

M2REG-DEFINT-2425-ASM-SET 4-MATH

Suggested solutions

1. Let $u = e^x - e^{-x}$. Then $du = (e^x + e^{-x}) dx$. 1M

$$\begin{aligned} \int_{\ln 2}^{\ln 4} \frac{e^x + e^{-x}}{\sqrt{e^x - e^{-x}}} dx &= \int_{\frac{3}{2}}^{\frac{15}{4}} \frac{du}{\sqrt{u}} \\ &= \left[2u^{\frac{1}{2}} \right]_{\frac{3}{2}}^{\frac{15}{4}} \\ &= \sqrt{15} - \sqrt{6} \end{aligned} \quad \begin{array}{l} 1A \\ 1A \end{array}$$

$$\begin{aligned} 2. \quad \int_{\frac{\pi}{9}}^{\frac{5\pi}{36}} \frac{\sqrt{3} + \tan 3x}{1 - \sqrt{3} \tan 3x} dx &= \int_{\frac{\pi}{9}}^{\frac{5\pi}{36}} \frac{\tan \frac{\pi}{3} + \tan 3x}{1 - \tan \frac{\pi}{3} \tan 3x} dx \\ &= \int_{\frac{\pi}{9}}^{\frac{5\pi}{36}} \tan \left(3x + \frac{\pi}{3} \right) dx \\ &= \int_{\frac{\pi}{9}}^{\frac{5\pi}{36}} \frac{\sin \left(3x + \frac{\pi}{3} \right)}{\cos \left(3x + \frac{\pi}{3} \right)} dx \end{aligned} \quad \begin{array}{l} 1M \end{array}$$

Let $u = \cos \left(3x + \frac{\pi}{3} \right)$. Then $du = -3 \sin \left(3x + \frac{\pi}{3} \right) dx$. 1M

$$\begin{aligned} \int_{\frac{\pi}{9}}^{\frac{5\pi}{36}} \frac{\sqrt{3} + \tan 3x}{1 - \sqrt{3} \tan 3x} dx &= -\frac{1}{3} \int_{-\frac{1}{2}}^{-\frac{\sqrt{2}}{2}} \frac{du}{u} \\ &= -\frac{1}{3} \left[\ln |u| \right]_{-\frac{1}{2}}^{-\frac{\sqrt{2}}{2}} \\ &= -\frac{1}{6} \ln 2 \end{aligned} \quad \begin{array}{l} 1A \\ 1A \end{array}$$

$$\begin{aligned} 3. \quad (a) \quad f(-x) &= \sum_{i=0}^{77} \sin^{2i+1}(-x) \\ &= - \sum_{i=0}^{77} \sin^{2i+1} x \\ &= -f(x) \end{aligned}$$

Thus, $f(x)$ is an odd function.

1

$$(b) \quad \int_{-\frac{\pi}{9}}^{\frac{\pi}{9}} f(x) dx = 0 \quad 1A$$

$$\begin{aligned} 4. \quad \frac{\tan(-x)^3 + \sin(-x)^3}{(-x)^2 + \cos(-x)} &= \frac{-\tan x^3 - \sin x^3}{x^2 + \cos x} \\ \frac{\tan x^3 + \sin x^3}{x^2 + \cos x} &\text{ is an odd function.} \end{aligned} \quad \begin{array}{l} 1M \end{array}$$

Thus, $\int_{-\frac{2\pi}{9}}^{\frac{2\pi}{9}} \frac{\tan x^3 + \sin x^3}{x^2 + \cos x} dx = 0$. 1A

5. $(-x)^3 \sin \left[(-x)^2 + e^{(-x)^4} \right] = -x^3 \sin \left(x^2 + e^{x^4} \right)$
 $x^3 \sin \left(x^2 + e^{x^4} \right)$ is an odd function. 1M
Thus, $\int_{-1}^1 x^3 \sin \left(x^2 + e^{x^4} \right) dx = 0.$ 1A

6. $\ln \frac{4 + \sin(-x)}{4 - \sin(-x)} = \ln \frac{4 - \sin x}{4 + \sin x} = -\ln \frac{4 + \sin x}{4 - \sin x}$
 $\ln \frac{4 + \sin x}{4 - \sin x}$ is an odd function. 1M
Thus, $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \ln \frac{4 + \sin x}{4 - \sin x} dx = 0.$ 1A

7. Let $u = \tan x$. Then $du = \sec^2 x dx$. 1M

$$\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{dx}{\cos^2 x \sqrt{4 - \tan^2 x}} = \int_{-\sqrt{3}}^{\sqrt{3}} \frac{du}{\sqrt{4 - u^2}}$$

Let $u = 2 \sin \theta$. Then $du = 2 \cos \theta d\theta$. 1M

$$\begin{aligned} \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{dx}{\cos^2 x \sqrt{4 - \tan^2 x}} &= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{2 \cos \theta}{\sqrt{4 - 4 \sin^2 \theta}} d\theta \\ &= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} d\theta \\ &= \left[\theta \right]_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \\ &= \frac{2\pi}{3} \end{aligned}$$

1A

8. $\frac{-2(-x)^5}{1 + (-x)^6} = -\frac{2x^5}{1 + x^6}$
 $\frac{-2x^5}{1 + x^6}$ is an odd function. 1M
 $\int_{-1}^1 \frac{x^2 - 2x^5 + x^8}{1 + x^6} dx = \int_{-1}^1 \frac{-2x^5}{1 + x^6} dx + \int_{-1}^1 \frac{x^2(1 + x^6)}{1 + x^6} dx$
 $= 0 + \int_{-1}^1 x^2 dx$ 1M
 $= \left[\frac{x^3}{3} \right]_{-1}^1$
 $= \frac{2}{3}$ 1A

9. $\sin(\sin(-x)) = -\sin(\sin x)$

$\sin(\sin x)$ is an odd function.

1M

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} [\tan^2 x + \sin(\sin x)] dx = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan^2 x dx + \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin(\sin x) dx$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\sec^2 x - 1) dx + 0$$

1M

$$= \left[\tan x - x \right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$= 2 - \frac{\pi}{2}$$

1A

10. (a) $f(-x) = (-x)^4 \sin(-x)$

$$= -x^4 \sin x$$

$$= -f(x)$$

Thus, $f(x)$ is an odd function.

1

(b) $\int_{-a}^a (1 - x^4 \sin x) dx = \frac{\pi}{2}$

$$\int_{-a}^a dx - \int_{-a}^a x^2 \sin x dx = \frac{\pi}{2}$$

$$\left[x \right]_{-a}^a - 0 = \frac{\pi}{2}$$

1M

$$2a = \frac{\pi}{2}$$

$$a = \frac{\pi}{4}$$

1A

11. (a) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan^2 x dx = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\sec^2 x - 1) dx$

1M

$$= \left[\tan x - x \right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$= 2 - \frac{\pi}{2}$$

1A

(b) $\frac{(-x) \tan^2(-x)}{1 + (-x)^2} = -\frac{x \tan^2 x}{1 + x^2}$

$\frac{x \tan^2 x}{1 + x^2}$ is an odd function.

1M

Thus, $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{x \tan^2 x}{1 + x^2} dx = 0.$

1A

12. $\tan^5(-x) = -\tan^5 x$

$\tan^5 x$ is an odd function.

$$\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} (\tan^5 x + \sin^2 x) \, dx = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \tan^5 x \, dx + \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \sin^2 x \, dx$$

$$= 0 + \frac{1}{2} \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} (1 - \cos 2x) \, dx$$

1M+1M

$$= \frac{1}{2} \left[x - \frac{\sin 2x}{2} \right]_{-\frac{\pi}{6}}^{\frac{\pi}{6}}$$

$$= \frac{\pi}{6} - \frac{\sqrt{3}}{4}$$

1A

$$\begin{aligned}
 13. \quad (a) \quad \frac{1}{4 \cos 2x + 5} &= \frac{1}{4(2 \cos^2 x - 1) + 5} \\
 &= \frac{1}{8 \cos^2 x + 1} \\
 &= \frac{\sec^2 x}{8 + \sec^2 x}
 \end{aligned}$$

1

(b) Let $u = \tan x$. Then $du = \sec^2 x \, dx$.

1M

$$\begin{aligned}
 \int_0^{\frac{\pi}{4}} \frac{1}{4 \cos 2x + 5} \, dx &= \int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{8 + \sec^2 x} \, dx \\
 &= \int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{9 + \tan^2 x} \, dx \\
 &= \int_0^1 \frac{du}{u^2 + 9}
 \end{aligned}$$

1M

Let $u = 3 \tan \theta$. Then $du = 3 \sec^2 \theta \, d\theta$.

$$\begin{aligned}
 \int_0^{\frac{\pi}{4}} \frac{1}{4 \cos 2x + 5} \, dx &= \int_0^{\tan^{-1} \frac{1}{3}} \frac{3 \sec^2 \theta}{9 \tan^2 \theta + 9} \, d\theta \\
 &= \frac{1}{3} \int_0^{\tan^{-1} \frac{1}{3}} d\theta \\
 &= \frac{1}{3} \left[\theta \right]_0^{\tan^{-1} \frac{1}{3}} \\
 &= \frac{1}{3} \tan^{-1} \frac{1}{3}
 \end{aligned}$$

1A

(c) Let $u = -x$. Then $du = -dx$.

$$\begin{aligned}
 &\int_{-a}^a f(x) \ln(1 + e^x) \, dx \\
 &= \int_{-a}^0 f(x) \ln(1 + e^x) \, dx + \int_0^a f(x) \ln(1 + e^x) \, dx \\
 &= - \int_a^0 f(-u) \ln(1 + e^{-u}) \, du + \int_0^a f(x) \ln(1 + e^x) \, dx \\
 &= - \int_a^0 [-f(x)] \ln(1 + e^{-x}) \, dx + \int_0^a f(x) \ln(1 + e^x) \, dx \\
 &= - \int_0^a f(x) \ln \left(\frac{e^x + 1}{e^x} \right) \, dx + \int_0^a f(x) \ln(1 + e^x) \, dx \\
 &= \int_0^a f(x) [\ln e^x - \ln(1 + e^x) + \ln(1 + e^x)] \, dx \\
 &= \int_0^a x f(x) \, dx
 \end{aligned}$$

1M

1M

1M

1

(d) Let $g(x) = \frac{4 \sin 2x}{(4 \cos 2x + 5)^2}$.

$$g(-x) = \frac{4 \sin[2(-x)]}{(4 \cos[2(-x)] + 5)^2} = -\frac{4 \sin 2x}{(4 \cos 2x + 5)^2} = -g(x)$$

1M

Thus, $g(x)$ is an odd function.

$$\text{Note that } \frac{d}{dx} \left[\frac{1}{2(4 \cos 2x + 5)} \right] = \frac{4 \sin 2x}{(4 \cos 2x + 5)^2}.$$

$$\begin{aligned}
& \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{4 \sin 2x}{(4 \cos 2x + 5)^2} \, dx \\
&= \int_0^{\frac{\pi}{4}} \frac{4x \sin 2x}{(4 \cos 2x + 5)^2} \, dx && 1\text{M} \\
&= \left[\frac{x}{2(4 \cos 2x + 5)} \right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \frac{dx}{2(4 \cos 2x + 5)} && 1\text{M} \\
&= \frac{1}{2} \left(\frac{\pi}{4} \right) \left(\frac{1}{0 + 5} \right) - \frac{1}{2} \left(\frac{1}{3} \tan^{-1} \frac{1}{3} \right) && 1\text{M} \\
&= \frac{\pi}{40} - \frac{1}{6} \tan^{-1} \frac{1}{3} && 1\text{A}
\end{aligned}$$