

M2REG-AOI-2425-ASM-SET 3-MATH**Suggested solutions**

1. Required volume

$$= \pi \int_1^2 [(e^y - 1) - (-2)]^2 dy - \pi \int_1^2 [(2y - 1) - (-2)]^2 dy \quad 1M+1M$$

$$= \pi \int_1^2 [(e^y + 1)^2 - (2y + 1)^2] dy$$

$$= \pi \int_1^2 (e^{2y} + 2e^y - 4y^2 - 4y) dy$$

$$= \pi \left[\frac{e^{2y}}{2} + 2e^y - \frac{4y^3}{3} - 2y^2 \right]_1^2 \quad 1M$$

$$= \pi \left(\frac{e^4}{2} + \frac{3e^2}{2} - 2e - \frac{46}{3} \right) \quad 1A$$

2. Required volume

$$= \pi \int_1^4 \left(2 - \frac{\sqrt{y}}{2} \right)^2 dy - \pi \int_1^4 (2 - \sqrt{y})^2 dy \quad 1M+1M$$

$$= \pi \int_1^4 \left(2\sqrt{y} - \frac{3y}{4} \right) dy$$

$$= \pi \left[\frac{4y^{\frac{3}{2}}}{3} - \frac{3y^2}{8} \right]_1^4 \quad 1M$$

$$= \frac{89\pi}{24} \quad 1A$$

$$3. \text{ Volume} = \pi \int_{-3}^3 \frac{9 - y^2}{2} dy - \pi \int_{-2}^2 \frac{4 - y^2}{2} dy \quad 2M+1A$$

$$= \pi \left[\frac{9y}{2} - \frac{y^3}{6} \right]_{-3}^3 - \pi \left[2y - \frac{y^3}{6} \right]_{-2}^2 \quad 1M$$

$$= \frac{38\pi}{3} \quad 1A$$

4. (a) $y = 4y - y^2$ 1M
 $y^2 - 3y = 0$
 $y = 0$ or 3
The coordinates of P are $(\sqrt{3}, 3)$. 1A
- (b) Required volume

$$= \pi \int_0^3 (4y - y^2) dy - \pi \int_0^3 y dy$$
 1M+1M

$$= \pi \int_0^3 (3y - y^2) dy$$

$$= \pi \left[\frac{3y^2}{2} - \frac{y^3}{3} \right]_0^3$$
 1M

$$= \frac{9\pi}{2}$$
 1A
5. (a) Solve $\begin{cases} y = -3x - 3 \\ y = 6 \end{cases}$, the coordinates of A are $(-3, 6)$. 1A
Solve $\begin{cases} y = x + 3 \\ y = -3x - 3 \end{cases}$, the coordinates of B are $(-\frac{3}{2}, \frac{3}{2})$. 1A
Solve $\begin{cases} y = x + 3 \\ y = 6 \end{cases}$, the coordinates of C are $(3, 6)$. 1A
- (b) Required volume

$$= \pi \int_{-3}^{-\frac{3}{2}} [8 - (-3x - 3)]^2 dx + \pi \int_{-\frac{3}{2}}^3 [8 - (x + 3)]^2 dx - \pi(8 - 6)^2(3 + 3)$$
 1M+1M

$$= \pi \int_{-3}^{-\frac{3}{2}} (9x^2 + 66x + 121) dx + \pi \int_{-\frac{3}{2}}^3 (x^2 - 10x + 25) dx - 24\pi$$

$$= \pi \left[3x^3 + 33x^2 + 121x \right]_{-3}^{-\frac{3}{2}} + \pi \left[\frac{x^3}{3} - 5x^2 + 25x \right]_{-\frac{3}{2}}^3 - 24\pi$$
 1M

$$= \frac{189\pi}{2}$$
 1A

6. (a) $\frac{dy}{dx} = (1-x)^{-\frac{1}{2}}(-1)$ 1M
 $= -\frac{1}{\sqrt{1-x}}$
 At A (0, 2), $\frac{dy}{dx} = -\frac{1}{\sqrt{1-0}} = -1$. Slope of $L = \frac{-1}{-1} = 1$. 1M
 The equation of L is $y = x + 2$. 1A
- (b) $L: x = y - 2$ and $C: x = 1 - \frac{y^2}{4}$.
 Required volume $= \pi \int_0^2 [1 - (y - 2)]^2 dy - \pi \int_0^2 \left[1 - \left(1 - \frac{y^2}{4}\right)\right]^2 dy$ 1M+1A
 $= \pi \int_0^2 (9 - 6y + y^2) dy - \pi \int_0^2 \frac{y^4}{16} dy$
 $= \pi \left[\frac{y^3}{3} - 3y^2 + 9y \right]_0^2 - \pi \left[\frac{y^5}{80} \right]_0^2$ 1M
 $= \frac{124\pi}{15}$ 1A
7. (a) $6 - x^2 = 4 - x$ 1M
 $0 = x^2 - x - 2$
 $x = 2$ or -1
 When $x = 2$, $y = 4 - 2 = 2$; when $x = -1$, $y = 4 - (-1) = 5$.
 The coordinates of A and B are (2, 2) and (-1, 5) respectively. 1A+1A
- (b) Volume $= \pi \int_{-1}^2 [(6 - x^2) - 2]^2 dx - \pi \int_{-1}^2 [(4 - x) - 2]^2 dx$ 1M+1M
 $= \pi \int_{-1}^2 (x^4 - 8x^2 + 16) dx - \pi \int_{-1}^2 (4 - 4x + x^2) dx$
 $= \pi \left[\frac{x^5}{5} - \frac{8x^3}{3} + 16x \right]_{-1}^2 - \pi \left[4x - 2x^2 + \frac{x^3}{3} \right]_{-1}^2$ 1M
 $= \frac{108\pi}{5}$ 1A

8. (a) $\frac{d}{dx}(e^x \cos x) = e^x \cos x - e^x \sin x$ 1A
- (b) $\int e^x \cos 2x \, dx = e^x \cos 2x + 2 \int e^x \sin 2x \, dx$ 1M
- $$= e^x \cos 2x + 2e^x \sin 2x - 4 \int e^x \cos 2x \, dx$$
- 1M
- $$5 \int e^x \cos 2x \, dx = e^x \cos 2x + 2e^x \sin 2x + \text{constant}$$
- $$\int e^x \cos 2x \, dx = \frac{e^x \cos 2x}{5} + \frac{2e^x \sin 2x}{5} + \text{constant}$$
- 1A
- (c) $e^x(\sin 2x + 1) = e^x(\cos 2x + 1)$
- $$\sin 2x = \cos 2x$$
- $$\tan 2x = 1$$
- $$x = \frac{\pi}{8} \quad \text{or} \quad \frac{5\pi}{8}$$
- 1A
- Volume = $\pi \int_{\frac{\pi}{8}}^{\frac{5\pi}{8}} [e^{2x}(\sin 2x + 1)^2 - e^{2x}(\cos 2x + 1)^2] \, dx$ 1M+1A
- $$= \pi \int_{\frac{\pi}{8}}^{\frac{5\pi}{8}} e^{2x}(\sin^2 2x - \cos^2 2x) \, dx + 2\pi \int_{\frac{\pi}{8}}^{\frac{5\pi}{8}} e^{2x}(\sin 2x - \cos 2x) \, dx$$
- $$= -\pi \int_{\frac{\pi}{8}}^{\frac{5\pi}{8}} e^{2x} \cos 4x \, dx + 2\pi \int_{\frac{\pi}{8}}^{\frac{5\pi}{8}} e^{2x}(\sin 2x - \cos 2x) \, dx$$
- $$= \frac{\pi}{2} \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} e^u \cos 2u \, du + \pi \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} e^u(\sin u - \cos u) \, du \quad \text{where } u = 2x$$
- 1M
- $$= \frac{\pi}{2} \left[\frac{e^u \cos 2u}{5} + \frac{2e^u \sin 2u}{5} \right]_{\frac{\pi}{4}}^{\frac{5\pi}{4}} - \pi \left[e^u \cos u \right]_{\frac{\pi}{4}}^{\frac{5\pi}{4}}$$
- 1M
- $$= \frac{\pi}{5} \left(e^{\frac{\pi}{4}} - e^{\frac{5\pi}{4}} \right) + \frac{\pi}{\sqrt{2}} \left(e^{\frac{5\pi}{4}} + e^{\frac{\pi}{4}} \right)$$
- 1A

9. (a) $1 - \cos 4\theta + 2 \cos 2\theta \sin^2 2\theta$
 $= 1 - (1 - 2 \sin^2 2\theta) + 2(2 \cos^2 \theta - 1) \sin^2 2\theta$ 1M
 $= 4 \cos^2 \theta \sin^2 2\theta$ 1
- (b) $\int_0^{n\pi} \cos^2 \theta \sin^2 2\theta \, d\theta = \frac{1}{4} \int_0^{n\pi} (1 - \cos 4\theta + 2 \cos 2\theta \sin^2 2\theta) \, d\theta$ 1M
 $= \frac{1}{4} \left[\theta - \frac{\sin 4\theta}{4} \right]_0^{n\pi} + \frac{1}{4} \int_0^{n\pi} \sin^2 2\theta \, d \sin 2\theta$ 1M
 $= \frac{n\pi}{4} + \frac{1}{4} \left[\frac{\sin^3 2\theta}{3} \right]_0^{n\pi}$
 $= \frac{n\pi}{4}$ 1
- (c) (i) Required area
 $= \int_0^\pi \cos \theta \sin 2\theta \, d\theta$ 1M
 $= \frac{1}{2} \int_0^\pi (\sin x + \sin 3x) \, dx$ 1M
 $= \frac{1}{2} \left[-\cos x - \frac{\cos 3x}{3} \right]_0^\pi$
 $= \frac{4}{3}$ 1A
- (ii) Required volume
 $= \pi \int_0^\pi \cos^2 \sin^2 2\theta \, d\theta$ 1M
 $= \pi \times \frac{\pi}{4}$ 1M
 $= \frac{\pi^2}{4}$ 1A

10. (a) (i) The coordinates of A and B are (a^2, a) and $(b^2, -b)$ respectively.
Required equation is

$$y - a = \frac{a + b}{a^2 - b^2}(x - a^2) \quad 1M$$

$$y - a = \frac{1}{a - b}(x - a^2)$$

$$x - (a - b)y - ab = 0 \quad 1A$$

- (ii) Required area

$$= \int_{-b}^a [(a - b)y + ab - y^2] dy \quad 1M$$

$$= \left[\frac{(a - b)y^2}{2} + aby - \frac{y^3}{3} \right]_{-b}^a \quad 1M$$

$$= \frac{1}{6} [3(a - b)a^2 + 6a^2b - 2a^3] - [3(a - b)b^2 - 6ab^2 + 2b^3]$$

$$= \frac{1}{6}(a^3 + 3a^2b + 3ab^2 + b^3)$$

$$= \frac{1}{6}(a + b)^3 \quad 1$$

- (b) (i) $4 - (a - b)(0) - ab = 0$

$$ab = 4 \quad 1$$

- (ii) Let A be the area of the shaded region.

$$A = \frac{1}{6}(a + b)^3$$

$$= \frac{1}{6} \left(a + \frac{4}{a} \right)^3$$

$$\frac{dA}{da} = \frac{1}{2} \left(a + \frac{4}{a} \right)^2 \left(1 - \frac{4}{a^2} \right) \quad 1M$$

When $\frac{dA}{da} = 0$,

$$1 - \frac{4}{a^2} = 0$$

$$a^2 = 4$$

$$a = 2 \quad \text{or} \quad -2 \text{ (rejected)}$$

a	$0 < a < 2$	$a > 2$
$\frac{dA}{da}$	–	+

1M

The area of the shaded region is a minimum when $a = 2$.

1A

- (c) When $a = 2$, we have $b = 2$ and the equation of AB is $x = 4$.

Required volume

$$= \pi \int_{-2}^2 [4 - (-1)]^2 dy - \pi \int_{-2}^2 [y^2 - (-1)^2] dy \quad 1M$$

$$= 25\pi \int_{-2}^2 dy - \pi \int_{-2}^2 (y^4 + 2y^2 + 1) dy$$

$$= 25\pi \left[y \right]_{-2}^2 - \pi \left[\frac{y^5}{5} + \frac{2y^3}{3} + y \right]_{-2}^2 \quad 1M$$

$$= \frac{1088\pi}{15} \quad 1A$$