M2REG-AOI-2425-ASM-SET 1-MATH
Suggested solutions

1. Required area $= \int_1^3 \frac{3}{x+1}\, dx$ 1M

$= \Big[3 \ln|x+1|\Big]_1^3$ 1M

$= 3 \ln 2$ 1A

2. (a) Let $u = x^2 - 2x + 4$. Then $du = 2(x-1)\, dx$. 1M

$$\int \frac{x-1}{x^2-2x+4}\, dx = \frac{1}{2}\int \frac{du}{u}$$
$$= \frac{1}{2}\ln|u| + \text{constant}$$
$$= \frac{1}{2}\ln|x^2-2x+4| + \text{constant}$$ 1A

(b) Required area

$= -\int_{-1}^{1} \frac{x-1}{x^2-2x+4}\, dx + \int_1^5 \frac{x-1}{x^2-2x+4}\, dx$ 1M

$= -\left[\frac{1}{2}\ln|x^2-2x+4|\right]_{-1}^{1} + \left[\frac{1}{2}\ln|x^2-2x+4|\right]_1^5$ 1M

$= \frac{1}{2}\ln 19 + \frac{1}{2}\ln 7 - \ln 3$ 1A

3. $3 = -x(x+4)$

$0 = -x^2 - 4x - 3$

$x = -1$ or -3 1A

Required area $= \int_{-3}^{-1} [-x(x+4) - 3]\, dx$ 1M

$= \int_{-3}^{-1} (-x^2 - 4x - 3)\, dx$

$= \left[-\frac{x^3}{3} - 2x^2 - 3x\right]_{-3}^{-1}$ 1M

$= \frac{4}{3}$ 1A

4. (a) $\int x\sin x\,dx = -x\cos x + \int \cos x\,dx$ 1M

$= -x\cos x + \sin x + \text{constant}$ 1A

(b) $x\sin x = x$

$\sin x = 1$ or $x = 0$

$x = \frac{\pi}{2}$ or 0 1A

Required area $= \int_0^{\frac{\pi}{2}} (x - x\sin x)\,dx$ 1M

$= \left[\frac{x^2}{2} + x\cos x - \sin x\right]_0^{\frac{\pi}{2}}$ 1M

$= \frac{\pi^2}{8} - 1$ 1A

5. (a) $g(x) = (-x)^2 - 4(-x) - 5$ 1M

$= x^2 + 4x - 5$

$x^2 + 4x - 5 = 0$

$x = -5$ or 1

The x-intercepts of the graph of $y = g(x)$ are -5 and 1. 1A

(b) Required area

$= \int_0^1 [(x^2 + 4x - 5) - (x^2 - 4x - 5)]\,dx - \int_1^5 (x^2 + 4x - 5)\,dx$ 1M+1M

$= \int_0^1 8x\,dx - \int_1^5 (x^2 + 4x - 5)\,dx$

$= \left[4x^2\right]_0^1 - \left[\frac{x^3}{3} + 2x^2 - 5x\right]_1^5$ 1M

$= \frac{220}{3}$ 1A

6. $g(x) = f\left(x - \frac{\pi}{6}\right)$ 1M

$= \cos\left(2x - \frac{\pi}{3}\right)$

$0 = \cos 2x - \cos\left(2x - \frac{\pi}{3}\right)$

$0 = -2\sin\left(2x - \frac{\pi}{6}\right)\sin\frac{\pi}{6}$ 1M

$2x - \frac{\pi}{6} = 0 \quad \text{or} \quad \pi$

$x = \frac{\pi}{12} \quad \text{or} \quad \frac{7\pi}{12}$ 1A

Required area

$= \int_0^{\frac{\pi}{12}} \left[\cos 2x - \cos\left(2x - \frac{\pi}{3}\right)\right] \mathrm{d}x + \int_{\frac{\pi}{12}}^{\frac{7\pi}{12}} \left[\cos\left(2x - \frac{\pi}{3}\right) - \cos 2x\right] \mathrm{d}x$

$+ \int_{\frac{7\pi}{12}}^{\pi} \left[\cos 2x - \cos\left(2x - \frac{\pi}{3}\right)\right] \mathrm{d}x$ 1M+1M

$= \left[\frac{1}{2}\sin 2x - \frac{1}{2}\sin\left(2x - \frac{\pi}{3}\right)\right]_0^{\frac{\pi}{12}} + \left[\frac{1}{2}\sin\left(2x - \frac{\pi}{3}\right) - \frac{1}{2}\sin 2x\right]_{\frac{\pi}{12}}^{\frac{7\pi}{12}} + \left[\frac{1}{2}\sin 2x - \frac{1}{2}\sin\left(2x - \frac{\pi}{3}\right)\right]_{\frac{7\pi}{12}}^{\pi}$ 1M

$= 2$ 1A

7. (a) $1 - \cos x = 2\sin^2 x$

$1 - \cos x - 2(1 - \cos^2 x) = 0$ 1M

$2\cos^2 x - \cos x - 1 = 0$

$\cos x = -\frac{1}{2} \quad \text{or} \quad 1$

$x = \frac{2\pi}{3} \quad \text{or} \quad \frac{4\pi}{3} \quad \text{or} \quad 0 \quad \text{or} \quad 2\pi$

Thus, we have $\alpha = \frac{2\pi}{3}$ and $\beta = \frac{4\pi}{3}$. 1A+1A

(b) Required area

$= \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} (1 - \cos x - 2\sin^2 x)\, \mathrm{d}x$ 1M

$= \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} [1 - \cos x - (1 - \cos 2x)]\, \mathrm{d}x$ 1M

$= \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} (\cos 2x - \cos x)\, \mathrm{d}x$

$= \left[\frac{\sin 2x}{2} - \sin x\right]_{\frac{2\pi}{3}}^{\frac{4\pi}{3}}$ 1M

$= \frac{3\sqrt{3}}{2}$ 1A

8. (a) $\sin\left(2x - \frac{\pi}{3}\right) - \sin 2x = 0$

$2\cos\left(2x - \frac{\pi}{6}\right)\sin\left(-\frac{\pi}{6}\right) = 0$ 1M

$2x - \frac{\pi}{6} = \frac{\pi}{2}$ or $\frac{3\pi}{2}$

$x = \frac{\pi}{3}$ or $\frac{5\pi}{6}$ 1A

When $x = \frac{\pi}{3}$, $y = \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$; when $x = \frac{5\pi}{6}$, $y = \sin\frac{5\pi}{3} = -\frac{\sqrt{3}}{2}$.

The coordinates of P and Q are $\left(\frac{\pi}{3}, \frac{\sqrt{3}}{2}\right)$ and $\left(\frac{5\pi}{6}, -\frac{\sqrt{3}}{2}\right)$ respectively. 1A+1A

(b) Area $= \int_{\frac{\pi}{3}}^{\frac{5\pi}{6}} \left[\sin\left(2x - \frac{\pi}{3}\right) - \sin 2x\right] \mathrm{d}x$ 1M

$= \frac{1}{2}\left[-\cos\left(2x - \frac{\pi}{3}\right) + \cos 2x\right]_{\frac{\pi}{3}}^{\frac{5\pi}{6}}$ 1M

$= 1$ 1A

9. $x^2 + (3x + 18) = 36$

$x^2 + 3x - 18 = 0$

$x = -6$ or 3 1A

Required area

$= \int_{-6}^{3} (\sqrt{36 - x^2} - \sqrt{3x + 18})\,\mathrm{d}x + \int_{-6}^{3} [-\sqrt{3x + 18} - (-\sqrt{36 - x^2})]\,\mathrm{d}x$ 1M

$= 2\int_{-6}^{3} (\sqrt{36 - x^2} - \sqrt{3x + 18})\,\mathrm{d}x$

Let $x = 6\sin\theta$. Then $\mathrm{d}x = 6\cos\theta\,\mathrm{d}\theta$. 1M

Required area

$= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{6}} \sqrt{36 - 36\sin^2\theta} \cdot 6\cos\theta\,\mathrm{d}\theta - \frac{4}{9}\left[(3x + 18)^{\frac{3}{2}}\right]_{-6}^{3}$

$= 72\int_{-\frac{\pi}{2}}^{\frac{\pi}{6}} \cos^2\theta\,\mathrm{d}\theta - 36\sqrt{3}$

$= 36\int_{-\frac{\pi}{2}}^{\frac{\pi}{6}} (1 + \cos 2\theta)\,\mathrm{d}\theta - 36\sqrt{3}$ 1M

$= 36\left[\theta + \frac{\sin 2\theta}{2}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{6}} - 36\sqrt{3}$ 1M

$= 24\pi - 27\sqrt{3}$ 1A

10. (a) $x = (x^2)^2$ 1M

$0 = x(x^3 - 1)$

$x = 0 \quad \text{or} \quad 1$

The coordinates of P are $(1,\ 1)$. 1A

(b) Required area $= \int_0^1 (\sqrt{x} - x^2)\,\mathrm{d}x$ 1M

$= \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{3}x^3\right]_0^1$ 1M

$= \frac{1}{3}$ 1A

11. (a) $y\text{-intercept} = \dfrac{12}{-3} - \dfrac{12}{3} - 1 = -9$ 1A

$$0 = \frac{12}{x-3} - \frac{12}{x+3} - 1$$

$$(x-3)(x+3) = 12[(x+3)-(x-3)]$$

$$x^2 - 81 = 0$$

$$x = \pm 9$$

x-intercepts are 9 and -9. 1A

(b) (i) $f'(x) = -\dfrac{12}{(x-3)^2} + \dfrac{12}{(x+3)^2}$ 1A

$f''(x) = \dfrac{24}{(x-3)^3} - \dfrac{24}{(x+3)^3}$ 1A

(ii) When $f'(x) = 0$,

$\dfrac{12}{(x-3)^2} = \dfrac{12}{(x+3)^2}$ 1M

$$x - 3 = -(x+3) \quad \text{or} \quad x - 3 = x + 3 \text{ (rejected)}$$

$$x = 0$$

x	$x < -3$	$-3 < x < 0$	$0 < x < 3$	$x > 3$
$f'(x)$	+	+	−	−

1M

The maximum point is $(0,\ -9)$ and there is no minimum point. 1A

When $f''(x) = 0$,

$$\frac{24}{(x-3)^3} = \frac{24}{(x+3)^3}$$

$$x + 3 = x - 3$$

There is no solution.

Thus, there is no point of inflexion. 1A

(c) $x = -3$ and $x = 3$ are vertical asymptote. 1A

Since $y = -1 + \dfrac{12}{x-3} - \dfrac{12}{x+3}$, $y = -1$ is a horizontal asymptote. 1A

(d) $A = \displaystyle\int_9^k \left(\frac{12}{x-3} - \frac{12}{x+3} - 1 - (-1) \right) \mathrm{d}x$ 1M

$= \Big[12 \ln|x-3| - 12 \ln|x+3| \Big]_9^k$ 1M

$$= 12 \ln \frac{k-3}{6} - 12 \ln \frac{k+3}{12}$$

$$= 12 \ln \frac{k-3}{k+3} + 12 \ln 2$$

Since $k > 9$, $0 < \dfrac{k-3}{k+3} < 1$ and $\ln \dfrac{k-3}{k+3} < 0$.

We have $A < 12 \ln 2$. 1

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