$ 1A

10. (a) $\frac{\mathrm{d}y}{\mathrm{d}x} = -\sin x$ 1M

The equation of L is

$$y - \cos h = -\sin h(x-h)$$

$$y = -\sin h(x-h) + \cos h$$

$A = \int_0^h [-\sin h(x-h) + \cos h - \cos x]\, \mathrm{d}x$ 1M

$= \left[-\sin h \left(\frac{x^2}{2} - hx \right) + (\cos h)x - \sin x \right]_0^h$ 1M

$= \frac{1}{2}h^2 \sin h + h\cos h - \sin h$ 1A

(b) $\frac{\mathrm{d}h}{\mathrm{d}t} = 2^{-t}\ln 2$ 1M

$\frac{\mathrm{d}A}{\mathrm{d}t} = \left(h\sin h + \frac{1}{2}h^2\cos h + \cos h - h\sin h - \cos h \right) \frac{\mathrm{d}h}{\mathrm{d}t}$ 1M

$= \frac{1}{2}h^2 \cos h \frac{\mathrm{d}h}{\mathrm{d}t}$

$= \frac{1}{2}(2^{-2t})\cos(2^{-t}) \cdot (-2^{-t})\ln 2$ 1M

$= -\frac{\ln 2}{2}(\cos 2^{-t})2^{-3t}$

Put $t = 1$.

$\frac{\mathrm{d}A}{\mathrm{d}t} = -\frac{\ln 2}{2}\cos\frac{1}{2} \cdot 2^{-3}$

$= -\frac{1}{16}(\ln 2)\cos\frac{1}{2}$ 1A

11. (a) $y = e^{-\frac{x}{3}+2}$

$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{3}e^{-\frac{x}{3}+2}$ 1M

Consider the slope of L.

$$-\frac{1}{3}e^{-\frac{6}{3}+2} = \frac{0-1}{c-6}$$ 1M

$$c = 9$$ 1A

(b) The equation of L is $y = -\frac{x}{3} + 3$. 1M

Required area

$$= \int_6^9 \left(e^{-\frac{x}{3}+2} - \left(-\frac{x}{3} + 3\right)\right) \mathrm{d}x \quad \text{1M}$$

$$= \left[-3e^{-\frac{x}{3}+2} + \frac{x^2}{6} - 3x\right]_6^9 \quad \text{1M}$$

$$= \frac{3}{2} - \frac{3}{e} \quad \text{1A}$$

12. $\cos x = \cos 2x$ 1M

$\cos x = 2\cos^2 x - 1$

$0 = 2\cos^2 x - \cos x - 1$ 1A

$\cos x = 1 \quad \text{or} \quad -\frac{1}{2}$

For $0 \le x < \frac{3\pi}{2}$, we have $x = 0$ or $\frac{2\pi}{3}$ or $\frac{4\pi}{3}$. 1A

Required area

$$= \int_0^{\frac{2\pi}{3}} (\cos x - \cos 2x)\, \mathrm{d}x + \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} (\cos 2x - \cos x)\, \mathrm{d}x \quad \text{1M}$$

$$= \left[\sin x - \frac{\sin 2x}{2}\right]_0^{\frac{2\pi}{3}} + \left[\frac{\sin 2x}{2} - \sin x\right]_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \quad \text{1A}$$

$$= \frac{9\sqrt{3}}{4} \quad \text{1A}$$

13. (a)

$$f(x) = \frac{(3-2x)^2}{(x+1)^2}$$

$$= 4 - \frac{20}{x+1} + \frac{25}{(x+1)^2}$$

$$f'(x) = \frac{20}{(x+1)^2} - \frac{50}{(x+1)^3}$$

$$= \frac{10(2x-3)}{(x+1)^3} \quad \text{1A}$$

$$f''(x) = \frac{20}{(x+1)62} - \frac{50}{(x+1)^3}$$

$$= \frac{-10(4x-11)}{(x+1)^4} \quad \text{1}$$

(b) When $f'(x) = 0$, $x = \frac{3}{2}$.

x	$x < -1$	$-1 < x < \frac{3}{2}$	$x > \frac{3}{2}$
$f'(x)$	+	−	+

1M

No maximum point.

Minimum point is $\left(\frac{3}{2},\ 0\right)$. 1A

When $f''(x) = 0$, $x = \dfrac{11}{4}$.

x	$x < -1$	$-1 < x < \dfrac{11}{4}$	$x > \dfrac{11}{4}$
$f''(x)$	$+$	$+$	$-$

Point of inflexion is $\left(\dfrac{11}{4}, \dfrac{4}{9}\right)$. 1A

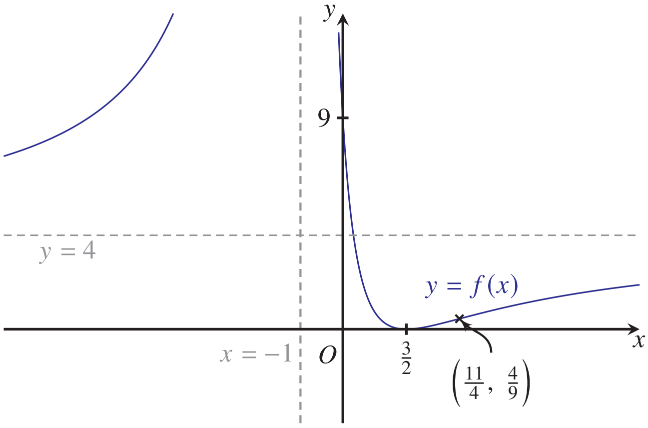
(c) Vertical asymptote is $x = -1$. 1A

$$f(x) = 4 - \frac{20}{x+1} + \frac{25}{(x+1)^2}$$ 1M

Horizontal asymptote is $y = 4$. 1A

(d) (Shape and asymptotes) 1A

(Special points and all correct) 1A



(e) Required area

$$= \int_0^{\frac{3}{2}} f(x)\,\mathrm{d}x$$ 1A

$$= \int_0^{\frac{3}{2}} \left(4 - \frac{20}{x+1} + \frac{25}{(x+1)^2}\right) \mathrm{d}x$$

$$= \left[4x - 20\ln|x+1| - \frac{25}{x+1}\right]_0^{\frac{3}{2}}$$ 1M

$$= 21 - 20\ln\frac{5}{2}$$ 1A

14. (a) Vertical asymptote is $x = -3$. 1A

$$f(x) = \frac{-x^3 + 6x^2}{(x+3)^2}$$

$$= -x + 12 - \frac{63}{x+3} + \frac{81}{(x+3)^2}$$ 1M

Oblique asymptote is $y = -x + 12$. 1A

(b) $f'(x) = -1 + \dfrac{63}{(x+3)^2} - \dfrac{162}{(x+3)^3}$ 1M

$f''(x) = \dfrac{-126}{(x+3)^3} + \dfrac{486}{(x+3)^4}$

$= \dfrac{-18(7x-6)}{(x+3)^4}$ 1A

(c) $f'(x) = -1 + \dfrac{63}{(x+3)^2} - \dfrac{162}{(x+3)^3}$

$= \dfrac{-x(x+12)(x-3)}{(x+3)^3}$

When $f'(x) = 0$, $x = -12$ or 0 or 3.

x	$x < -12$	$-12 < x < -3$	$-3 < x < 0$	$0 < x < 3$	$x > 3$
$f'(x)$	$-$	$+$	$-$	$+$	$-$

1M

H has two minimum points at $x = -12$ and $x = 0$.

H has one maximum point at $x = 3$.

H has three turning points.

The claim is disagreed. 1A

(d) When $f''(x) = 0$, $x = \dfrac{6}{7}$.

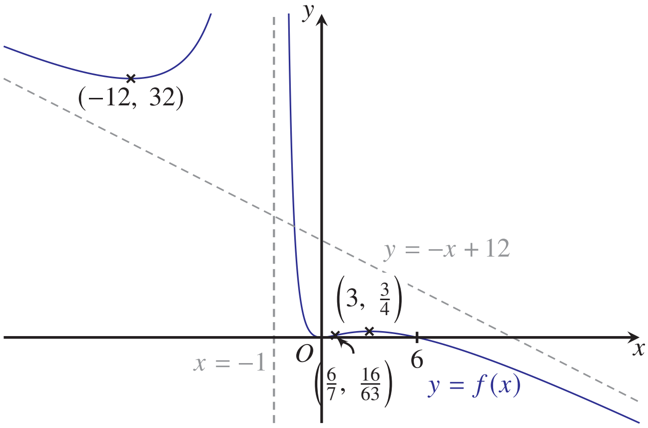
x	$x < -3$	$-3 < x < \frac{6}{7}$	$x > \frac{6}{7}$
$f''(x)$	$+$	$+$	$-$

1M

Point of inflexion is $\left(\dfrac{6}{7}, \dfrac{16}{63}\right)$. 1A

(e) (Maximum point, minimum point, point of inflexion, intercepts) 1A

(All correct) 1A



(f) Required area

$= \displaystyle\int_0^6 \left(-x + 12 - \frac{63}{x+3} + \frac{81}{(x+3)^2}\right) dx$ 1M

$= \left[-\dfrac{x^2}{2} + 12x - 63\ln|x+3| - \dfrac{81}{x+3}\right]_0^6$ 1M

$= 72 - 63\ln 3$ 1A

15. (a) No vertical asymptote.

$$f(x) = x + 6 + \frac{11x + 2}{x^2 + 1} \qquad \text{1M}$$

Oblique asymptote is $y = x + 6$. 1A

(b) $$f'(x) = \frac{3(x+2)^2(x^2+1) - (x+2)^3(2x)}{(x^2+1)^2} \qquad \text{1M}$$

$$= \frac{(x+2)^2(x-1)(x-3)}{(x^2+1)^2}$$

When $f'(x) = 0$, $x = -2$ or 1 or 3. 1M

x	$x < -2$	$-2 < x < 1$	$1 < x < 3$	$x > 3$
$f'(x)$	$+$	$+$	$-$	$+$

1M

Minimum point is $\left(3, \dfrac{25}{2}\right)$. 1A

Maximum point is $\left(1, \dfrac{27}{2}\right)$. 1A

(c) The coordinates of P and Q are $(0,\ 8)$ and $\left(1, \dfrac{27}{2}\right)$ respectively.

Equation of PQ is

$$\frac{y - \frac{27}{2}}{x - 1} = \frac{8 - \frac{27}{2}}{0 - 1}$$

$$y = \frac{11x}{2} + 8 \qquad \text{1M}$$

Required area

$$= \int_0^1 \left[f(x) - \left(\frac{11x}{2} + 8 \right) \right] \mathrm{d}x \qquad \text{1M}$$

$$= \int_0^1 \left(-\frac{9x}{2} - 2 + \frac{11x}{x^2+1} + \frac{2}{x^2+1} \right) \mathrm{d}x$$

Let $u = x^2 + 1$. Then $\mathrm{d}u = 2x\,\mathrm{d}x$. 1M

Let $x = \tan\theta$. Then $\mathrm{d}x = \sec^2\theta\,\mathrm{d}\theta$. 1M

Required area

$$= \left[-\frac{9x^2}{4} - 2x \right]_0^1 + \frac{11}{2} \int_1^2 \frac{\mathrm{d}u}{u} + 2 \int_0^{\frac{\pi}{4}} \frac{\sec^2\theta}{\tan^2\theta + 1}\,\mathrm{d}\theta$$

$$= -\frac{17}{4} + \frac{11}{2} \Big[\ln|u| \Big]_1^2 + 2 \Big[\theta \Big]_0^{\frac{\pi}{4}}$$

$$= -\frac{17}{4} + \frac{11}{2} \ln 2 + \frac{\pi}{2} \qquad \text{1A}$$

16. (a) Vertical asymptote is $x = -1$. 1A

$$f(x) = x + \frac{1}{x+1} + \frac{3}{(x+1)^2} \qquad \text{1M}$$

Oblique asymptote is $y = x$. 1A

(b) (i) $f'(x) = 1 - \dfrac{1}{(x+1)^2} - \dfrac{6}{(x+1)^3}$ 1M

$= \dfrac{(x-1)(x^2+4x+6)}{(x+1)^3}$

When $f'(x) = 0$, $x = 1$.

x	$x < -1$	$-1 < x < 1$	$x > 1$
$f'(x)$	$+$	$-$	$+$

1M

C has a minimum point when $x = 1$. 1

(ii) Required coordinates are $\left(1,\ \dfrac{9}{4}\right)$. 1A

(c) (i) $b = \dfrac{0+0+0+4}{(0+1)^2}$

$b = 4$ 1A

(ii) Required area

$= \displaystyle\int_0^1 [4 - f(x)]\,\mathrm{d}x$ 1M

$= \displaystyle\int_0^1 \left(4 - x - \frac{1}{x+1} - \frac{3}{(x+1)^2}\right)\mathrm{d}x$ 1M

$= \left[4x - \dfrac{x^2}{2} - \ln|x+1| + \dfrac{3}{x+1}\right]_0^1$ 1M

$= 2 - \ln 2$ 1A

17. (a) $f(x) = \dfrac{(x-6)^3}{(x+6)^2}$

$= x - 30 + \dfrac{432}{x+6} - \dfrac{1728}{(x+6)^2}$

$f'(x) = 1 - \dfrac{432}{(x+6)^2} + \dfrac{3456}{(x+6)^3}$ 1M

$= \dfrac{(x+30)(x-6)^2}{(x+6)^3}$ 1A

$f''(x) = \dfrac{864}{(x+6)^3} - \dfrac{10\,368}{(x+6)^4}$

$= \dfrac{864(x-6)}{(x+6)^4}$ 1A

(b) When $f'(x) = 0$, $x = -30$ or 6.

x	$x < -30$	$-30 < x < -6$	$-6 < x < 6$	$x > 6$
$f'(x)$	$+$	$-$	$+$	$+$

1M

Maximum point is $(-30,\ -81)$.

H has 1 turning point. 1A

(c) Vertical asymptote is $x = -6$. 1A

$f(x) = x - 30 + \dfrac{432}{x+6} - \dfrac{1728}{(x+6)^2}$ 1M

Oblique asymptote is $y = x - 30$. 1A

(d) (i) We have $f(3) = -\frac{1}{3}$ and $f'(3) = \frac{11}{27}$. 1M

Required equation is

$$\frac{y + \frac{1}{3}}{x - 3} = \frac{11}{27}$$

$$y = \frac{11x}{27} - \frac{14}{9}$$ 1A

(ii) Required area

$$= \int_3^{\frac{42}{11}} \left[\left(\frac{11x}{27} - \frac{14}{9} \right) - f(x) \right] \mathrm{d}x + \int_{\frac{42}{11}}^{6} [-f(x)] \, \mathrm{d}x$$ 1M

$$= \int_3^{\frac{42}{11}} \left(\frac{11x}{27} - \frac{14}{9} \right) \mathrm{d}x - \int_3^{6} \left(x - 30 + \frac{432}{x+6} - \frac{1728}{(x+6)^2} \right) \mathrm{d}x$$

$$= \left[\frac{11x^2}{54} - \frac{14x}{9} \right]_3^{\frac{42}{11}} - \left[\frac{x^2}{2} - 30x + 432 \ln |x+6| + \frac{1728}{x+6} \right]_3^6$$ 1M

$$= \frac{1368}{11} + 432 \ln \frac{3}{4}$$ 1A

18. (a) $$f(x) = \frac{(x-1)^3}{(x+1)^2}$$

$$= x - 5 + \frac{12}{x+1} - \frac{8}{(x+1)^2}$$

$$f'(x) = 1 - \frac{12}{(x+1)^2} + \frac{16}{(x+1)^3}$$ 1M

$$= \frac{(x-1)^2(x+5)}{(x+1)^3}$$ 1

$$f''(x) = \frac{24}{(x+1)^3} - \frac{48}{(x+1)^4}$$

$$= \frac{24(x-1)}{(x+1)^4}$$ 1A

(b) When $f'(x) = 0$, $x = 1$ or -5.

x	$x < -5$	$-5 < x < -1$	$-1 < x < 1$	$x > 1$
$f'(x)$	+	−	+	+

1M

Maximum point is $\left(-5, -\frac{27}{2}\right)$. 1A

No minimum point.

When $f''(x) = 0$, $x = 1$.

x	$x < -1$	$-1 < x < 1$	$x > 1$
$f''(x)$	−	−	+

1M

Point of inflexion is $(1, 0)$. 1A

(c) (i) $f'(3) = \frac{1}{2}$

Required equation is

$$\frac{y-\frac{1}{2}}{x-3}=\frac{1}{2}$$

$$y=\frac{x}{2}-1$$ 1A

(ii) $f(x)=\dfrac{(x-1)^3}{(x+1)^2}$

For $1<x<3$, we have $(x-1)^3>0$ and $(x+1)^2>2$.

Thus, $f(x)>0$ and G lies above the x-axis on $(1,\ 3)$. 1

For $1<x<3$, $f''(x)=\dfrac{24(x-1)}{(x+1)^4}>0$.

G is concave upwards on the interval $(1,\ 3)$.

L is a tangent to G at $x=3$.

Thus, G lies above L. 1

(iii) Required area

$$=\int_1^3 f(x)\,\mathrm{d}x-\int_2^3\left(\frac{x}{2}-1\right)\mathrm{d}x$$ 1M

$$=\int_1^3\left(x-5+\frac{12}{x+1}-\frac{8}{(x+1)^2}\right)\mathrm{d}x-\int_2^3\left(\frac{x}{2}-1\right)\mathrm{d}x$$

$$=\left[\frac{x^2}{2}-5x+12\ln|x+1|+\frac{8}{x+1}\right]_1^3-\left[\frac{x^2}{4}-x\right]_2^3$$ 1M

$$=12\ln 2-\frac{33}{4}$$ 1A

19. (a) $f(x)=\dfrac{x^2+cx-5}{x+3}$

$$=x+c-3+\frac{4-3c}{x+3}$$

Oblique asymptote is $y=x+c-3$. 1M

$$1=8+c-3$$

$$c=-4$$ 1A

(b) $f(x)=x-7+\dfrac{16}{x+3}$

$$f'(x)=1-\frac{16}{(x+3)^2}$$ 1M

$$=\frac{(x+7)(x-1)}{(x+3)^2}$$ 1A

(c) When $f'(x)=0$, $x=-7$ or 1. 1M

x	$x<-7$	$-7<x<-3$	$-3<x<1$	$x>1$
$f'(x)$	$+$	$-$	$-$	$+$

1M

Maximum point is $(-7,\ -18)$. 1A

Minimum point is $(1,\ -2)$. 1A

(d) Area of R

$= \int_0^m (x + 9 - f(x))\,\mathrm{d}x$ 1M

$= \int_0^m \left(16 - \frac{16}{x+3}\right)\mathrm{d}x$

$= \Big[16x - 16\ln|x+3|\Big]_0^m$ 1M

$= 16m - 16\ln(m+3) + 16\ln 3$ 1A

$< 16m - 16\ln 3 + 16\ln 3$

$= 16m$

The area of R is less than $16m$. 1A

20. (a)

$$y = \sqrt{24 - x^2}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}(24 - x^2)^{-\frac{1}{2}}(-2x)$$ 1M

$$= \frac{-x}{\sqrt{24 - x^2}}$$

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=3\sqrt{2}} = \frac{-3\sqrt{2}}{\sqrt{24 - (3\sqrt{2})^2}}$$

$$= -\sqrt{3}$$

The equation of L is

$$\frac{y - \sqrt{6}}{x - 3\sqrt{2}} = -3\sqrt{3}$$ 1M

$$y = -\sqrt{3}x + 4\sqrt{6}$$ 1A

(b) (i) $\left(3\sqrt{k - x^2}\right)^2 = \left(-\sqrt{3}x + 4\sqrt{6}\right)^2$ 1M

$$9(k - x^2) = 3x^2 - 24\sqrt{2}x + 96$$

$$0 = 12x^2 - 24\sqrt{2}x + (96 - 9k)$$

L is a tangent to C_1.

$$(24\sqrt{2})^2 - 4(12)(96 - 9k) = 0$$

$$432k - 3456 = 0$$

$$k = 8$$

Put $k = 8$.

$$0 = 12x^2 - 24\sqrt{2}x + 24$$

$$0 = 12(x - \sqrt{2})^2$$

$$x = \sqrt{2}$$

L is the tangent to C_1 at $x = \sqrt{2}$. 1

(ii) $(3\sqrt{8-x^2})^2 = (\sqrt{24-x^2})^2$ 1M

$$9(8-x^2) = 24-x^2$$
$$x^2 = 6$$
$$x = \sqrt{6} \quad \text{or} -\sqrt{6} \text{ (rejected)}$$

Required coordinates are $\left(\sqrt{6},\ 3\sqrt{2}\right)$. 1A

(iii) Required area

$$= \int_2^{\sqrt{6}} \left(-\sqrt{3}x + 4\sqrt{6} - 3\sqrt{8-x^2}\right) dx + \int_{\sqrt{6}}^{3\sqrt{2}} \left(-\sqrt{3}x + 4\sqrt{6} - \sqrt{24-x^2}\right) dx$$ 1M+1A

$$= \int_2^{3\sqrt{2}} \left(-\sqrt{3}x + 4\sqrt{6}\right) dx - 3\int_2^{\sqrt{6}} \sqrt{8-x^2}\, dx - \int_{\sqrt{6}}^{3\sqrt{2}} \sqrt{24-x^2}\, dx$$

Consider $\int_2^{\sqrt{6}} \sqrt{8-x^2}\, dx$. Let $x = \sqrt{8}\sin\theta$. Then $dx = \sqrt{8}\cos\theta\, d\theta$. 1M

$$\int_2^{\sqrt{6}} \sqrt{8-x^2}\, dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sqrt{8-8\sin^2\theta} \cdot \sqrt{8}\cos\theta\, d\theta$$
$$= 8\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \cos^2\theta\, d\theta$$
$$= 4\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (1+\cos 2\theta)\, d\theta$$ 1M
$$= 4\left[\theta + \frac{\sin 2\theta}{2}\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$
$$= -2 + \sqrt{3} + \frac{\pi}{3}$$

Consider $\int_{\sqrt{6}}^{3\sqrt{2}} \sqrt{24-x^2}\, dx$.
Let $x = \sqrt{24}\sin\beta$. Then $dx = \sqrt{24}\cos\beta\, d\beta$.

$$\int_{\sqrt{6}}^{3\sqrt{2}} \sqrt{24-x^2}\, dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sqrt{24-24\sin^2\beta} \cdot \sqrt{24}\cos\beta\, d\beta$$
$$= 24\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos^2\beta\, d\beta$$
$$= 12\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (1+\cos 2\beta)\, d\beta$$
$$= 12\left[\beta + \frac{\sin 2\beta}{2}\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$
$$= 2\pi$$

Required area

$$= \left[-\frac{\sqrt{3}x^2}{2} + 4\sqrt{6}x\right]_2^{3\sqrt{2}} - 3\left(-2 + \sqrt{3} + \frac{\pi}{3}\right) - 2\pi$$
$$= 6 + 14\sqrt{3} - 8\sqrt{6} - 3\pi$$ 1A

21. (a) $f'(x) = -2xe^{-x} + (p - x^2)(-e^{-x})$

$= (-p - 2x + x^2)e^{-x}$ 1A

$f''(x) = (-2 + 2x)e^{-x} + (-p - 2x + x^2)(-e^{-x})$

$= (-2 + p + 4x - x^2)e^{-x}$ 1A

(b) H has two points of inflexion, then $f''(x) = 0$ has at least two distinct real roots.
Since $e^{-x} \neq 0$, the equation $-2 + p + 4x - x^2 = 0$ has two distinct real roots.

$\Delta = 4^2 - 4(-1)(-2 + p) > 0$ 1M

$p > -\dfrac{9}{4}$ 1A

(c) (i) Let α and β be the x-coordinates of the two points of inflexion respectively.
We have $\alpha + \beta = 4$ and $\alpha\beta = 2 - p$. 1M

$(\alpha - \beta)^2 = (2\sqrt{5})^2$

$(\alpha + \beta)^2 - 4\alpha\beta = 20$ 1M

$4^2 - 4(2 - p) = 20$

$p = 3$ 1

We have $f'(x) = (x^2 - 2x - 3)e^{-x} = (x - 3)(x + 1)e^{-x}$.
When $f'(x) = 0$, $x = 3$ or -1.
Since $f''(3) = 4e^{-3} > 0$ and $f''(-1) = -4e < 0$,
the maximum point of H is $(-1,\ 2e)$;
the minimum point of H is $\left(3,\ -6e^{-3}\right)$. 1A

(ii) Area $= \displaystyle\int_{-\sqrt{3}}^{\sqrt{3}} (3 - x^2)e^{-x}\,\mathrm{d}x$ 1M

$= \left[(x^2 - 3)e^{-x}\right]_{-\sqrt{3}}^{\sqrt{3}} - 2\displaystyle\int_{-\sqrt{3}}^{\sqrt{3}} xe^{-x}\,\mathrm{d}x$ 1M

$= 0 + 2\left[xe^{-x}\right]_{-\sqrt{3}}^{\sqrt{3}} - 2\displaystyle\int_{-\sqrt{3}}^{\sqrt{3}} e^{-x}\,\mathrm{d}x$

$= 2\sqrt{3}(e^{-\sqrt{3}} + e^{\sqrt{3}}) + 2\left[e^{-x}\right]_{-\sqrt{3}}^{\sqrt{3}}$ 1M

$= 2\sqrt{3}(e^{-\sqrt{3}} + e^{\sqrt{3}}) + 2(e^{-\sqrt{3}} - e^{\sqrt{3}})$ 1A

22. (a) $f(x) = \dfrac{(x - 6)(x + 3)}{x + 6}$

$= x - 9 + \dfrac{36}{x + 6}$

$f'(x) = 1 - \dfrac{36}{(x + 6)^2}$ 1M

$= \dfrac{x(x + 12)}{(x + 6)^2}$ 1A

(b) When $f'(x) = 0$, $x = 0$ or -12. 1M

x	$x < -12$	$-12 < x < -6$	$-6 < x < 0$	$x > 0$
$f'(x)$	$+$	$-$	$-$	$+$

1M

Maximum point is $(-12,\ -27)$. 1A

Minimum point is $(0,\ -3)$. 1A

(c) Vertical asymptote is $x = -6$. 1A

$$f(x) = x - 9 + \frac{36}{x+6}$$ 1M

Oblique asymptote is $y = x - 9$.

G has only two asymptotes.

The claim is agreed. 1A

(d) Required area

$= \int_{-3}^{6} -f(x)\,\mathrm{d}x$ 1M

$= \int_{-3}^{6} \left(-x + 9 - \frac{36}{x+6}\right) \mathrm{d}x$

$= \left[-\frac{x^2}{2} + 9x - 36 \ln|x+6| \right]_{-3}^{6}$ 1M

$= \frac{135}{2} - 72 \ln 2$ 1A

23. (a) $h(x) = x^2 e^{2x^3}$

$h'(x) = 2xe^{2x^3} + x^2 e^{2x^3}(6x^2)$ 1M

$\quad = 2xe^{2x^3}(1 + 3x^3)$ 1A

$h''(x) = 2e^{2x^3}(6x^2)(x + 3x^4) + 2e^{2x^3}(1 + 12x^3)$

$\quad = 2e^{2x^3}(1 + 18x^3 + 18x^6)$ 1A

(b) When $h'(x) = 0$, $x = 0$ or $-\frac{1}{\sqrt[3]{3}}$. 1M

x	$x < -\frac{1}{\sqrt[3]{3}}$	$-\frac{1}{\sqrt[3]{3}} < x < 0$	$x > 0$
$h'(x)$	$+$	$-$	$+$

1M

The maximum point is $\left(-\frac{1}{\sqrt[3]{3}},\ \frac{1}{\sqrt[3]{9e^2}}\right)$.

The minimum point is $(0,\ 0)$. 1A

(c) (i) The coordinates of P are $\left(1,\ e^2\right)$.

$h'(1) = 2e^2(1+3) = 8e^2$ 1M

Required equation is

$$y - e^2 = 8e^2(x - 1)$$

$$y = 8e^2 x - 7e^2$$ 1A

(ii) Note that $h''(x) = 2e^{2x^3}(1 + 18x^3 + 18x^6) > 0$ for $x > 0$ and L is a tangent to G at $x = 1$.

G is concave upwards for $x > 0$.

Thus, G lies above L. 1

Let $u = 2x^3$. Then $\mathrm{d}u = 6x^2\,\mathrm{d}x$. 1M

Required area

$$= \int_0^1 [x^2e^{2x^3} - (8e^2x - 7e^2)]\,\mathrm{d}x \qquad \text{1M}$$

$$= \int_0^1 x^2e^{2x^3}\,\mathrm{d}x - \int_0^1 (8e^2x - 7e^2)\,\mathrm{d}x$$

$$= \frac{1}{6}\int_0^2 e^u\,\mathrm{d}u - \left[4e^2x^2 - 7e^2x\right]_0^1$$

$$= \frac{1}{6}\left[e^u\right]_0^2 + 3e^2$$

$$= \frac{19e^2 - 1}{6} \qquad \text{1A}$$

24. (a) Vertical asymptote is $x = -2$. 1A

$$f(x) = x - 10 + \frac{48}{x+2} - \frac{64}{(x+2)^2} \qquad \text{1M}$$

Oblique asymptote is $y = x - 10$. 1A

(b) $f'(x) = 1 - \dfrac{48}{(x+2)^2} + \dfrac{128}{(x+2)^3}$ 1M

$$f''(x) = \frac{96}{(x+2)^3} - \frac{384}{(x+2)^4}$$

$$= \frac{96(x-2)}{(x+2)^4} \qquad \text{1A}$$

(c) $f'(x) = \dfrac{(x-2)^2(x+10)}{(x+2)^3}$

When $f'(x) = 0$, $x = 2$ or -10.

x	$x < -10$	$-10 < x < -2$	$-2 < x < 2$	$x > 2$
$f'(x)$	+	−	+	+

1M

There is only one turning point at $x = -10$, which is a maximum point.

The claim is disagreed. 1A

(d) When $f''(x) = 0$, $x = 2$.

x	$x < -2$	$-2 < x < 2$	$x > 2$
$f''(x)$	−	−	+

1M

Point of inflexion is $(2,\ 0)$. 1A

(e) Required area

$$= \int_0^2 \left(x - 10 + \frac{48}{x+2} - \frac{64}{(x+2)^2} \right) \mathrm{d}x$$ 1M

$$= \left[\frac{x^2}{2} - 10x + 48 \ln |x+2| + \frac{64}{x+2} \right]_0^2$$ 1M

$$= 34 - 48 \ln 2$$ 1A

25. (a) $$f'(x) = \frac{2x(x-3) - (x^2+a)(1)}{(x-3)^2}$$ 1M

$$= \frac{x^2 - 6x - a}{(x-3)^2}$$

$$f'(5) = 0 = \frac{5^2 - 6(5) - a}{(5-3)^2}$$ 1M

$$a = -5$$ 1A

(b) $$f'(x) = \frac{(x-1)(x-5)}{(x-3)^2}$$

$f'(x) = 0$ when $x = 1$ or 5. 1M

x	$x < 1$	$1 < x < 3$	$3 < x < 5$	$x > 5$
$f'(x)$	$+$	$-$	$-$	$+$

1M

The maximum point of C is $(1,\ 2)$. 1A

(c) The vertical asymptote is $x = 3$. 1A

$$f(x) = \frac{x^2 - 5}{x - 3}$$

$$= x + 3 + \frac{4}{x-3}$$ 1M

The oblique asymptote is $y = x + 3$. 1A

(d) $$\text{Area} = \int_{-\sqrt{5}}^{\sqrt{5}} f(x)\,\mathrm{d}x$$ 1M+1A

$$= \int_{-\sqrt{5}}^{\sqrt{5}} \left(x + 3 + \frac{4}{x-3} \right) \mathrm{d}x$$

$$= \left[\frac{x^2}{2} + 3x + 4 \ln |x-3| \right]_{-\sqrt{5}}^{\sqrt{5}}$$ 1A

$$= \left(\frac{5}{2} + 3\sqrt{5} + 4 \ln\left(3 - \sqrt{5}\right) \right) - \left(\frac{5}{2} - 3\sqrt{5} + 4 \ln\left(\sqrt{5} + 3\right) \right)$$

$$= 6\sqrt{5} + 4 \ln \frac{3 - \sqrt{5}}{3 + \sqrt{5}}$$ 1A

26. (a) Vertical asymptote is $x = -\frac{1}{2}$. 1A

$$f(x) = \frac{x^2 - 4x}{2x + 1}$$

$$= \frac{x}{2} - \frac{9}{4} + \frac{9}{4(2x+1)}$$ 1M

Oblique asymptote is $y = \frac{x}{2} - \frac{9}{4}$. 1A

(b) $f'(x) = \frac{1}{2} - \frac{9}{4}(2x+1)^{-2}(2)$ 1M

$= \frac{1}{2} - \frac{9}{2(2x+1)^2}$ 1A

(c) $\frac{1}{2} - \frac{9}{2(2x+1)^2} = 0$ 1M

$(2x+1)^2 = 9$

$x = -2 \quad \text{or} \quad 1$

x	$x < -2$	$-2 < x < -\frac{1}{2}$	$-\frac{1}{2} < x < 1$	$x > 1$
$f'(x)$	+	−	−	+

1M

Maximum point is $(-2, -4)$ and minimum point is $(1, -1)$. 1A+1A

(d) $0 = \frac{x^2 - 4x}{2x+1}$

$x = 0 \quad \text{or} \quad 4$

Area $= \int_0^4 [-f(x)]\,\mathrm{d}x$ 1M+1A

$= \int_0^4 \left(\frac{9}{4} - \frac{x}{2} - \frac{9}{4(2x+1)}\right)\mathrm{d}x$

$= \left[\frac{9x}{4} - \frac{x^2}{4} - \frac{9}{8}\ln|2x+1|\right]_0^4$ 1M

$= 5 - \frac{9}{8}\ln 9$ 1A

27. (a) $\int \sin^2\theta\cos^2\theta\,\mathrm{d}\theta$

$= \frac{1}{4}\int \sin^2 2\theta\,\mathrm{d}\theta$ 1M

$= \frac{1}{8}\int (1 - \cos 4\theta)\,\mathrm{d}\theta$

$= \frac{\theta}{8} - \frac{\sin 4\theta}{32} + \text{constant}$ 1A

(b) Volume

$= \pi\int_0^1 \left[6x(1-x^2)^{\frac{1}{4}}\right]^2 \mathrm{d}x$ 1M

$= 36\pi\int_0^1 x^2(1-x^2)^{\frac{1}{2}}\,\mathrm{d}x$

Let $x = \sin\theta$. Then $\mathrm{d}x = \cos\theta\,\mathrm{d}\theta$. 1M

Volume

$$= 36\pi \int_0^{\frac{\pi}{2}} \sin^2\theta(1-\sin^2\theta)^{\frac{1}{2}} \cos\theta \, d\theta$$

$$= 36\pi \int_0^{\frac{\pi}{2}} \sin^2\theta \cos^2\theta \, d\theta$$

$$= 36\pi \left[\frac{\theta}{8} - \frac{\sin 4\theta}{32} \right]_0^{\frac{\pi}{2}} \qquad \text{1M}$$

$$= \frac{9\pi^2}{4} \qquad \text{1A}$$

28. (a) $\dfrac{dy}{dx} = (1-x)^{-\frac{1}{2}}(-1)$ 1M

$$= -\frac{1}{\sqrt{1-x}}$$

At $A\,(0,\ 2)$, $\dfrac{dy}{dx} = -\dfrac{1}{\sqrt{1-0}} = -1$. Slope of $L = \dfrac{-1}{-1} = 1$. 1M

The equation of L is $y = x + 2$. 1A

(b) L: $x = y - 2$ and C: $x = 1 - \dfrac{y^2}{4}$.

Required volume $= \pi \displaystyle\int_0^2 [1-(y-2)]^2 \, dy - \pi \int_0^2 \left[1 - \left(1 - \frac{y^2}{4}\right)\right]^2 dy$ 1M+1A

$$= \pi \int_0^2 (9 - 6y + y^2) \, dy - \pi \int_0^2 \frac{y^4}{16} \, dy$$

$$= \pi \left[\frac{y^3}{3} - 3y^2 + 9y \right]_0^2 - \pi \left[\frac{y^5}{80} \right]_0^2 \qquad \text{1M}$$

$$= \frac{124\pi}{15} \qquad \text{1A}$$

29. (a) $\displaystyle\int \cos^2\theta \, d\theta = \frac{1}{2} \int (1 + \cos 2\theta) \, d\theta$ 1M

$$= \frac{\theta}{2} + \frac{\sin 2\theta}{4} + \text{constant} \qquad \text{1A}$$

(b) Let $x = 2\tan\theta$. Then $dx = 2\sec^2\theta \, d\theta$. 1M

Required volume

$$= \pi \int_0^2 \left(\frac{1}{4+x^2} \right)^2 dx \qquad \text{1M}$$

$$= \pi \int_0^{\frac{\pi}{4}} \frac{2\sec^2\theta}{(4+4\tan^2\theta)^2} \, d\theta$$

$$= \frac{\pi}{8} \int_0^{\frac{\pi}{4}} \cos^2\theta \, d\theta$$

$$= \frac{\pi}{8} \left[\frac{\theta}{2} + \frac{\sin 2\theta}{4} \right]_0^{\frac{\pi}{4}} \qquad \text{1M}$$

$$= \frac{\pi(\pi+2)}{64} \qquad \text{1A}$$

30. (a) $\cos^4 x = \left(\dfrac{1+\cos 2x}{2}\right)^2$ 1M

$$= \frac{1}{4} + \frac{1}{2}\cos 2x + \frac{1}{4}\cos^2 2x$$

$$= \frac{1}{4} + \frac{1}{2}\cos 2x + \frac{1}{4}\left(\frac{1+\cos 4x}{2}\right)$$

$$= \frac{1}{8}(3 + 4\cos 2x + \cos 4x)$$ 1

(b) Let $x = \sin\theta$. Then $\mathrm{d}x = \cos\theta\,\mathrm{d}\theta$. 1M

Required volume

$$= \pi\int_0^1 (x^2 - 1 - \sqrt{1-x^2})^2\,\mathrm{d}x$$ 1M

$$= \pi\int_0^1 [(x^2-1)^2 + (1-x^2)]\,\mathrm{d}x - 2\pi\int_0^1 (x^2-1)\sqrt{1-x^2}\,\mathrm{d}x$$

$$= \pi\int_0^1 (x^4 - 3x^2 + 2)\,\mathrm{d}x - 2\pi\int_0^{\frac{\pi}{2}} (-\cos^2\theta)\sqrt{1-\sin^2\theta}\cos\theta\,\mathrm{d}\theta$$

$$= \pi\left[\frac{x^5}{5} - x^3 + 2x\right]_0^1 + 2\pi\int_0^{\frac{\pi}{2}} \cos^4\theta\,\mathrm{d}\theta$$ 1M

$$= \frac{6\pi}{5} + \frac{\pi}{4}\int_0^{\frac{\pi}{2}} (3 + 4\cos 2\theta + \cos 4\theta)\,\mathrm{d}\theta$$ 1M

$$= \frac{6\pi}{5} + \frac{\pi}{4}\left[3\theta + 2\sin 2\theta + \frac{\sin 4\theta}{4}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{6\pi}{5} + \frac{3\pi^2}{8}$$ 1A

31. (a) $g'(x) = e^x(\cos x - \sin x)$ 1A

When $g'(x) = 0$,

$$\cos x - \sin x = 0$$

$$\tan x = 1$$

$$x = \frac{\pi}{4}$$

x	$0 < x < \frac{\pi}{4}$	$\frac{\pi}{4} < x < \pi$
$g'(x)$	$+$	$-$

Thus, G has only one maximum point. 1

(b) $\displaystyle\int e^{2x}\cos 2x\,\mathrm{d}x = \frac{e^{2x}\cos 2x}{2} + \int e^{2x}\sin 2x\,\mathrm{d}x$ 1M

$$= \frac{e^{2x}\cos 2x}{2} + \frac{e^{2x}\sin 2x}{2} - \int e^{2x}\cos 2x\,\mathrm{d}x$$

$$2\int e^{2x}\cos 2x\,\mathrm{d}x = \frac{e^{2x}(\sin 2x + \cos 2x)}{2} + \text{constant}$$ 1M

$$\int e^{2x}\cos 2x\,\mathrm{d}x = \frac{e^{2x}(\sin 2x + \cos 2x)}{4} + \text{constant}$$ 1A

(c) Volume $= \pi \int_0^{\frac{\pi}{4}} (e^x \cos x)^2 \, dx$ 1M

$= \frac{\pi}{2} \int_0^{\frac{\pi}{4}} e^{2x} \cos 2x \, dx + \frac{\pi}{2} \int_0^{\frac{\pi}{4}} e^{2x} \, dx$ 1M

$= \frac{\pi}{2} \left[\frac{e^{2x}(\sin 2x + \cos 2x)}{4} \right]_0^{\frac{\pi}{4}} + \frac{\pi}{2} \left[\frac{e^{2x}}{2} \right]_0^{\frac{\pi}{4}}$

$= \frac{3\pi(e^{\frac{\pi}{2}} - 1)}{8}$ 1A

32. (a) $\int x^k \ln x \, dx = \frac{x^{k+1} \ln x}{k+1} - \frac{1}{k+1} \int x^k \, dx$ 1M

$= \frac{x^{k+1} \ln x}{k+1} - \frac{x^{k+1}}{(k+1)^2} + \text{constant}$ 1A

(b) (i) Required area

$= \int_1^4 \sqrt{x} \ln x \, dx$ 1M

$= \left[\frac{x^{\frac{3}{2}} \ln x}{\left(\frac{3}{2}\right)} - \frac{x^{\frac{3}{2}}}{\left(\frac{3}{2}\right)^2} \right]_1^4$ 1M

$= \frac{16}{3} \ln 4 - \frac{28}{9}$

$= \frac{32}{3} \ln 2 - \frac{28}{9}$ 1A

(ii) Required volume

$= \pi \int_1^4 (\sqrt{x} \ln x)^2 \, dx$ 1M

$= \pi \int_1^4 x (\ln x)^2 \, dx$

$= \frac{\pi}{2} \left[x^2 (\ln x)^2 \right]_1^4 - \pi \int_1^4 x \ln x \, dx$ 1M

$= 8\pi (\ln 4)^2 - \pi \left[\frac{x^2 \ln x}{2} - \frac{x^2}{2^2} \right]_1^4$

$= 8\pi (\ln 4)^2 - 8\pi \ln 14 + \frac{15\pi}{4}$

$= 32\pi (\ln 2)^2 - 16\pi \ln 2 + \frac{15\pi}{4}$ 1A

33. (a) $V = \pi \int_0^b [16 - (y-4)^2] \, dy$ 1M

$= \pi \left[16y - \frac{(y-4)^3}{3} \right]_0^h$ 1A

$= \frac{\pi h^2 (12 - h)}{3}$ 1A

(b) $V = \frac{\pi}{3}(12h^2 - h^3)$

$\frac{dV}{dt} = \frac{\pi}{3}(24h - 3h^2)\frac{dh}{dt}$ 1M

$= \pi h(8 - h)\frac{dh}{dt}$ 1A

When $h = 3$,

$\pi = \pi(3)(8-3)\frac{dh}{dt}$

$\frac{dh}{dt} = \frac{1}{15}$ 1A

Required rate is $\frac{1}{15}$ cm/s.

34. (a) Required volume $= \pi \int_0^h \left(\frac{y^2}{9} + r\right)^2 dy$ 1M

$= \pi \int_0^h \left(\frac{y^4}{81} + \frac{2ry^2}{9} + r^2\right) dy$

$= \pi \left[\frac{y^5}{5} + \frac{2ry^3}{27} + r^2 y\right]_0^h$

$= \pi \left(\frac{h^5}{405} + \frac{2rh^3}{27} + r^2 h\right)$ 1

(b) (i) When $0 \le h \le 6$.

$V = \pi \left[\frac{h^5}{405} + \frac{2(8)h^3}{27} + (8)^2 h\right]$

$\frac{dV}{dt} = \pi \left(\frac{h^4}{81} + \frac{16h^2}{9} + 64\right)\frac{dh}{dt}$ 1A

When $h = 3$.

$4\pi = \pi \left[\frac{3^4}{81} + \frac{16(3)^2}{9} + 64\right]\frac{dh}{dt}$

$\frac{dh}{dt} = \frac{4}{81}$

Required rate is $\frac{4}{81}$ cm/s. 1A

(ii) Base radius of the frustum $= \frac{6^2}{9} + 8$

$= 12$ cm 1A

Let H cm be the height of the frustum.

$2H = \sqrt{20^2 - 12^2}$

$H = 8$ 1A

When $6 \le h \le T$.

$V = \pi \left[\frac{6^5}{405} + \frac{16(6)^3}{27} + 64(6)\right] + \frac{1}{3}\pi(12)^2(16)\left[1 - \left(\frac{22-h}{16}\right)^3\right]$ 1M+1M+1A

$= \frac{6496}{5}\pi - \frac{3}{16}\pi(22-h)^3$ 1

(iii) When $6 \le h \le T$.

$$\frac{dV}{dt} = -\frac{9}{16}\pi(22-h)^2(-1)\frac{dh}{dt}$$

$$= \frac{9}{16}\pi(22-h)^2\frac{dh}{dt} \qquad \text{1A}$$

When $h = 10$.

$$4\pi - k = \frac{9}{16}\pi(22-10)^2\left(\frac{1}{27}\right)$$

$$k = \pi \qquad \text{1A}$$

35. (a) (i) $$\frac{d}{dx}\sqrt{1-x^2} = \frac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x)$$

$$= -\frac{x}{\sqrt{1-x^2}} \qquad \text{1A}$$

(ii) $$y = \sin^{-1} x$$

$$\sin y = x$$

$$\cos y \frac{dy}{dx} = 1 \qquad \text{1A+1A}$$

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

$$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}} \qquad 1$$

(b) Use the result of (a).

$$\int (\sin^{-1} x)^2\, dx = x(\sin^{-1} x) - 2\int x\sin^{-1} x \cdot \frac{1}{\sqrt{1-x^2}}\, dx \qquad \text{1M}$$

$$= x(\sin^{-1} x)^2 + 2\sqrt{1-x^2}\sin^{-1} x - 2\int \sqrt{1-x^2}\cdot\frac{1}{\sqrt{1-x^2}}\, dx \qquad \text{1M}$$

$$= x(\sin^{-1} x)^2 + 2\sqrt{1-x^2}\sin^{-1} x - 2x + \text{constant} \qquad 1$$

(c) (i) Required capacity $= \pi\int_0^1 (\sin^{-1} y)^2\, dy$ 1M

$$= \pi\left[y(\sin^{-1} y)^2 + 2\sqrt{1-y^2}\sin^{-1} -2y\right]_0^1$$

$$= \frac{\pi^3}{4} - 2\pi \qquad \text{1A}$$

(ii) Let h units and V cubic units be the depth and the volume of the wine respectively.

$$\frac{\sqrt{3}}{2} = \sin x$$

$$x = \frac{\pi}{3} \qquad \text{1A}$$

Note that $V = \pi \int_0^h (\sin^{-1} y)^2 \, \mathrm{d}y$.

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}h} \cdot \frac{\mathrm{d}h}{\mathrm{d}t}$$

$$= \pi (\sin^{-1} h)^2 \cdot \frac{\mathrm{d}h}{\mathrm{d}t} \qquad \text{1M}$$

$$\frac{\pi^3}{3600} = \pi \left(\sin^{-1} \frac{\sqrt{3}}{2} \right)^2 \cdot \left. \frac{\mathrm{d}h}{\mathrm{d}t} \right|_{h=\frac{\sqrt{3}}{2}} \qquad \text{1M}$$

$$= \pi \left(\frac{\pi}{3} \right)^2 \cdot \left. \frac{\mathrm{d}h}{\mathrm{d}t} \right|_{h=\frac{\sqrt{3}}{2}}$$

$$\left. \frac{\mathrm{d}h}{\mathrm{d}t} \right|_{h=\frac{\sqrt{3}}{2}} = \frac{1}{400}$$

Required rate is $\frac{1}{400}$ units/s. 1A

36. (a) (i) $\cos^4 x \sin^2 x = \cos^2 x \left(\frac{2 \sin x \cos x}{2} \right)^2$

$$= \left(\frac{1 + \cos 2x}{2} \right) \left(\frac{\sin^2 2x}{4} \right) \qquad \text{1M}$$

$$= \frac{\sin^2 2x + \cos 2x \sin^2 2x}{8} \qquad \text{1}$$

(ii) Let $x = \frac{\pi}{2} - u$. Then $\mathrm{d}x = -\mathrm{d}u$.

$$\int_0^{\frac{\pi}{2}} \sin^4 x \cos^2 x \, \mathrm{d}x = -\int_{\frac{\pi}{2}}^0 \sin^4 \left(\frac{\pi}{2} - u \right) \cos^2 \left(\frac{\pi}{2} - u \right) \mathrm{d}u \qquad \text{1M}$$

$$= \int_0^{\frac{\pi}{2}} \cos^4 u \sin^2 u \, \mathrm{d}u$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin^2 2u + \cos 2u \sin^2 2u}{8} \, \mathrm{d}u$$

$$= \frac{1}{16} \int_0^{\frac{\pi}{2}} (1 - \cos 4u) \, \mathrm{d}u + \frac{1}{16} \int_0^{\frac{\pi}{2}} \sin^2 2u \, \mathrm{d}(\sin 2u) \qquad \text{1M+1M}$$

$$= \frac{1}{16} \left[u - \frac{\sin 4u}{4} \right]_0^{\frac{\pi}{2}} + \frac{1}{16} \left[\frac{\sin^3 2u}{3} \right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{32} \qquad \text{1A}$$

(b) (i) Let $x = k - u$. Then $\mathrm{d}x = -\mathrm{d}u$.

$$\int_0^k f(x) \, \mathrm{d}x = \int_0^{\frac{k}{2}} f(x) \, \mathrm{d}x + \int_{\frac{k}{2}}^k f(x) \, \mathrm{d}x$$

$$= \int_0^{\frac{k}{2}} f(x) \, \mathrm{d}x - \int_{\frac{k}{2}}^0 f(k - u) \, \mathrm{d}u \qquad \text{1M}$$

$$= \int_0^{\frac{k}{2}} f(x) \, \mathrm{d}x + \int_0^{\frac{k}{2}} f(x) \, \mathrm{d}x$$

$$= 2 \int_0^{\frac{k}{2}} f(x) \, \mathrm{d}x \qquad \text{1}$$

(ii) Let $x = k - w$. Then $\mathrm{d}x = -\,\mathrm{d}w$.

$$\int_0^k x f(x)\,\mathrm{d}x = -\int_k^0 (k-w) f(k-w)\,\mathrm{d}w \qquad \text{1M}$$

$$= k\int_0^k f(w)\,\mathrm{d}w - \int_0^k w f(w)\,\mathrm{d}w$$

$$= k\int_0^k f(x)\,\mathrm{d}x - \int_0^k x f(x)\,\mathrm{d}x$$

$$2\int_0^k x f(x)\,\mathrm{d}x = k\int_0^k f(x)\,\mathrm{d}x$$

$$\int_0^k x f(x)\,\mathrm{d}x = \frac{k}{2}\int_0^k f(x)\,\mathrm{d}x$$

$$= k\int_0^{\frac{k}{2}} f(x)\,\mathrm{d}x \qquad \text{1}$$

(iii) Let $f(x) = \cos^4 x \sin^2 x$.

Then $f(\pi - x) = \cos^4(\pi - x)\sin^2(\pi - x) = \cos^4 x \sin^2 x = f(x)$. 1M

Volume $= \pi \int_0^{\pi} x \sin^4 x \cos^2 x\,\mathrm{d}x$ 1M

$$= \pi^2 \int_0^{\frac{\pi}{2}} \sin^4 x \cos^2 x\,\mathrm{d}x$$

$$= \frac{\pi^3}{32} \qquad \text{1A}$$

37. (a) Vertical asymptote is $x = -2$. 1A

$$f(x) = x - 7 + \frac{27}{x+2} - \frac{27}{(x+2)^2} \qquad \text{1M}$$

Oblique asymptote is $y = x - 7$. 1A

(b) $$f'(x) = 1 - \frac{27}{(x+2)^2} + \frac{54}{(x+2)^3} \qquad \text{1M}$$

$$f''(x) = \frac{54}{(x+2)^3} - \frac{162}{(x+2)^4}$$

$$= \frac{54(x-1)}{(x+2)^4} \qquad \text{1A}$$

(c) $$f'(x) = 1 - \frac{27}{(x+2)^2} + \frac{54}{(x+2)^3}$$

$$= \frac{(x-1)^2(x+8)}{(x+2)^3}$$

When $f'(x) = 0$, $x = 1$ or -8.

x	$x < -8$	$-8 < x < -2$	$-2 < x < 1$	$x > 1$
$f'(x)$	$+$	$-$	$+$	$+$

1M

Maximum point is $\left(-8,\ -\dfrac{81}{4}\right)$.

G has exactly one turning point. 1A

(d) When $f''(x) = 0$, $x = 1$.

x	$x < -2$	$-2 < x < 1$	$x > 1$
$f''(x)$	$-$	$-$	$+$

1M

Point of inflexion is $(1,\ 0)$. 1A

(e) Let $u = x + 2$. Then $\mathrm{d}u = \mathrm{d}x$.

Required volume

$$= \pi \int_0^1 [f(x)]^2 \,\mathrm{d}x \qquad \text{1M}$$

$$= \pi \int_2^3 \frac{(u-3)^6}{u^4} \,\mathrm{d}u$$

$$= \pi \int_2^3 \frac{u^6 - 18u^5 + 135u^4 - 540u^3 + 1215u^2 - 1458u + 729}{u^4} \,\mathrm{d}u \qquad \text{1M}$$

$$= \pi \int_2^3 \left(u^2 - 18u + 135 - \frac{540}{u} + \frac{1215}{u^2} - \frac{1458}{u^3} + \frac{729}{u^4} \right) \mathrm{d}u$$

$$= \pi \left[\frac{u^3}{3} - 9u^2 + 135u - 540 \ln|u| - \frac{1215}{u} + \frac{729}{u^2} - \frac{243}{u^3} \right]_2^3 \qquad \text{1M}$$

$$= \pi \left(\frac{5255}{24} + 540 \ln \frac{2}{3} \right) \qquad \text{1A}$$

38. (a) $\sin 16\theta \sec\theta \sec 2\theta \sec 4\theta \sec 8\theta$

$= (2\sin 8\theta \cos 8\theta) \sec\theta \sec 2\theta \sec 4\theta \sec 8\theta$ 1M

$= 2(2\sin 4\theta \cos 4\theta) \sec\theta \sec 2\theta \sec 4\theta$

$= 4(2\sin 2\theta \cos 2\theta) \sec\theta \sec 2\theta$

$= 8(2\sin\theta \cos\theta) \sec\theta$

$= 16 \sin\theta$ 1

(b) Use the result of (a).

$y = \sqrt{\sin 16x \sec^2 x \sec 2x \sec 4x \sec 8x}$

$= \sqrt{16 \sin x \sec x}$ 1M

$= 4\sqrt{\tan x}$

Let $u = \cos x$. Then $\mathrm{d}u = -\sin x \,\mathrm{d}x$. 1M

Required capacity $= \pi \int_0^d 16 \tan x \,\mathrm{d}x$ 1M+1A

$= -16\pi \int_1^{\cos d} \frac{\mathrm{d}u}{u}$

$= -16\pi \Big[\ln|u| \Big]_1^{\cos d}$ 1M

$= -16\pi \ln(\cos d)$ 1

(c) (i) Required capacity $= -16\pi \ln[\cos(0.1)]$

$\approx 0.252 \text{ m}^3$ 1A

(ii) Let p m be the depth of water when the volume of water in the container is $0.08\ \mathrm{m}^3$.

$$-16\pi \ln(\cos p) = 0.08$$
$$p \approx 0.0564$$ 1A

Let k m and V m^3 be the depth and the volume of the water at time t s respectively.

$$V = -16\pi \ln(\cos k)$$
$$\frac{\mathrm{d}V}{\mathrm{d}t} = 16\pi \tan k \frac{\mathrm{d}k}{\mathrm{d}t}$$ 1M
$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{k=p} = 16\pi \tan p \cdot \left.\frac{\mathrm{d}k}{\mathrm{d}t}\right|_{k=p}$$
$$\left.\frac{\mathrm{d}k}{\mathrm{d}t}\right|_{k=p} \approx 0.001\,06$$

Required rate is $0.001\,06$ m/s. 1A

39. (a) Vertical asymptote is $x = -2$. 1A

$$f(x) = x + 5 + \frac{4}{x+2}$$ 1M

Oblique asymptote is $y = x + 5$. 1A

(b) $f'(x) = 1 - \dfrac{4}{(x+2)^2}$ 1M

$$= \frac{x(x+4)}{(x+2)^2}$$

When $f'(x) = 0$, $x = 0$ or -4. 1M

x	$x < -4$	$-4 < x < -2$	$-2 < x < 0$	$x > 0$
$f'(x)$	+	−	−	+

1M

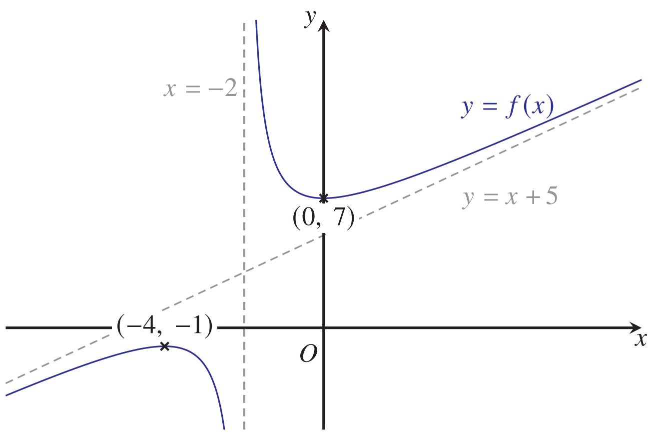
Maximum point is $(-4,\ -1)$. 1A

Minimum point is $(0,\ 7)$. 1A

(c) (Shape) 1A

(Asymptote) 1A

(All correct) 1A



(d) $\dfrac{x^2 + 7x + 14}{x + 2} = 8$

$x^2 - x - 2 = 0$

$x = 2 \quad \text{or} \quad -1$

Let $u = x + 2$. Then $\mathrm{d}u = \mathrm{d}x$.

Required volume

$= \pi \displaystyle\int_{-1}^{2} [f(x) - 8]^2 \,\mathrm{d}x$ 1M

$= \pi \displaystyle\int_{1}^{4} \left[(u-5) + \frac{4}{u}\right]^2 \mathrm{d}u$

$= \pi \displaystyle\int_{1}^{4} \left[(u-5)^2 + \frac{8(u-5)}{u} + \frac{16}{u^2}\right] \mathrm{d}u$

$= \pi \displaystyle\int_{1}^{4} \left[(u-5)^2 + 8 - \frac{40}{u} + \frac{16}{u^2}\right] \mathrm{d}u$

$= \pi \left[\dfrac{(u-5)^3}{3} + 8u - 40\ln|u| - \dfrac{16}{u}\right]_1^4$ 1M

$= \pi(57 - 80\ln 2)$ 1A

40. (a) Vertical asymptote is $x = -2$. 1A

$f(x) = x - 8 + \dfrac{16}{x + 2}$ 1M

Oblique asymptote is $y = x - 8$. 1A

(b) $f'(x) = 1 - \dfrac{16}{(x+2)^2}$ 1M

$= \dfrac{(x+6)(x-2)}{(x+2)^2}$

When $f'(x) = 0$, $x = 2$ or -6.

x	$x < -6$	$-6 < x < -2$	$-2 < x < 2$	$x > 2$
$f'(x)$	+	−	−	+

1M

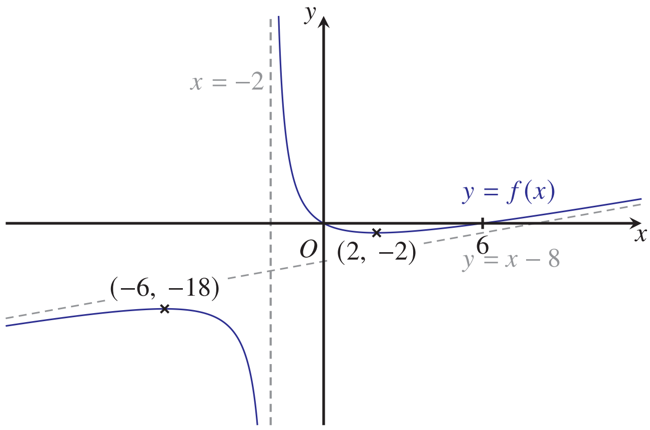
Maximum point is $(-6,\ -18)$. 1A

Minimum point is $(2,\ -2)$. 1A

(c) (Shape) 1A

(Asymptotes) 1A

(All correct) 1A



(d) $$\frac{x^2 - 6x}{x + 2} = 7$$

$x^2 - 13x - 14 = 0$

$x = 14 \quad \text{or} \quad -1$

Let $u = x + 2$. Then $\mathrm{d}u = \mathrm{d}x$.

Required volume

$= \pi \int_{-1}^{14} [f(x) - 7]^2 \, \mathrm{d}x$ 1M

$= \pi \int_{1}^{16} \left[u - 17 + \frac{16}{u}\right]^2 \mathrm{d}u$

$= \pi \int_{1}^{16} \left[(u - 17)^2 + \frac{32(u - 17)}{u} + \frac{256}{u^2}\right] \mathrm{d}u$

$= \pi \int_{1}^{16} \left[(u - 17)^2 + 32 - \frac{544}{u} + \frac{256}{u^2}\right] \mathrm{d}u$

$= \pi \left[\frac{(u - 17)^3}{3} + 32u - 544 \ln |u| - \frac{256}{u}\right]_1^{16}$ 1M

$= \pi(2085 - 2176 \ln 2)$ 1A

41. (a) Vertical asymptote is $x = 4$. 1A

$$f(x) = x + 1 + \frac{4}{x - 4}$$ 1M

Oblique asymptote is $y = x + 1$. 1A

(b) $f'(x) = 1 - \dfrac{4}{(x - 4)^2}$ 1M

$= \dfrac{(x - 2)(x - 6)}{(x - 4)^2}$ 1A

(c) When $f'(x) = 0$, $x = 2$ or $x = 6$. 1A

x	$x < 2$	$2 < x < 4$	$4 < x < 6$	$x > 6$
$f'(x)$	+	−	−	+

1M

Maximum point is $(2, 1)$. 1A

Minimum point is $(6, 9)$. 1A

(d) Let $u = x - 4$. Then $\mathrm{d}u = \mathrm{d}x$.

Required volume

$= \pi \int_0^3 [f(x)]^2 \, \mathrm{d}x$ 1M

$= \pi \int_{-4}^{-1} \left[(u+5) + \frac{4}{u} \right]^2 \mathrm{d}u$ 1M

$= \pi \int_{-4}^{-1} \left[(u+5)^2 + \frac{8(u+5)}{u} + \frac{16}{u^2} \right] \mathrm{d}u$

$= \pi \int_{-4}^{-1} \left[(u+5)^2 + 8 + \frac{40}{u} + \frac{16}{u^2} \right] \mathrm{d}u$

$= \pi \left[\frac{(u+5)^3}{3} + 8u + 40 \ln |u| - \frac{16}{u} \right]_{-4}^{-1}$ 1M

$= (57 - 80 \ln 2)\pi$ 1A

42. (a) Let $u = x^2 - 2x + 4$. Then $\mathrm{d}u = (2x - 2) \, \mathrm{d}x$. 1M

$$\int_0^1 \frac{x-1}{x^2 - 2x + 4} \, \mathrm{d}x = \frac{1}{2} \int_4^3 \frac{\mathrm{d}u}{u}$$

$= \frac{1}{2} \left[\ln |u| \right]_4^3$ 1A

$= \frac{1}{2} \ln \frac{3}{4}$ 1A

(b) (i) $x^2 - 2x + 4 = (x-1)^2 + 3$ 1A

(ii) Let $x - 1 = \sqrt{3} \tan \theta$. Then $\mathrm{d}x = \sqrt{3} \sec^2 \theta \, \mathrm{d}\theta$. 1M

$$\int_0^1 \frac{1}{x^2 - 2x + 4} \, \mathrm{d}x = \int_{-\frac{\pi}{6}}^{0} \frac{\sqrt{3} \sec^2 \theta}{3 \tan^2 \theta + 3} \, \mathrm{d}\theta$$

$= \frac{\sqrt{3}}{3} \int_{-\frac{\pi}{6}}^{0} \mathrm{d}\theta$ 1M

$= \frac{\sqrt{3}}{3} \left[\theta \right]_{-\frac{\pi}{6}}^{0}$

$= \frac{\sqrt{3}\pi}{18}$ 1A

(c) (i) $$\frac{A}{x+2} - \frac{B(x-4)}{x^2 - 2x + 4} = \frac{A(x^2 - 2x + 4) - B(x+2)(x-4)}{(x+2)(x^2 - 2x + 4)}$$

$$= \frac{(A-B)x^2 + 2(B-A)x + (4A + 8B)}{x^3 + 8}$$

We have $A - B = 0$, $B - A = 0$ and $4A = 8B = 12$.

Solving, we have $A = 1$ and $B = 1$. 1A

(ii) Required volume $= \pi \int_{-1}^{0} \left(\frac{1}{\sqrt{8 - x^3}} \right)^2 \mathrm{d}x$ 1M

$= \pi \int_{-1}^{0} \frac{\mathrm{d}X}{8 - x^3}$

Let $u = -x$. Then $\mathrm{d}u = -\,\mathrm{d}x$. 1M

$$
\begin{aligned}
\text{Required volume} &= \pi \int_0^1 \frac{\mathrm{d}u}{u^3+8} \\
&= \frac{\pi}{12}\int_0^1 \left(\frac{1}{u+2} - \frac{u-4}{u^2-2u+4}\right)\mathrm{d}u && \text{1M} \\
&= \frac{\pi}{12}\int_0^1 \frac{\mathrm{d}u}{u+2} - \frac{\pi}{12}\left(\int_0^1 \frac{u-1}{u^2-2u+4}\,\mathrm{d}u - \int_0^1 \frac{3}{u^2-2u+4}\,\mathrm{d}u\right) && \text{1M} \\
&= \frac{\pi}{12}\Big[\ln|u+2|\Big]_0^1 - \frac{\pi}{12}\left(\frac{1}{2}\ln\frac{3}{4}\right) + \frac{\pi}{4}\left(\frac{\sqrt{3}\pi}{18}\right) \\
&= \frac{\pi \ln 3}{24} + \frac{\sqrt{3}\pi^2}{24} && \text{1A}
\end{aligned}
$$

43. (a) Vertical asymptote is $x = -1$. 1A

$$f(x) = x - 5 + \frac{12}{x+1} - \frac{8}{(x+1)^2} \qquad \text{1M}$$

Oblique asymptote is $y = x - 5$. 1A

(b) $f'(x) = 1 - \dfrac{12}{(x+1)^2} + \dfrac{16}{(x+1)^3}$ 1M

$$
\begin{aligned}
f''(x) &= \frac{24}{(x+1)^3} - \frac{48}{(x+1)^4} \\
&= \frac{24(x-1)}{(x+1)^4} && \text{1A}
\end{aligned}
$$

(c)
$$
\begin{aligned}
f'(x) &= 1 - \frac{12}{(x+1)^2} + \frac{16}{(x+1)^3} \\
&= \frac{(x-1)^2(x+5)}{(x+1)^3}
\end{aligned}
$$

When $f'(x) = 0$, $x = 1$ or -5.

x	$x < -5$	$-5 < x < -1$	$-1 < x < 1$	$x > 1$
$f'(x)$	$+$	$-$	$+$	$+$

1M

Maximum point is $\left(-5,\ -\dfrac{27}{2}\right)$.

Γ has exactly one turning point.

The claim is agreed. 1A

(d) When $f''(x) = 0$, $x = 1$.

x	$x < -1$	$-1 < x < 1$	$x > 1$
$f''(x)$	$-$	$-$	$+$

1M

Point of inflexion is $(1,\ 0)$. 1A

(e) Let $u = x + 1$. Then $\mathrm{d}u = \mathrm{d}x$.

Required volume

$= \pi \int_0^1 [f(x)]^2 \, dx$ 1M

$= \pi \int_1^2 \frac{(u-2)^6}{u^4} \, du$

$= \pi \int_1^2 \frac{u^6 - 12u^5 + 60u^4 - 160u^3 + 240u^2 - 192u + 64}{u^4} \, du$ 1M

$= \pi \int_1^2 \left(u^2 - 12u + 60 - \frac{160}{u} + \frac{240}{u^2} - \frac{192}{u^3} + \frac{64}{u^4} \right) du$

$= \pi \left[\frac{u^3}{3} - 6u^2 + 60u - 160 \ln |u| - \frac{240}{u} + \frac{96}{u^2} - \frac{64}{3u^3} \right]_1^2$ 1M

$= (111 - 160 \ln 2)\pi$ 1A

44. (a) -6 1A

(b) (i) Vertical asymptote is $x = 10$. 1A

(ii) $f(x) = x - 6 + \frac{36}{x-10}$

$f'(x) = 1 - \frac{36}{(x-10)^2}$ 1M

$= \frac{(x-4)(x-16)}{(x-10)^2}$

When $f'(x) = 0$, $x = 4$ or 16. 1A

x	$x < 4$	$4 < x < 10$	$10 < x < 16$	$x > 16$
$f'(x)$	+	−	−	+

1M

Maximum point is $(4, -8)$. 1A

Minimum point is $(16, 16)$. 1A

(c) $x - 6 + \frac{36}{x-10} = 24$

$(x-6)(x-10) + 36 = 24(x-10)$

$x^2 - 40x + 336 = 0$

$x = 12$ or 28 1A

Let $u = x - 10$. Then $du = dx$.

Required volume

$= \pi \int_{12}^{28} [f(x) - 24]^2 \, dx$ 1M

$= \pi \int_{2}^{18} \left[(u - 20) + \frac{36}{u} \right]^2 du$

$= \pi \int_{2}^{18} \left[(u - 20)^2 + \frac{72(u - 20)}{u} + \frac{1296}{u^2} \right] du$ 1M

$= \pi \int_{2}^{18} \left[(u - 20)^2 + 72 - \frac{1440}{u} + \frac{1296}{u^2} \right] du$

$= \pi \left[\frac{(u - 20)^3}{3} + 72u - 1440 \ln |u| - \frac{1296}{u} \right]_2^{18}$ 1M

$= \left(\frac{11\,008}{3} - 2880 \ln 3 \right) \pi$ 1A

45. (a) The vertical asymptote is $x = -1$. 1A

$$f(x) = x - 8 + \frac{27}{x + 1} - \frac{27}{(x + 1)^2}$$ 1M

The oblique asymptote is $y = x - 8$. 1A

(b) $f'(x) = 1 - \frac{27}{(x + 1)^2} + \frac{54}{(x + 1)^3}$ 1M

$= \frac{(x + 7)(x - 2)^2}{(x + 1)^3}$

When $f'(x) = 0$, $x = 2$ or -7.

x	$x < -7$	$-7 < x < -1$	$-1 < x < 2$	$x > 2$
$f'(x)$	+	−	+	+

1M

G has exactly one turning point, which is a maximum point at $x = -7$. 1

(c) $f''(x) = \frac{54}{(x + 1)^3} - \frac{162}{(x + 1)^4}$

$= \frac{54(x - 2)}{(x + 1)^4}$ 1A

When $f''(x) = 0$, $x = 2$.

x	$x < -1$	$-1 < x < 2$	$x > 2$
$f''(x)$	−	−	+

1M

The point of inflexion is $(2,\ 0)$. 1A

(d) Let $u = x + 1$. Then $du = dx$.

Required volume

$$= \pi \int_0^2 \left[\frac{(x-2)^3}{(x+1)^2} \right]^2 \mathrm{d}x \qquad \text{1M}$$

$$= \pi \int_1^3 \frac{(u-3)^6}{u^4} \,\mathrm{d}u$$

$$= \pi \int_1^3 \frac{u^6 - 18u^5 + 135u^4 - 540u^3 + 1215u^2 - 1458u + 729}{u^4} \,\mathrm{d}u \qquad \text{1M}$$

$$= \pi \int_1^3 (u^2 - 18u + 135 - 540u^{-1} + 1215u^{-2} - 1458u^{-3} + 729u^{-4}) \,\mathrm{d}u$$

$$= \pi \left[\frac{u^3}{3} - 9u^2 + 135u - 540 \ln|u| - 1215u^{-1} + \frac{729}{u^2} - \frac{243}{u^3} \right]_1^3$$

$$= \pi \left(\frac{1808}{3} - 540 \ln 3 \right) \qquad \text{1A}$$

46. (a) Required volume $= \pi \int_0^k (12y - y^2) \,\mathrm{d}y$ 1M+1A

$$= \pi \left[6y^2 - \frac{y^3}{3} \right]_0^k$$

$$= \pi \left(6k^2 - \frac{k^3}{3} \right) \qquad 1$$

(b) (i) When $0 \le h \le$.

$$V = \pi \left[6(h+2)^2 - \frac{(h+2)^3}{3} \right] - \pi \left[6(2)^2 - \frac{2^3}{3} \right] \qquad \text{1M}$$

$$= \pi \left[6(h+2)^2 - \frac{1}{3}(h+2)^3 \right] - \frac{64\pi}{3} \qquad 1$$

(ii) When $4 \le h \le 11$.

$$V = \pi(6)^2(h-4) + \pi \left[6(4+2)^2 - \frac{1}{3}(4+2)^3 \right] - \frac{64\pi}{3} \qquad \text{1M}$$

$$= 36\pi(h-4) + \frac{368\pi}{3}$$

$$\frac{\mathrm{d}V}{\mathrm{d}t} = 36\pi \cdot \frac{\mathrm{d}h}{\mathrm{d}t} \qquad \text{1A}$$

$$-3 = 36\pi \cdot \frac{\mathrm{d}h}{\mathrm{d}t}$$

$$\frac{\mathrm{d}h}{\mathrm{d}t} = -\frac{1}{12\pi}$$

Required rate is $\frac{1}{12\pi}$ cm/s. 1A

(iii) When $0 \le h \le 4$.

$$V = \pi \left[6(h+2)^2 - \frac{1}{3}(h+2)^3 \right] - \frac{64\pi}{3}$$

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \pi \left[12(h+2)\frac{\mathrm{d}h}{\mathrm{d}t} - (h+2)^2 \frac{\mathrm{d}h}{\mathrm{d}t} \right] \qquad \text{1A}$$

When $h = 1$, $\dfrac{dV}{dt} = 5 - 3 = 2$.

$$2 = \pi\left[12(1+2)\frac{dh}{dt} - (1+2)^2\frac{dh}{dt}\right] \qquad \text{1M}$$

$$\frac{dh}{dt} = \frac{2}{27\pi}$$

Required rate is $\dfrac{2}{27\pi}$ cm/s. 1A

47. (a) (i) $h + 1 = x^2 + 1$

$$x = \sqrt{h} \quad \text{or} \quad -\sqrt{h} \text{ (rejected)}$$

The coordinates of A are $\left(\sqrt{h},\ h+1\right)$. 1A

(ii)
$$y = x^2 + 1$$

$$\frac{dy}{dx} = 2x \qquad \text{1A}$$

$$\left.\frac{dy}{dx}\right|_{x=\sqrt{h}} = 2\sqrt{h}$$

The equation of L is

$$\frac{y-(h+1)}{x-\sqrt{h}} = 2\sqrt{h} \qquad \text{1M}$$

$$y = 2\sqrt{h}x - h + 1 \qquad \text{1}$$

(b) (i) Consider the capacity of the cup.

$$\pi\int_0^{h+1}\left(\frac{y+h-1}{2\sqrt{h}}\right)^2 dy - \pi\int_1^{h+1}(y-1)\,dy = \frac{31}{10}\pi \qquad \text{1M+1A}$$

$$\frac{\pi}{4h}\left[\frac{(y+h-1)^3}{3}\right]_0^{h+1} - \pi\left[\frac{(y-1)^2}{2}\right]_1^{h+1} = \frac{31}{10}\pi \qquad \text{1M}$$

$$\frac{\pi}{6}h^2 - \frac{\pi}{12h}(h-1)^3 = \frac{31}{10}\pi$$

$$10h^3 - 5(h^3 - 3h^2 + 3h - 1) = 186h$$

$$5h^3 + 15h^2 - 201h + 5 = 0$$

$$(h-5)(5h^2 + 40h - 1) = 0 \qquad \text{1A}$$

$$h = 5 \quad \text{or} \quad h = \frac{-40 \pm \sqrt{40^2 - 4(5)(-1)}}{2(5)}$$

$$h = 5 \quad \text{or} \quad h = \frac{-40 \pm \sqrt{1620}}{10} \text{ (rejected)} \qquad \text{1A}$$

(ii) Let V cm^3, A cm^2 and H cm be the volume of water in the cup, the area of the water surface and the depth of water at time t s respectively.

$$V = \pi \int_1^{H+1} (y-1)\,\mathrm{d}y \qquad \text{1M}$$

$$= \pi \left[\frac{(y-1)^2}{2} \right]_1^{H+1}$$

$$= \frac{\pi}{2} H^2$$

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \pi H \frac{\mathrm{d}H}{\mathrm{d}t} \qquad \text{1M}$$

When $H = 3$ and $\dfrac{\mathrm{d}V}{\mathrm{d}t} = 1$.

$$1 = \pi(3) \frac{\mathrm{d}H}{\mathrm{d}T}$$

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{1}{3\pi} \qquad \text{1A}$$

When $y = H + 1$.

$$H + 1 = x^2 + 1$$

$$x^2 = H$$

We have $A = \pi H$.

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \pi \frac{\mathrm{d}H}{\mathrm{d}t} \qquad \text{1M}$$

When $H = 3$, $\dfrac{\mathrm{d}H}{\mathrm{d}t} = \dfrac{1}{3\pi}$.

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \pi \left(\frac{1}{3\pi} \right)$$

$$= \frac{1}{3}$$

Required rate is $\dfrac{1}{3}$ cm^2/s. 1A

48. (a) $x^2 - x + \dfrac{1}{2} = \left(x - \dfrac{1}{2} \right)^2 + \dfrac{1}{4}$ 1M

Let $x - \dfrac{1}{2} = \dfrac{1}{2} \tan\theta$. Then $\mathrm{d}x = \dfrac{1}{2} \sec^2\theta\,\mathrm{d}\theta$

$$\int \frac{\mathrm{d}x}{x^2 - x + \frac{1}{2}} = \frac{1}{2} \int \frac{\sec^2\theta}{\frac{1}{4}\tan^2\theta + \frac{1}{4}}\,\mathrm{d}\theta \qquad \text{1M}$$

$$= 2\theta + \text{constant}$$

$$= 2\tan^{-1}(2x - 1) + \text{constant} \qquad \text{1A}$$

(b) Let $u = \lambda - x$. Then $\mathrm{d}u = -\,\mathrm{d}x$.

$$\int_0^{\lambda} xf(x)\,\mathrm{d}x = -\int_{\lambda}^{0} (\lambda - u)f(\lambda - u)\,\mathrm{d}u \qquad \text{1M}$$

$$= \int_0^{\lambda} (\lambda - x)f(x)\,\mathrm{d}x \qquad \text{1M}$$

$$2\int_0^{\lambda} xf(x)\,\mathrm{d}x = \lambda\int_0^{\lambda} f(x)\,\mathrm{d}x$$

$$\int_0^{\lambda} xf(x)\,\mathrm{d}x = \frac{\lambda}{2}\int_0^{\lambda} f(x)\,\mathrm{d}x \qquad \text{1}$$

(c) Volume $= \pi\int_0^{\frac{\pi}{2}} \frac{x\sin x\cos x}{\sin^4 x + \cos^4 x}\,\mathrm{d}x$ 1M+1A

Let $f(x) = \dfrac{\sin x\cos x}{\sin^4 x + \cos^4 x}$.

$$f\left(\frac{\pi}{2} - x\right) = \frac{\sin\left(\frac{\pi}{2} - x\right)\cos\left(\frac{\pi}{2} - x\right)}{\sin^4\left(\frac{\pi}{2} - x\right) + \cos^4\left(\frac{\pi}{2} - x\right)} = \frac{\cos x\sin x}{\cos^4 x + \sin^4 x} = f(x).$$

Volume $= \dfrac{\pi^2}{4}\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x\cos x}{\sin^4 x + \cos^4 x}\,\mathrm{d}x$ 1M

Let $u = \sin^2 x$. Then $\mathrm{d}u = 2\sin x\cos x\,\mathrm{d}x$.

Volume $= \dfrac{\pi^2}{8}\displaystyle\int_0^{1} \frac{\mathrm{d}u}{u^2 + (1 - u)^2}$ 1M

$$= \frac{\pi^2}{16}\int_0^{1} \frac{\mathrm{d}u}{u^2 - u + \frac{1}{2}}$$

$$= \frac{\pi^2}{8}\Big[\tan^{-1}(2u - 1)\Big]_0^1 \qquad \text{1M}$$

$$= \frac{\pi^3}{16} \qquad \text{1A}$$

49. (a) Vertical asymptote is $x = -2$. 1A

$$f(x) = x - 6 + \frac{9}{x + 2} \qquad \text{1M}$$

Oblique asymptote is $y = x - 6$. 1A

(b) $f'(x) = 1 - \dfrac{9}{(x + 2)^2}$ 1M

$$= \frac{(x + 5)(x - 1)}{(x + 2)^2} \qquad \text{1A}$$

(c) When $f'(x) = 0$, $x = 1$ or -5. 1M

x	$x < -5$	$-5 < x < -2$	$-2 < x < 1$	$x > 1$
$f'(x)$	$+$	$-$	$-$	$+$

1M

Maximum point is $(-5,\ -14)$. 1A

Minimum point is $(1,\ -2)$. 1A

(d) Required volume

$$= \pi \int_0^1 \left(x - 6 + \frac{9}{x+2} \right)^2 \mathrm{d}x \qquad \text{1M}$$

$$= \pi \int_0^1 \left[(x-6)^2 + \frac{18(x-6)}{x+2} + \frac{81}{(x+2)^2} \right] \mathrm{d}x$$

$$= \pi \int_0^1 \left[(x-6)^2 + \left(18 - \frac{144}{x+2} \right) + \frac{81}{(x+2)^2} \right] \mathrm{d}x \qquad \text{1M}$$

$$= \pi \left[\frac{(x-6)^3}{3} + 18x - 144 \ln|x+2| - \frac{81}{x+2} \right]_0^1 \qquad \text{1M}$$

$$= \pi \left(\frac{371}{6} - 144 \ln \frac{3}{2} \right) \qquad \text{1A}$$

50. (a) (i)

$$f(x) = \frac{162(x-1)}{(x+1)^3}$$

$$= \frac{162}{(x+1)^2} - \frac{324}{(x+1)^3}$$

$$f'(x) = -\frac{324}{(x+1)^3} + \frac{972}{(x+1)^4}$$

$$= \frac{324(2-x)}{(x+1)^4} \qquad \text{1A}$$

$$f''(x) = \frac{972}{(x+1)^4} - \frac{3888}{(x+1)^5}$$

$$= \frac{972(x-3)}{(x+1)^5} \qquad \text{1A}$$

(ii) When $f'(x) = 0$, $x = 2$.

x	$x < -1$	$-1 < x < 2$	$x > 2$
$f'(x)$	+	+	−

1M

The maximum point is $(2,\ 6)$. 1A

No minimum point.

When $f''(x) = 0$, $x = 3$.

x	$x < -1$	$-1 < x < 3$	$x > 3$
$f''(x)$	+	−	+

1M

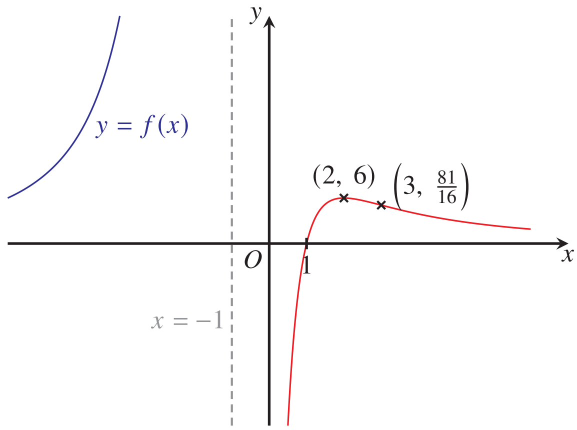
The point of inflexion is $\left(3,\ \frac{81}{16}\right)$. 1A

(b) Vertical asymptote is $x = -1$. 1A

Horizontal asymptote is $y = 0$. 1A

(c) (Shape, extrema and point of inflexion) 1A

(All correct) 1A



(d) Let $u = x + 1$. Then $du = dx$.

$$\text{Volume} = \pi \int_0^1 \left[\frac{162(x-1)}{(x+1)^3}\right]^2 dx \qquad \text{1M}$$

$$= 26\,244\pi \int_1^2 \frac{(u-2)^2}{u^6}\, du$$

$$= 26\,244\pi \int_1^2 (u^{-4} - 4u^{-5} + 4u^{-6})\, du \qquad \text{1M}$$

$$= 26\,244\left[\frac{u^{-3}}{-3} + u^{-4} - \frac{4u^{-5}}{5}\right]_1^2$$

$$= \frac{67\,797\pi}{20} \qquad \text{1A}$$

51. (a) $$f(x) = \frac{ax^2 + bx + 3}{x-1}$$

$$= ax + (a+b) + \frac{a+b+3}{x-1} \qquad \text{1M}$$

Oblique asymptote is $y = ax + (a+b)$.

We have $a = 4$ and $a + b = 1$.

Solving, we have $a = 4$ and $b = -3$. 1A

(b) $$f(x) = 4x + 1 + \frac{4}{x-1}$$

$$f'(x) = 4 - \frac{4}{(x-1)^2} \qquad \text{1M}$$

$$= \frac{4x(x-2)}{(x-1)^2} \qquad \text{1A}$$

(c) When $f'(x) = 0$, $x = 0$ or 2. 1M

x	$x < 0$	$0 < x < 1$	$1 < x < 2$	$x > 2$
$f'(x)$	+	−	−	+

1M

Maximum point is $(0, -3)$. 1A

Minimum point is $(2, 13)$. 1A

(d) Let $u = x - 1$. Then $du = dx$.

Required volume

$= \pi \int_2^4 (f(x) - 1)^2 \, dx$ 1M

$= \pi \int_2^4 \left(4x + \frac{4}{x-1}\right)^2 dx$

$= 16\pi \int_2^4 \left[x + \frac{2x}{x-1} + \frac{1}{(x-1)^2}\right] dx$

$= 16\pi \int_1^3 \left[(u+1)^2 + 2 + \frac{2}{u} + \frac{1}{u^2}\right] du$

$= 16\pi \left[\frac{(u+1)^3}{3} + 2u + 2\ln|u| - \frac{1}{u}\right]_1^3$ 1M

$= \frac{1120\pi}{3} + 32\pi \ln 3$ 1A

(e) Note that $f(3) = 15$ and $f'(3) = 3$.

Let $(\alpha, 0)$ be the coordinates of A.

$$\frac{0 - 15}{\alpha - 3} = 3$$

$$\alpha = -2$$

The coordinates of A are $(-2, 0)$. 1M

The area of $\triangle OAP$ is the least when P is nearest to the x-axis.

The coordinates of P are $(0, -3)$.

Least area of $\triangle OAP$

$= \frac{1}{2}(0+2)(0+3)$

$= 3$

Thus, the area of $\triangle OAP$ cannot be less than 3. 1A

END OF PAPER