M2REG-DEFINT-2425-ASM-SET 5-MATH
Suggested solutions

1. (a) Let $x = \tan\theta$. Then $\mathrm{d}x = \sec^2\theta\,\mathrm{d}\theta$. 1M

$$\int_0^1 \frac{\mathrm{d}x}{1+x^2} = \int_0^{\frac{\pi}{4}} \frac{\sec^2\theta}{1+\tan^2\theta}\,\mathrm{d}\theta \qquad \text{1A}$$
$$= \int_0^{\frac{\pi}{4}} \mathrm{d}\theta$$
$$= \Big[\theta\Big]_0^{\frac{\pi}{4}}$$
$$= \frac{\pi}{4} \qquad \text{1A}$$

(b) Let $u = \frac{\pi}{2} - x$. Then $\mathrm{d}u = -\,\mathrm{d}x$. 1M

$$\int_0^{\frac{\pi}{4}} \sin x\, f(\sin 2x)\,\mathrm{d}x = \int_{\frac{\pi}{2}}^{\frac{\pi}{4}} \sin\left(\frac{\pi}{2}-u\right) f\left[\sin 2\left(\frac{\pi}{2}-u\right)\right](-1)\,\mathrm{d}u$$
$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos u\, f(\sin 2u)\,\mathrm{d}u \qquad \text{1M}$$
$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos x\, f(\sin 2x)\,\mathrm{d}x \qquad \text{1}$$

$$\int_0^{\frac{\pi}{2}} \cos x\, f(\sin 2x)\,\mathrm{d}x = \int_0^{\frac{\pi}{4}} \cos x\, f(\sin 2x)\,\mathrm{d}x + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos x\, f(\sin 2x)\,\mathrm{d}x$$
$$= \int_0^{\frac{\pi}{4}} \cos x\, f(\sin 2x)\,\mathrm{d}x + \int_0^{\frac{\pi}{4}} \sin x\, f(\sin 2x)\,\mathrm{d}x$$
$$= \int_0^{\frac{\pi}{4}} (\cos x + \sin x) f(\sin 2x)\,\mathrm{d}x \qquad \text{1}$$

(c)
$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{2-\sin 2x}\,\mathrm{d}x = \int_0^{\frac{\pi}{4}} \frac{\cos x + \sin x}{2-\sin 2x}\,\mathrm{d}x \qquad \text{1M}$$
$$= \int_0^{\frac{\pi}{4}} \frac{\cos x + \sin x}{1+(1-\sin 2x)}\,\mathrm{d}x$$
$$= \int_0^{\frac{\pi}{4}} \frac{\cos x + \sin x}{1+(\cos^2 x + \sin^2 x - 2\sin x\cos x)}\,\mathrm{d}x \qquad \text{1M}$$
$$= \int_0^{\frac{\pi}{4}} \frac{\cos x + \sin x}{1+(\cos x - \sin x)^2} \qquad \text{1A}$$

Let $u = \cos x - \sin x$. Then $\mathrm{d}u = (-\sin x - \cos x)\,\mathrm{d}x$. 1M

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{2-\sin 2x}\,\mathrm{d}x = \int_0^{\frac{\pi}{4}} \frac{\cos x + \sin x}{1+(\cos x - \sin x)^2}\,\mathrm{d}x$$
$$= -\int_1^0 \frac{1}{1+u^2}\,\mathrm{d}u \qquad \text{1A}$$
$$= \int_0^1 \frac{\mathrm{d}u}{1+u^2}$$
$$= \frac{\pi}{4} \qquad \text{1A}$$

2. (a) Let $u = a - x$. Then $\mathrm{d}u = -\,\mathrm{d}x$. 1M

$$\int_0^a f(a-x)\,\mathrm{d}x = -\int_a^0 f(u)\,\mathrm{d}u$$
$$= \int_0^a f(u)\,\mathrm{d}u \qquad \text{1M}$$
$$= \int_0^a f(x)\,\mathrm{d}x \qquad \text{1}$$

(b) (i) $$\int_0^{\frac{\pi}{2}} \frac{\cos\theta}{\sin\theta+\cos\theta}\,\mathrm{d}\theta = \int_0^{\frac{\pi}{2}} \frac{\cos\left(\frac{\pi}{2}-\theta\right)}{\sin\left(\frac{\pi}{2}-\theta\right)+\cos\left(\frac{\pi}{2}-\theta\right)}\,\mathrm{d}\theta$$
$$= \int_0^{\frac{\pi}{2}} \frac{\sin\theta}{\sin\theta+\cos\theta}\,\mathrm{d}\theta \qquad \text{1}$$

(ii) $$\int_0^{\frac{\pi}{2}} \frac{\cos\theta}{\sin\theta+\cos\theta}\,\mathrm{d}\theta = \frac{1}{2}\left[\int_0^{\frac{\pi}{2}} \frac{\sin\theta}{\sin\theta+\cos\theta}\,\mathrm{d}\theta + \int_0^{\frac{\pi}{2}} \frac{\cos\theta}{\sin\theta+\cos\theta}\,\mathrm{d}\theta\right] \qquad \text{1M}$$
$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \mathrm{d}\theta$$
$$= \frac{1}{2}\Big[\theta\Big]_0^{\frac{\pi}{2}}$$
$$= \frac{\pi}{4} \qquad \text{1A}$$

(c) Let $x = \sin\theta$. Then $\mathrm{d}x = \cos\theta\,\mathrm{d}\theta$. 1M

$$\int_0^1 \frac{\mathrm{d}x}{x+\sqrt{1-x^2}} = \int_0^{\frac{\pi}{2}} \frac{\cos\theta}{\sin\theta+\sqrt{1-\sin^2\theta}}\,\mathrm{d}\theta$$
$$= \int_0^{\frac{\pi}{2}} \frac{\cos\theta}{\sin\theta+\cos\theta}\,\mathrm{d}\theta$$
$$= \frac{\pi}{4} \qquad \text{1A}$$

3. (a) Let $x = \sqrt{3}\tan\theta$. Then $dx = \sqrt{3}\sec^2\theta\, d\theta$. 1M

$$\int_0^1 \frac{1}{x^2+3}\,dx = \int_0^{\frac{\pi}{6}} \frac{\sqrt{3}\sec^2\theta}{3\tan^2\theta+3}\,d\theta$$

$$= \frac{\sqrt{3}}{3}\int_0^{\frac{\pi}{6}} d\theta \qquad \text{1M}$$

$$= \frac{\sqrt{3}}{3}\Big[\theta\Big]_0^{\frac{\pi}{6}}$$

$$= \frac{\sqrt{3}\pi}{18} \qquad \text{1A}$$

(b) (i)
$$\frac{1-\tan^2\theta}{1+\tan^2\theta} = \frac{1-\frac{\sin^2\theta}{\cos^2\theta}}{1+\frac{\sin^2\theta}{\cos^2\theta}}$$

$$= \frac{\cos^2\theta-\sin^2\theta}{\cos^2\theta+\sin^2\theta}$$

$$= \frac{\cos 2\theta}{1}$$

$$= \cos 2\theta \qquad \text{1}$$

(ii) Let $t = \tan\theta$. Then $dt = \sec^2\theta\, d\theta$ and $d\theta = \dfrac{1}{1+t^2}\,dt$. 1M

$$\int_0^{\frac{\pi}{4}} \frac{1}{\cos 2\theta+2}\,d\theta = \int_0^1 \frac{1}{\frac{1-t^2}{1+t^2}+2}\cdot\frac{1}{1+t^2}\,dt \qquad \text{1M}$$

$$= \int_0^1 \frac{1}{t^2+3}\,dt$$

$$= \frac{\sqrt{3}\pi}{18} \qquad \text{1A}$$

(c) Let $u = \dfrac{\pi}{4}-\theta$. Then $du = -\,d\theta$. 1M

$$\int_0^{\frac{\pi}{4}} \frac{1}{\cos 2\theta+2}\,d\theta = -\int_{\frac{\pi}{4}}^0 \frac{1}{\cos\left[\frac{\pi}{2}-2u\right]+2}\,du$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{\sin 2u+2}\,du$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{\sin 2\theta+2}\,d\theta \qquad \text{1}$$

(d) $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\sin 2\theta + \cos 2\theta + 4}{(\sin 2\theta + 2)(\cos 2\theta + 2)}\,\mathrm{d}\theta$

$\displaystyle= \int_0^{\frac{\pi}{4}} \frac{\sin 2\theta + 2}{(\sin 2\theta + 2)(\cos 2\theta + 2)}\,\mathrm{d}\theta + \int_0^{\frac{\pi}{4}} \frac{\cos 2\theta + 2}{(\sin 2\theta + 2)(\cos 2\theta + 2)}\,\mathrm{d}\theta$ 1M

$\displaystyle= \int_0^{\frac{\pi}{4}} \frac{1}{\cos 2\theta + 2}\,\mathrm{d}\theta + \int_0^{\frac{\pi}{4}} \frac{1}{\sin 2\theta + 2}\,\mathrm{d}\theta$

$\displaystyle= 2\int_0^{\frac{\pi}{4}} \frac{1}{\cos 2\theta + 2}\,\mathrm{d}\theta$

$\displaystyle= 2 \times \frac{\sqrt{3}\pi}{18}$ 1M

$\displaystyle= \frac{\sqrt{3}\pi}{9}$ 1A

4. (a) Let $u = \pi - x$. Then $\mathrm{d}u = -\,\mathrm{d}x$. 1M

$$\int_0^{\pi} x f(x)\,\mathrm{d}x = -\int_{\pi}^{0} (\pi - u) f(\pi - u)\,\mathrm{d}u$$

$$= \int_0^{\pi} (\pi - u) f(u)\,\mathrm{d}u \quad \text{1M}$$

$$= \int_0^{\pi} (\pi - x) f(x)\,\mathrm{d}x \quad \text{1}$$

$$\int_0^{\pi} x f(x)\,\mathrm{d}x = \frac{1}{2}\left[\int_0^{\pi} x f(x)\,\mathrm{d}x + \int_0^{\pi} (\pi - x) f(x)\,\mathrm{d}x\right]$$

$$= \frac{\pi}{2}\int_0^{\pi} f(x)\,\mathrm{d}x \quad \text{1}$$

(b) Let $u = \cos x$. Then $\mathrm{d}u = -\sin x\,\mathrm{d}x$. 1M

$$\int_0^{\pi} \frac{\sin x}{1 + \cos^2 x}\,\mathrm{d}x = -\int_1^{-1} \frac{\mathrm{d}u}{1 + u^2}$$

$$= \int_{-1}^{1} \frac{\mathrm{d}u}{1 + u^2}$$

Let $u = \tan\theta$. Then $\mathrm{d}u = \sec^2\theta\,\mathrm{d}\theta$. 1M

$$\int_0^{\pi} \frac{\sin x}{1 + \cos^2 x}\,\mathrm{d}x = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sec^2\theta}{1 + \tan^2\theta}\,\mathrm{d}\theta \quad \text{1A}$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \mathrm{d}\theta$$

$$= \Big[\theta\Big]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$= \frac{\pi}{2} \quad \text{1A}$$

(c) Let $f(x) = \dfrac{\cos\left(\frac{\pi}{6} - x\right) - \cos\left(\frac{\pi}{6} + x\right)}{3 + \cos 2x}$.

$$f(\pi - x) = \frac{\cos\left(\frac{\pi}{6} - \pi + x\right) - \cos\left(\frac{\pi}{6} + \pi - x\right)}{3 + \cos(2\pi - 2x)} = \frac{-\cos\left(\frac{\pi}{6} + x\right) + \cos\left(\frac{\pi}{6} - x\right)}{3 + \cos 2x} = f(x) \quad \text{1M}$$

$$\int_0^{\pi} \frac{x\left[\cos\left(\frac{\pi}{6} - x\right) - \cos\left(\frac{\pi}{6} + x\right)\right]}{3 + \cos 2x}\,\mathrm{d}x$$

$$= \frac{\pi}{2}\int_0^{\pi} \frac{\cos\left(\frac{\pi}{6} - x\right) - \cos\left(\frac{\pi}{6} + x\right)}{3 + \cos 2x}\,\mathrm{d}x \quad \text{1M}$$

$$= \frac{\pi}{2}\int_0^{\pi} \frac{-2\sin\frac{\pi}{6}\sin(-x)}{3 + \cos 2x}\,\mathrm{d}x \quad \text{1M}$$

$$= \frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{3 + (2\cos^2 x - 1)}\,\mathrm{d}x$$

$$= \frac{\pi}{4}\int_0^{\pi} \frac{\sin x}{1 + \cos^2 x}\,\mathrm{d}x$$

$$= \frac{\pi}{4}\cdot\frac{\pi}{2}$$

$$= \frac{\pi^2}{8} \quad \text{1A}$$

5. (a) (i) $\int \sin^4 x \, dx = \int \sin^3 x \sin x \, dx$

$= -\sin^3 x \cos x + 3 \int \sin^2 x \cos^2 x \, dx$ 1M

$= -\sin^3 x \cos x + 3 \int \sin^2 x (1 - \sin^2 x) \, dx$

$= -\sin^3 x \cos x + 3 \int \sin^2 x \, dx - 3 \int \sin^4 x \, dx$

$4 \int \sin^4 x \, dx = -\sin^3 x \cos x + 3 \int \sin^2 x \, dx$

$\int \sin^4 x \, dx = -\frac{1}{4} \cos x \sin^3 x + \frac{3}{4} \int \sin^2 x \, dx$ 1

(ii) $\int_0^{2\pi} \sin^4 x \, dx = -\frac{1}{4} \left[\cos x \sin^3 x \right]_0^{2\pi} + \frac{3}{4} \int_0^{2\pi} \sin^2 x \, dx$

$= \frac{3}{8} \int_0^{2\pi} (1 - \cos 2x) \, dx$ 1M

$= \frac{3}{8} \left[x - \frac{\sin 2x}{2} \right]_0^{2\pi}$ 1M

$= \frac{3\pi}{4}$ 1A

(b) (i) Let $u = a - x$. Then $du = -dx$. 1M

$\int_0^a x f(x) \, dx = -\int_a^0 (a - u) f(u) \, du$ 1M

$= \int_0^a (a - u) f(u) \, du$

$= a \int_0^x f(x) \, dx - \int_0^a x f(x) \, dx$

$2 \int_0^a x f(x) \, dx = a \int_0^a f(x) \, dx$

$\int_0^a x f(x) \, dx = \frac{a}{2} \int_0^a f(x) \, dx$ 1

(ii) Put $f(x) = \sin^4 x$. Then $f(2\pi - x) = \sin^4 (2\pi - x) = \sin^4 x = f(x)$. 1M

$\int_0^{2\pi} x \sin^4 x \, dx = \frac{2\pi}{2} \int_0^{2\pi} \sin^4 x \, dx$ 1M

$= \pi \left(\frac{3\pi}{4} \right)$

$= \frac{3\pi^2}{4}$ 1A

END OF PAPER