

M2REG-AOI-2425-ASM-SET 4-MATH

Suggested solutions

$$1. \quad (a) \quad \int_0^k \frac{1}{\sqrt{4-x}} dx = \int_k^3 \frac{1}{\sqrt{4-x}} dx \quad 1M+1M$$

$$\left[-2\sqrt{4-x} \right]_0^k = \left[-2\sqrt{4-x} \right]_k^3$$

$$4 - 2\sqrt{4-k} = 2\sqrt{4-k} - 2$$

$$\sqrt{4-k} = \frac{3}{2}$$

$$k = \frac{7}{4}$$

1A

(b) Required volume

$$= \pi \int_0^{\frac{7}{4}} \frac{1}{4-x} dx \quad 1M$$

$$= \pi \left[-\ln|4-x| \right]_0^{\frac{7}{4}} \quad 1M$$

$$= \pi \ln \frac{16}{9} \quad 1A$$

$$2. \quad (a) \quad y = x - 3 \quad 1A$$

$$(b) \quad \text{Required area} = \int_2^3 \left[\left(x - 3 + \frac{16}{x^2 - 4x + 7} \right) - (x - 3) \right] dx \quad 1M$$

$$= \int_2^3 \frac{16}{x^2 - 4x + 7} dx$$

$$= \int_2^3 \frac{16}{(x-2)^2 + 3} dx \quad 1M$$

$$\text{Let } x - 2 = \sqrt{3} \tan \theta. \text{ Then } dx = \sqrt{3} \sec^2 \theta d\theta. \quad 1M$$

$$\text{Required area} = \int_0^{\frac{\pi}{6}} \frac{16\sqrt{3} \sec^2 \theta}{(\sqrt{3} \tan \theta)^2 + 3} d\theta$$

$$= \frac{16\sqrt{3}}{3} \int_0^{\frac{\pi}{6}} \frac{\sec^2 \theta}{\sec^2 \theta} d\theta \quad 1M$$

$$= \frac{16\sqrt{3}}{3} \int_0^{\frac{\pi}{6}} d\theta$$

$$= \frac{16\sqrt{3}}{3} \left[\theta \right]_0^{\frac{\pi}{6}}$$

$$= \frac{8\sqrt{3}\pi}{9} \quad 1A$$

3. (a) Let $w = x + 1$. Then $dw = dx$. 1M

$$\text{Area} = \int_0^u \frac{x}{\sqrt{x+1}} dx$$

$$= \int_1^{u+1} \frac{w-1}{\sqrt{w}} dw$$
1A

$$= \int_1^{u+1} \left(w^{\frac{1}{2}} - w^{-\frac{1}{2}} \right) dw$$

$$= \left[\frac{2}{3} w^{\frac{3}{2}} - 2w^{\frac{1}{2}} \right]_1^{u+1}$$
1M

$$= \frac{2}{3}(u-2)\sqrt{u+1} + \frac{4}{3}$$
1A

- (b) Let $A = \frac{2}{3}(u-2)\sqrt{u+1} + \frac{4}{3}$.

$$\frac{dA}{dt} = \frac{dA}{du} \cdot \frac{du}{dt}$$

$$= \frac{2}{3} \left[(u-2) \cdot \frac{1}{2\sqrt{u+1}} + \sqrt{u+1} \right] \frac{du}{dt}$$
1M+1A

$$= \frac{u}{\sqrt{u+1}} \cdot \frac{du}{dt}$$

When $OQ = 8$,

$$\frac{dA}{dt} = \frac{8}{\sqrt{8+1}} \cdot 3$$

$$= 8$$

Required rate is 8 square units per second.

1A

4. (a) Let $x = 3 + 3 \sin \theta$. Then $dx = 3 \cos \theta d\theta$.

$$\int \sqrt{x(6-x)} dx = \int \sqrt{(3+3\sin\theta)(3-3\sin\theta)} 3 \cos \theta d\theta \quad 1M$$

$$= 9 \int \cos^2 \theta d\theta$$

$$= \frac{9}{2} \int (1 + \cos 2\theta) d\theta \quad 1A$$

$$= \frac{9\theta}{2} + \frac{9 \sin 2\theta}{4} + \text{constant} \quad 1A$$

$$= \frac{9}{2} \sin^{-1} \frac{x-3}{3} + \frac{9}{2} \sin \theta \cos \theta + \text{constant}$$

$$= \frac{9}{2} \sin^{-1} \frac{x-3}{3} + \frac{9}{2} \left(\frac{x-3}{3} \right) \sqrt{1 - \left(\frac{x-3}{3} \right)^2} + \text{constant} \quad 1M$$

$$= \frac{9}{2} \sin^{-1} \frac{x-3}{3} + \frac{x-3}{2} \sqrt{6x-x^2} + \text{constant} \quad 1A$$

- (b) In the first quadrant, $y^2 = \sqrt{6x-x^2} = \sqrt{x(6-x)}$. 1M

$$\text{Volume} = \pi \int_0^6 \sqrt{x(6-x)} dx \quad 1M$$

$$= \pi \left[\frac{9}{2} \sin^{-1} \frac{x-3}{3} + \frac{x-3}{2} \sqrt{6x-x^2} \right]_0^6 \quad 1M$$

$$= \frac{9\pi^2}{2} \quad 1A$$

5. (a) Volume = $\pi \int_4^{4+h} y \, dy$ 1M

$$= \pi \left[\frac{y^2}{2} \right]_4^{4+h}$$

$$= \frac{\pi}{2} (h^2 + 8h)$$

1

(b) (i) $V = V_1 + \frac{\pi}{2} (h^2 + 8h)$, where V_1 is the volume of the frustum and $h = H - 4$ for $4 \leq H \leq 12$.

$$\frac{dV}{dt} = \frac{\pi(2h+8)}{2} \frac{dh}{dt}$$

1A

$$8 = \frac{\pi[2(7-4)+8]}{2} \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{8}{7\pi}$$

Required rate is $\frac{8}{7\pi}$ cm/s.

1A

(ii) Consider the cone that makes up the frustum. Let the height of the cone be H_0 cm.

$$\frac{H_0 - 4}{H_0} = \frac{2}{4}$$

1M

$$H_0 = 8$$

1A

$$V = \frac{\pi}{3} (4^2)(8) \left[1 - \left(\frac{8-H}{8} \right)^3 \right]$$

1M+1A

$$= \frac{128\pi}{3} \left[1 - \left(1 - \frac{H}{8} \right)^3 \right]$$

1

(iii) Volume of the upper part = $\frac{\pi}{2} [8^2 + 8(8)] = 64\pi \text{ cm}^3$.

Volume of milk leaked = $\frac{\pi}{2} \times 130 = 65\pi \text{ cm}^3 > 64\pi \text{ cm}^3$.

Depth of remaining milk is less than 4 cm.

$$64\pi + \frac{128\pi}{3} \left[1 - \left(1 - \frac{4}{8} \right)^3 \right] - 65\pi = \frac{128\pi}{3} \left[1 - \left(1 - \frac{H}{8} \right)^3 \right]$$

1M

$$\left(1 - \frac{H}{8} \right)^3 = \frac{19}{128}$$

$$1 - \frac{H}{8} = \left(\frac{19}{128} \right)^{\frac{1}{3}}$$

1A

$$V = \frac{128\pi}{3} \left[1 - \left(1 - \frac{H}{8} \right)^3 \right]$$

$$\frac{dV}{dt} = \frac{128\pi}{3} \left[0 - 3 \left(1 - \frac{H}{8} \right)^2 \left(-\frac{1}{8} \right) \frac{dH}{dt} \right]$$

$$= 16\pi \left(1 - \frac{H}{8} \right)^2 \frac{dH}{dt}$$

1A

$$-\frac{\pi}{2} = 16\pi \left(\frac{19}{128} \right)^{\frac{2}{3}} \frac{dH}{dt}$$

$$\frac{dH}{dt} \approx -0.111$$

Required rate is 0.111 cm/s.

1A

6. (a) (i) Put $y = 0$ into $3x^2 - 2y - 4 = 0$, we have $x = \pm\sqrt{\frac{4}{3}}$. 1A
- The points $\left(\pm\sqrt{\frac{4}{3}}, 0\right)$ lies on the curve $3x^2 + 2y + k = 0$,
- $$3 \times \frac{4}{3} + 2(0) + k = 0$$
- $$k = -4$$
- 1A
- (ii) The equation of BC can be rewritten as $y = -\frac{3}{2}x^2 + 2$.
- The coordinates of vertex are $(0, 2)$. 1A
- (b) (i) When $0 \leq h \leq 2$,
- $$\begin{aligned} \text{Volume} &= \pi \int_0^h \left[\frac{2y+4}{3} - \frac{4-2y}{3} \right] dx \\ &= \pi \int_0^h \frac{4y}{3} dy \\ &= \pi \left[\frac{2y^2}{3} \right]_0^h \\ &= \frac{2\pi h^2}{3} \text{ cubic units} \end{aligned}$$
- 1M+1A
-
- 1M
-
- 1A
- (ii) When $2 < h \leq 6$,
- $$\begin{aligned} \text{Volume} &= \frac{2\pi(2)^2}{3} + \pi \int_2^h \frac{2y+4}{3} dy \\ &= \frac{8\pi}{3} + \pi \left[\frac{4y}{3} + \frac{y^2}{3} \right]_2^h \\ &= \pi \left(\frac{h^2}{3} + \frac{4h}{3} - \frac{4}{3} \right) \text{ cubic units} \end{aligned}$$
- 1M
-
- 1A
- (c) Let V cubic units be the volume of water in the cup at time t seconds.
- When $V = \frac{28\pi}{3}$, we have $V > \frac{8\pi}{3}$ and so $h > 2$. 1M
- $$\frac{28\pi}{3} = \pi \left(\frac{h^2}{3} + \frac{4h}{3} - \frac{4}{3} \right)$$
- $$h^2 + 4h - 32 = 0$$
- 1M
- $$h = 4 \quad \text{or} \quad -8 \text{ (rejected)}$$
- For $2 < h \leq 6$, $\frac{dV}{dt} = \pi \left(\frac{2h}{3} + \frac{4}{3} \right) \frac{dh}{dt}$ 1M
- When $\frac{dV}{dt} = \frac{8\pi}{3}$ and $h = 4$,
- $$\begin{aligned} \frac{8\pi}{3} &= \pi \left(\frac{2 \times 4}{3} + \frac{4}{3} \right) \frac{dh}{dt} \\ \frac{dh}{dt} &= \frac{2}{3} \end{aligned}$$
- Required rate is $\frac{2}{3}$ units per second. 1A

7. (a) $\sqrt{(x+4)^2 + y^2} + \sqrt{(x-4)^2 + y^2} = 10$

$$\sqrt{x^2 + y^2 + 8x + 16} = 10 - \sqrt{x^2 + y^2 - 8x + 16} \quad 1M$$

$$x^2 + y^2 + 8x + 16 = 100 + x^2 + y^2 - 8x + 16 - 20\sqrt{x^2 + y^2 - 8x + 16}$$

$$5\sqrt{x^2 + y^2 - 8x + 16} = 25 - 4x \quad 1M$$

$$25(x^2 + y^2 - 8x + 16) = 625 - 200x + 16x^2$$

$$9x^2 + 25y^2 = 225$$

$$\frac{x^2}{25} + \frac{y^2}{9} = 1 \quad 1$$

(b) (i) Let V cubic units be the volume of water.

$$V = \pi \int_{-3}^{-3+h} 25 \left(1 - \frac{y^2}{9} \right) dy \quad 1M$$

$$= \frac{25\pi}{9} \int_{-3}^{-3+h} (9 - y^2) dy$$

$$= \frac{25\pi}{9} \left[9y - \frac{y^3}{3} \right]_{-3}^{-3+h} \quad 1A$$

$$= \frac{25\pi}{27} (9h^2 - h^3) \quad 1A$$

(ii) $\frac{dV}{dh} = \frac{25\pi}{27} (18h - 3h^2) \frac{dh}{dt} \quad 1M$

$$30 - 5 = \frac{25\pi}{9} (6h - h^2) \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{9}{\pi(6h - h^2)} \quad 1A$$

$$= \frac{9}{-\pi(h-3)^2 + 9\pi} \quad 1M$$

Since $0 < h < 6$, $\frac{dh}{dt}$ is minimum when the denominator is maximum.

Required depth of water is 3 units. 1A

(iii) $V = \int \frac{-\pi}{800} (t + 100) dt \quad 1M$

$$= -\frac{\pi t^2}{1600} - \frac{\pi t}{8} + C$$

When $t = 0$, $h = 3$ and $V = \frac{25\pi}{27} [9(3)^2 - 3^3] = 50\pi$.

So, $C = 50\pi$. 1A

When $V = 0$,

$$-\frac{\pi t^2}{1600} - \frac{\pi t}{8} + 50\pi = 0 \quad 1M$$

$$t^2 + 200t - 80\,000 = 0$$

$$t = 200 \quad \text{or} \quad -400 \text{ (rejected)}$$

The time required is 200 minutes. 1A

8. (a) $f(-2) = (-2)^3 + p(-2)^2 + q(-2) + 1 = 9$ 1M

$$4p - 2q = 16$$

$$\frac{dy}{dx} = 3x^2 + 2px + q$$
 1A

$$3(-2)^2 + 2p(-2) + q = 0$$

$$4p - q = 12$$

Solving, we have $p = 2$ and $q = -4$. 1A

(b) $\frac{d^2y}{dx^2} = 6x + 4$. So, $\left. \frac{d^2y}{dx^2} \right|_{x=-2} = 6(-2) + 4 = -8 < 0$ 1M

So, P is a maximum point and is not a minimum point. 1A

(c) $\frac{dy}{dx} = 3x^2 + 4x - 4 = 0$ 1M

$$x = \frac{2}{3} \quad \text{or} \quad -2$$

When $x = \frac{2}{3}$, $y = -\frac{13}{27}$ and $\frac{d^2y}{dx^2} = 8 > 0$.

So, maximum point is $(-2, 9)$ and minimum point is $\left(\frac{2}{3}, -\frac{13}{27}\right)$. 1A

(d) When $\frac{d^2y}{dx^2} = 6x + 4 = 0$, $x = -\frac{2}{3}$.

x	$x < -\frac{2}{3}$	$x > -\frac{2}{3}$
$\frac{d^2y}{dx^2}$	-	+

1M

The point of inflexion is $\left(-\frac{2}{3}, \frac{115}{27}\right)$. 1A

(e) The equation of L is $y = 9$.

$$x^3 + 2x^2 - 4x + 1 = 9$$
 1M

$$x^3 + 2x^2 - 4x - 8 = 0$$

$$(x + 2)^2(x - 2) = 0$$

$$x = \pm 2$$

$$\text{Area} = \int_{-2}^2 [9 - (x^3 + 2x^2 - 4x + 1)] dx$$
 1M

$$= \left[-\frac{x^4}{4} - \frac{2x^3}{3} + 2x^2 + 8x \right]_{-2}^2$$
 1M

$$= \frac{64}{3}$$
 1A