

1. (a) $\frac{1}{4}(\cos x + \cos 3x + \cos 7x + \cos 9x) = \frac{1}{4}(\cos x + \cos 9x) + \frac{1}{4}(\cos 3x + \cos 7x)$
 $= \frac{1}{2} \cos 5x \cos 4x + \frac{1}{2} \cos 5x \cos 2x$ 1M
 $= \frac{\cos 5x}{2}(\cos 4x + \cos 2x)$
 $= \cos 5x(\cos 3x \cos x)$ 1M
 $= \cos x \cos 3x \cos 5x$ 1
- (b) $\int_0^{\frac{\pi}{4}} 8 \cos x \cos 3x \cos 5x \, dx = 2 \int_0^{\frac{\pi}{4}} (\cos x + \cos 3x + \cos 7x + \cos 9x) \, dx$ 1M
 $= 2 \left[\sin x + \frac{\sin 3x}{3} + \frac{\sin 7x}{7} + \frac{\sin 9x}{9} \right]_0^{\frac{\pi}{4}}$ 1A
 $= 2 \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{6} - \frac{\sqrt{2}}{14} + \frac{\sqrt{2}}{18} \right) - 0$
 $= \frac{82\sqrt{2}}{63}$ 1A
2. (a) $\frac{1}{\sqrt{x+2} - \sqrt{x-1}} = \frac{\sqrt{x+2} + \sqrt{x-1}}{(x+2) - (x-1)}$
 $= \frac{\sqrt{x+2} + \sqrt{x-1}}{3}$ 1A
- (b) $\int_2^7 \frac{dx}{\sqrt{x+2} - \sqrt{x-1}} = \frac{1}{3} \int_2^7 (\sqrt{x+2} + \sqrt{x-1}) \, dx$ 1M
 $= \frac{1}{3} \left[\frac{2(x+2)^{\frac{3}{2}}}{3} + \frac{2(x-1)^{\frac{3}{2}}}{3} \right]_2^7$ 1A
 $= \frac{1}{3} \left[(18 + 4\sqrt{6}) - \left(\frac{16}{3} + \frac{2}{3} \right) \right]$
 $= 4 + \frac{4\sqrt{6}}{3}$ 1A
3. $\int \sin 3x \cos 8x \, dx = \frac{1}{2} \int (\sin 11x - \sin 5x) \, dx$ 1M
 $= -\frac{\cos 11x}{22} + \frac{\cos 5x}{10} + \text{常數}$ 1A

$$\begin{aligned}
4. \quad & \int_1^4 \frac{3x}{1+e^x} dx - 3 \int_4^1 \frac{xe^x}{1+e^x} dx \\
&= \int_1^4 \frac{3x}{1+e^x} + 3 \int_1^4 \frac{xe^x}{1+e^x} dx && 1M \\
&= \int_1^4 \frac{3x(1+e^x)}{1+e^x} dx \\
&= \int_1^4 3x dx \\
&= \left[\frac{3x^2}{2} \right]_1^4 \\
&= \frac{45}{2} && 1A
\end{aligned}$$

$$\begin{aligned}
5. \quad & \int_0^{\frac{\pi}{4}} \sin^4 x dx = \frac{1}{4} \int_0^{\frac{\pi}{4}} (1 - \cos 2x)^2 dx && 1M \\
&= \frac{1}{4} \int_0^{\frac{\pi}{4}} (1 - 2 \cos 2x + \cos^2 2x) dx \\
&= \frac{1}{4} \int_0^{\frac{\pi}{4}} \left[1 - 2 \cos 2x + \frac{1}{2}(1 + \cos 4x) \right] dx && 1M \\
&= \frac{1}{4} \left[\frac{3x}{2} - \sin 2x + \frac{\sin 4x}{8} \right]_0^{\frac{\pi}{4}} \\
&= \frac{1}{4} \left(\frac{3\pi}{8} - 1 + 0 \right) - \frac{1}{4}(0) \\
&= \frac{3\pi}{32} - \frac{1}{4} && 1A
\end{aligned}$$

$$\begin{aligned}
6. \quad (a) \quad & \frac{\cos x + \cos 9x}{\cos 5x} = \frac{2 \cos 5x \cos 4x}{\cos 5x} && 1M \\
&= 2 \cos 4x && 1
\end{aligned}$$

$$\begin{aligned}
(b) \quad & \int_0^{\frac{\pi}{16}} \frac{(\cos x + \cos 9x) \sin 4x}{\cos 5x} dx = \int_0^{\frac{\pi}{16}} 2 \cos 4x \sin 4x dx && 1M \\
&= \int_0^{\frac{\pi}{16}} \sin 8x dx && 1M \\
&= -\frac{1}{8} \left[\cos 8x \right]_0^{\frac{\pi}{16}} && 1A \\
&= -\frac{1}{8}(0 - 1) \\
&= \frac{1}{8} && 1A
\end{aligned}$$

$$7. \quad (a) \quad \frac{d}{dx} [\ln(x + \sqrt{x^2 + 9})] = \frac{1 + \frac{1}{2}(x^2 + 9)^{-\frac{1}{2}}(2x)}{x + \sqrt{x^2 + 9}} \quad 1M$$

$$= \frac{\sqrt{x^2 + 9} + x}{(x + \sqrt{x^2 + 9})\sqrt{x^2 + 9}} \\ = \frac{1}{\sqrt{x^2 + 9}} \quad 1A$$

$$(b) \quad \int_0^1 \frac{dx}{\sqrt{x^2 + 9}} = \left[\ln(x + \sqrt{x^2 + 9}) \right]_0^1 \quad 1M$$

$$= \ln(1 + \sqrt{10}) - \ln 3 \quad 1A$$

$$8. \quad \int_{-\pi}^{\frac{\pi}{2}} \frac{e^{2x} - \pi e^x - 2\pi^2}{e^x + \pi} dx = \int_{-\pi}^{\frac{\pi}{2}} \frac{e^x(e^x + \pi) - 2\pi(e^x + \pi)}{e^x + \pi} dx \quad 1M$$

$$= \int_{-\pi}^{\frac{\pi}{2}} (e^x - 2\pi) dx$$

$$= \left[e^x - 2\pi x \right]_{-\pi}^{\frac{\pi}{2}}$$

$$= e^{\frac{\pi}{2}} - e^{-\pi} - 3\pi^2 \quad 1A$$

$$9. \quad \int_0^2 \frac{1}{\sqrt{2x+1} - \sqrt{2x}} dx = \int_0^2 \left(\frac{1}{\sqrt{2x+1} - \sqrt{2x}} \times \frac{\sqrt{2x+1} + \sqrt{2x}}{\sqrt{2x+1} + \sqrt{2x}} \right) dx \quad 1M$$

$$= \int_0^2 (\sqrt{2x+1} + \sqrt{2x}) dx$$

$$= \left[\frac{(2x+1)^{\frac{3}{2}}}{3} + \frac{(2x)^{\frac{3}{2}}}{3} \right]_0^2 \quad 1A$$

$$= \frac{5\sqrt{5} + 7}{3} \quad 1A$$

$$10. \quad \int_{-2}^2 (3x+2)(7x+5) dx = \int_{-2}^2 (21x^2 + 29x + 10) dx \quad 1M$$

$$= \left[7x^3 + \frac{29x^2}{2} + 10x \right]_{-2}^2$$

$$= 152 \quad 1A$$

$$11. \quad \int_0^{\frac{\pi}{4}} \tan^2 x dx - \int_{\frac{\pi}{4}}^0 dx = \int_0^{\frac{\pi}{4}} \tan^2 x dx + \int_0^{\frac{\pi}{4}} dx \quad 1A$$

$$= \int_0^{\frac{\pi}{4}} \sec^2 x dx$$

$$= \left[\tan x \right]_0^{\frac{\pi}{4}}$$

$$= 1 \quad 1A$$

$$12. \quad (a) \quad x + h + \frac{k}{x+1} = \frac{(x+h)(x+1) + k}{x+1} \\ = \frac{x^2 + (h+1)x + (h+k)}{x+1}$$

故此， $h+1=0$ 及 $h+k=1$ 。 1M

求解後，可得 $h=-1$ 及 $k=2$ 。 1A

$$(b) \quad \int_0^1 \frac{x^2+1}{x+1} dx = \int_0^1 \left(x-1 + \frac{2}{x+1} \right) dx \quad 1M$$

$$= \left[\frac{x^2}{2} - x + 2 \ln|x+1| \right]_0^1 \quad 1A$$

$$= \left(\frac{1}{2} - 1 + 2 \ln 2 \right) - (0 - 0 + 2 \ln 1) \\ = 2 \ln 2 - \frac{1}{2} \quad 1A$$

$$13. \quad (a) \quad \int_0^1 (x+1)^4 dx = \left[\frac{(x+1)^5}{5} \right]_0^1 \quad 1A$$

$$= \frac{2^5}{5} - \frac{1}{5} \\ = \frac{31}{5} \quad 1A$$

$$\int_0^1 (x+1)^5 dx = \left[\frac{(x+1)^6}{6} \right]_0^1 \\ = \frac{2^6}{6} - \frac{1}{6} \\ = \frac{21}{2} \quad 1A$$

$$(b) \quad \int_0^1 30x(x+1)^4 dx = 30 \int_0^1 [(x+1)(x+1)^4 - (x+1)^4] dx \quad 1M$$

$$= 30 \int_0^1 (x+1)^5 dx - 30 \int_0^1 (x+1)^4 dx \\ = 30 \times \frac{21}{2} - 30 \times \frac{31}{5} \\ = 129 \quad 1A$$

$$14. \quad \int_0^{n\pi} (\cos \theta + 1) d\theta = \left[\sin \theta + \theta \right]_0^{n\pi} \quad 1M$$

$$= \sin n\pi + n\pi \\ = n\pi \quad 1$$

$$15. \quad (a) \quad \int_0^4 \sqrt{2x+1} dx = \left[\frac{(2x+1)^{\frac{3}{2}}}{3} \right]_0^4 \quad 1A$$

$$= 9 - \frac{1}{3} \\ = \frac{26}{3} \quad 1A$$

$$\begin{aligned}\int_0^4 \frac{dx}{\sqrt{2x+1}} &= \left[\sqrt{2x+1} \right]_0^4 \\ &= 3 - 1 \\ &= 2\end{aligned}\quad 1A$$

$$\begin{aligned}\text{(b)} \quad \int_0^4 \frac{6x \, dx}{\sqrt{2x+1}} &= 3 \int_0^4 \frac{2x+1-1}{\sqrt{2x+1}} \, dx \\ &= 3 \int_0^4 \sqrt{2x+1} \, dx - 3 \int_0^4 \frac{dx}{\sqrt{2x+1}} \\ &= 3 \times \frac{26}{3} - 3 \times 2 \\ &= 20\end{aligned}\quad \begin{array}{l} 1M \\ 1A \end{array}$$

$$\begin{aligned}16. \quad & \int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1}{2(1+\sin x)} \, dx + \int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos 2x}{2(1+\sin x)} \, dx \\ &= \int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1 + (1 - 2\sin^2 x)}{2(1+\sin x)} \, dx \\ &= \int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{(1+\sin x)(1-\sin x)}{1+\sin x} \, dx \\ &= \int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} (1 - \sin x) \, dx \\ &= \left[x + \cos x \right]_{-\frac{\pi}{6}}^{\frac{\pi}{2}} \\ &= \frac{2\pi}{3} - \frac{\sqrt{3}}{2}\end{aligned}\quad \begin{array}{l} 1M \\ 1A \\ 1A \end{array}$$

$$17. \quad \text{(a)} \quad \frac{d}{dx} [\ln(1+e^x)] = \frac{e^x}{1+e^x} \quad 1A$$

$$\begin{aligned}\text{(b)} \quad \text{(i)} \quad \int_0^1 \frac{e^x}{1+e^x} \, dx &= \left[\ln(1+e^x) \right]_0^1 \\ &= \ln(1+e) - \ln 2\end{aligned}\quad \begin{array}{l} 1M \\ 1A \end{array}$$

$$\begin{aligned}\text{(ii)} \quad \int_0^1 \frac{dx}{1+e^{-x}} &= \int_0^1 \frac{e^x}{e^x+1} \, dx \\ &= \ln(1+e) - \ln 2\end{aligned}\quad \begin{array}{l} 1M \\ 1A \end{array}$$

$$\begin{aligned}\text{(iii)} \quad \int_0^1 \frac{dx}{1+e^x} &= \int_0^1 \left(1 - \frac{e^x}{1+e^x} \right) \, dx \\ &= \left[x - \ln(1+e^x) \right]_0^1 \\ &= [1 - \ln(1+e)] + \ln 2 \\ &= 1 - \ln(1+e) + \ln 2\end{aligned}\quad \begin{array}{l} 1M \\ 1A \end{array}$$

$$\begin{aligned}
 18. \quad \int_0^{2n\pi} \sin \theta \, d\theta &= \left[-\cos \theta \right]_0^{2n\pi} && 1M \\
 &= -\cos(2n\pi) + 1 \\
 &= 0 && 1A
 \end{aligned}$$

$$\begin{aligned}
 19. \quad (a) \quad \sin(2x + x) &= \sin 2x \cos x + \cos 2x \sin x && 1M \\
 &= 2 \sin x \cos^2 x + (1 - 2 \sin^2 x) \sin x && 1M \\
 &= 2 \sin x (1 - \sin^2 x) + \sin x - 2 \sin^3 x \\
 &= 3 \sin x - 4 \sin^3 x && 1
 \end{aligned}$$

$$\begin{aligned}
 (b) \quad \int_0^{\frac{\pi}{6}} 8 \sin^3 x \, dx &= \int_0^{\frac{\pi}{6}} (6 \sin x - 2 \sin 3x) \, dx && 1M \\
 &= \left[-6 \cos x + \frac{2 \cos 3x}{3} \right]_0^{\frac{\pi}{6}} && 1A \\
 &= \left(-3\sqrt{3} + 0 \right) - \left(-6 + \frac{2}{3} \right) \\
 &= \frac{16}{3} - 3\sqrt{3} && 1A
 \end{aligned}$$

$$\begin{aligned}
 20. \quad \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos 2x}{\cos x + \sin x} \, dx &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos^2 x - \sin^2 x}{\cos x + \sin x} \, dx \\
 &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\cos x - \sin x) \, dx && 1M \\
 &= \left[\sin x + \cos x \right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} && 1A \\
 &= (1 + 0) - \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \right) \\
 &= 1 - \sqrt{2} && 1A
 \end{aligned}$$

$$\begin{aligned}
 21. \quad \int_1^2 \frac{1+2x}{2x-1} \, dx &= \int_1^2 \left(\frac{2x-1}{2x-1} + \frac{2}{2x-1} \right) \, dx && 1M \\
 &= \left[x + \ln |2x-1| \right]_1^2 && 1A \\
 &= (2 + \ln 3) - (1 + \ln 1) \\
 &= 1 + \ln 3 && 1A
 \end{aligned}$$

$$\begin{aligned}
22. \quad \int_h^k (x-h)(x-k) \, dx &= \int_h^k (x^2 - (h+k)x + hk) \, dx && 1M \\
&= \left[\frac{x^3}{3} - \frac{(h+k)x^2}{2} + hkx \right]_h^k \\
&= \frac{2(k^3 - h^3) - 3(h+k)(k^2 - h^2) + 6hk(k-h)}{6} \\
&= \frac{h^3 - 3h^2k + 3hk^2 - k^3}{6} \\
&= \frac{(h-k)^3}{6} && 1
\end{aligned}$$

$$\begin{aligned}
23. \quad \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{1 + \cos 2x} \, dx &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{1 + (2 \cos^2 x - 1)} \, dx && 1M \\
&= \frac{1}{2} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec^2 x \, dx && 1A \\
&= \frac{1}{2} \left[\tan x \right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} \\
&= \frac{\sqrt{3}}{3} && 1A
\end{aligned}$$

$$\begin{aligned}
24. \quad \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{dx}{\tan \frac{x}{2} + \cot \frac{x}{2}} &= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{dx}{\tan \frac{x}{2} + \cot \frac{x}{2}} \times \frac{\sin \frac{x}{2} \cos \frac{x}{2}}{\sin \frac{x}{2} \cos \frac{x}{2}} \, dx \\
&= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\sin \frac{x}{2} \cos \frac{x}{2}}{\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2}} \, dx \\
&= \frac{1}{2} \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin x \, dx && 1A \\
&= -\frac{1}{2} \left[\cos x \right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} \\
&= \frac{1}{4} && 1A
\end{aligned}$$

$$\begin{aligned}
25. \quad \int_{-\pi}^{-\frac{\pi}{3}} (\sin 2x \cos x - \cos 2x \sin x) \, dx &= \int_{-\pi}^{-\frac{\pi}{3}} \sin(2x - x) \, dx && 1M \\
&= \left[-\cos x \right]_{-\pi}^{-\frac{\pi}{3}} \\
&= -\cos\left(-\frac{\pi}{3}\right) + \cos(-\pi) \\
&= -\frac{3}{2} && 1A
\end{aligned}$$

$$\begin{aligned}
26. \quad \int_0^4 f(x) \, dx + \int_5^2 f(x) \, dx &= \int_0^2 f(x) \, dx + \int_2^4 f(x) \, dx - \int_2^5 f(x) \, dx & 1M \\
&= \int_0^2 f(x) \, dx - \left(\int_4^2 f(x) \, dx + \int_2^5 f(x) \, dx \right) \\
&= \int_0^2 f(x) \, dx - \int_4^5 f(x) \, dx & 1M \\
&= k & 1A
\end{aligned}$$

$$\begin{aligned}
27. \quad \int_1^3 (x-2) \left(4 + \frac{3}{x^2} \right) dx &= \int_1^3 (4x - 8 + 3x^{-1} - 6x^{-2}) \, dx & 1M \\
&= \left[2x^2 - 8x + 3 \ln |x| + 6x^{-1} \right]_1^3 \\
&= 3 \ln 3 - 4 & 1A
\end{aligned}$$

$$\begin{aligned}
28. \quad \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (e^x + e^{-x})^2 \cos x \, dx &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (e^{2x} + e^{-2x} + 2) \cos x \, dx \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (e^{2x} + e^{-2x}) \cos x \, dx + 2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x \, dx & 1M \\
&= k + 2 \left[\sin x \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} & 1A \\
&= k + 2[1 - (-1)] \\
&= k + 4 & 1A
\end{aligned}$$

$$\begin{aligned}
29. \quad (a) \quad \frac{A}{x+1} + \frac{B}{x+3} &= \frac{A(x+3) + B(x+1)}{(x+1)(x+3)} \\
&= \frac{(A+B)x + (3A+B)}{x^2 + 4x + 3} \\
\text{故此, } A+B &= 0 \text{ 及 } 3A+B = 1. \\
\text{求解後, 可得 } A &= \frac{1}{2} \text{ 及 } B = -\frac{1}{2}. & 1M \\
& & 1A
\end{aligned}$$

$$\begin{aligned}
(b) \quad \int_1^2 \frac{dx}{x^2 + 4x + 3} &= \frac{1}{2} \int_1^2 \left(\frac{1}{x+1} - \frac{1}{x+3} \right) dx & 1M \\
&= \frac{1}{2} \left[\ln |x+1| - \ln |x+3| \right]_1^2 & 1A \\
&= \frac{1}{2} [(\ln 3 - \ln 5) - (\ln 2 - \ln 4)] \\
&= \frac{1}{2} \ln \frac{6}{5} & 1A
\end{aligned}$$

30. (a) 設 $u = 1 - e^{-x}$ 。則 $du = e^{-x} dx$ 。 1M

$$\int_{\ln 2}^{\ln 3} \frac{e^{-x}}{1 - e^{-x}} dx = \int_{\frac{1}{2}}^{\frac{2}{3}} \frac{du}{u}$$
 1M

$$= \left[\ln |u| \right]_{\frac{1}{2}}^{\frac{2}{3}}$$
 1A

$$= \ln \frac{2}{3} - \ln \frac{1}{2}$$

$$= \ln \frac{4}{3}$$
 1A

$$\begin{aligned} \text{(b)} \quad \frac{A}{u} + \frac{B}{u-1} &= \frac{A(u-1) + Bu}{u(u-1)} \\ &= \frac{(A+B)u - A}{u(u-1)} \end{aligned}$$

故此， $A + B = 0$ 及 $-A = 1$ 。 1M

求解後，可得 $A = -1$ 及 $B = 1$ 。 1A

$$\text{(c)} \quad \int_{\ln 2}^{\ln 3} \frac{1}{e^x(e^x - 1)} dx = \int_{\ln 2}^{\ln 3} \left(\frac{1}{e^x - 1} - \frac{1}{e^x} \right) dx$$
 1M

$$= \int_{\ln 2}^{\ln 3} \left(\frac{e^{-x}}{1 - e^{-x}} - e^{-x} \right) dx$$

$$= \ln \frac{4}{3} + \left[e^{-x} \right]_{\ln 2}^{\ln 3}$$
 1M+1A

$$= \ln \frac{4}{3} + \left(\frac{1}{3} - \frac{1}{2} \right)$$

$$= \ln \frac{4}{3} - \frac{1}{6}$$
 1A

31. 設 $x = \sqrt{3} \tan \theta$ 。則 $dx = \sqrt{3} \sec^2 \theta d\theta$ 。 1M

$$\int_{-3}^3 \frac{1}{(x^2 + 3)^{\frac{3}{2}}} dx = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{\sqrt{3} \sec^2 \theta}{(3 \tan^2 \theta + 3)^{\frac{3}{2}}} d\theta$$
 1A

$$= \frac{1}{3} \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{\sec^2 \theta}{\sec^3 \theta} d\theta$$

$$= \frac{1}{3} \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \cos \theta d\theta$$

$$= \frac{1}{3} \left[\sin \theta \right]_{-\frac{\pi}{3}}^{\frac{\pi}{3}}$$

$$= \frac{\sqrt{3}}{3}$$
 1A

$$\begin{aligned}
32. \quad (a) \quad & \frac{A(x-1)}{x^2-2x+2} + \frac{B(x+1)}{x^2+2x+2} \\
&= \frac{A(x-1)(x^2+2x+2) + B(x+1)(x^2-2x+2)}{(x^2-2x+2)(x^2+2x+2)} \\
&= \frac{(A+B)x^2 + (A-B)x + (-2A+2B)}{x^4+4}
\end{aligned}$$

可得 $A+B=0$ 、 $A-B=1$ 及 $-2A+2B=-2$ 。 1M

求解後，可得 $A=\frac{1}{2}$ 及 $B=-\frac{1}{2}$ 。 1A

(b) 設 $u = x^2 - 2x + 2$ 。則 $du = (2x - 2) dx$ 。 1M

設 $w = x^2 + 2x + 2$ 。則 $dw = (2x + 2) dx$ 。

$$\int_0^1 \frac{x^2-2}{x^4+4} dx = \int_0^1 \left(\frac{x-1}{2(x^2-2x+2)} - \frac{x+1}{2(x^2+2x+2)} \right) dx \quad 1M$$

$$\begin{aligned}
&= \frac{1}{4} \int_2^1 \frac{du}{u} - \frac{1}{4} \int_2^5 \frac{dw}{w} \\
&= \frac{1}{4} \left[\ln |u| \right]_2^1 - \frac{1}{4} \left[\ln |w| \right]_2^5 \quad 1M
\end{aligned}$$

$$= -\frac{1}{4} \ln 5 \quad 1A$$

$$33. \quad \text{設 } u = \sqrt{x} + 1 \text{。則 } du = \frac{1}{2\sqrt{x}} dx \text{。} \quad 1M$$

$$\int_1^9 \frac{dx}{x + \sqrt{x}} = 2 \int_2^4 \frac{1}{u} du \quad 1M$$

$$= 2 \left[\ln |u| \right]_2^4 \quad 1A$$

$$\begin{aligned}
&= 2(\ln 4 - \ln 2) \\
&= 2 \ln 2 \quad 1A
\end{aligned}$$

$$\begin{aligned}
34. \quad (a) \quad & \frac{A}{x} + \frac{B}{x-1} = \frac{A(x-1) + Bx}{x(x-1)} \\
&= \frac{(A+B)x - A}{x(x-1)}
\end{aligned}$$

可得 $-A=1$ 及 $A+B=0$ 。 1M

求解後，可得 $A=-1$ 及 $B=1$ 。 1A

(b) 設 $u = 1 + e^x$ 。則 $du = e^x dx$ 。 1M

$$\int_0^1 \frac{dx}{1+e^x} = \int_2^{1+e} \frac{du}{u(u-1)} \quad 1M$$

$$\begin{aligned}
&= \int_2^{1+e} \left(-\frac{1}{u} + \frac{1}{u-1} \right) du \\
&= \left[-\ln |u| + \ln |u-1| \right]_2^{1+e} \quad 1M
\end{aligned}$$

$$= 1 + \ln 2 - \ln(1+e) \quad 1A$$

$$35. (a) \int \frac{1}{e^{4u}} du = \int e^{-4u} du \quad 1M$$

$$= -\frac{1}{4e^{4u}} + \text{常數} \quad 1A$$

$$(b) \text{ 設 } u = \sqrt{x} \circ \text{ 則 } du = \frac{1}{2\sqrt{x}} dx \circ \quad 1M$$

$$\int_1^{16} \frac{1}{\sqrt{x}e^{4\sqrt{x}}} dx = \int_1^4 \frac{2}{e^{4u}} du \quad 1A$$

$$= 2 \left[-\frac{1}{4e^{4u}} \right]_1^4 \quad 1M$$

$$= \frac{1}{2e^4} - \frac{1}{2e^{16}} \quad 1A$$

$$36. (a) \sin^4 x + \cos^4 x = (\sin^2 x + \cos^2 x)^2 - 2 \sin^2 x \cos^2 x \quad 1M$$

$$= 1 - \frac{1}{2} (2 \sin x \cos x)^2$$

$$= 1 - \frac{1}{2} \sin^2 2x \quad 1A$$

$$(b) \quad t = \tan 2x$$

$$1 = 2 \sec^2 2x \frac{dx}{dt} \quad 1M$$

$$\frac{dx}{dt} = \frac{1}{2(1 + \tan^2 2x)}$$

$$= \frac{1}{2(1 + t^2)} \quad 1$$

$$\int_0^{\frac{\pi}{8}} \frac{1}{\sin^4 x + \cos^4 x} dx = \int_0^{\frac{\pi}{8}} \frac{1}{1 - \frac{1}{2} \sin^2 2x} dx \quad 1M$$

$$= \int_0^1 \frac{1}{1 - \frac{1}{2} \left(\frac{t}{\sqrt{1+t^2}} \right)^2} \cdot \frac{1}{2(1+t^2)} dt$$

$$= \int_0^1 \frac{1}{2(1+t^2) - t^2} dt$$

$$= \int_0^1 \frac{1}{2+t^2} dt$$

$$\text{設 } t = \sqrt{2} \tan \theta \circ \text{ 則 } dt = \sqrt{2} \sec^2 \theta d\theta \circ \quad 1M$$

$$\int_0^{\frac{\pi}{8}} \frac{1}{\sin^4 x + \cos^4 x} dx = \int_0^{\tan^{-1} \frac{\sqrt{2}}{2}} \frac{\sqrt{2} \sec^2 \theta}{2 + 2 \tan^2 \theta} d\theta$$

$$= \frac{\sqrt{2}}{2} \int_0^{\tan^{-1} \frac{\sqrt{2}}{2}} d\theta$$

$$= \frac{\sqrt{2}}{2} \left[\theta \right]_0^{\tan^{-1} \frac{\sqrt{2}}{2}}$$

$$= \frac{\sqrt{2}}{2} \tan^{-1} \frac{\sqrt{2}}{2} \quad 1A$$

37. 設 $u = \sqrt{x}$ 。則 $du = \frac{1}{2\sqrt{x}} dx$ 。 1M

$$\int_1^4 \frac{f(\sqrt{x})}{\sqrt{x}} dx = 2 \int_1^2 f(u) du \quad 1M$$

$$= 2(9)$$

$$= 18$$

1A

38. 設 $u = x^4$ 。則 $du = 4x^3 dx$ 。 1M

$$\int_{-1}^0 x^3 f(x^4) dx = \frac{1}{4} \int_1^0 f(u) du \quad 1M$$

$$= \frac{1}{4}(-4)$$

$$= -1$$

1A

39. (a) 設 $u = 1 + x^3$ 。則 $du = 3x^2 dx$ 。 1M

$$\int_{\frac{1}{2}}^2 \frac{x^2}{1+x^3} dx = \frac{1}{3} \int_{\frac{9}{8}}^9 \frac{du}{u} \quad 1M$$

$$= \frac{1}{3} \left[\ln |u| \right]_{\frac{9}{8}}^9 \quad 1A$$

$$= \frac{1}{3} \left(\ln 9 - \ln \frac{9}{8} \right)$$

$$= \ln 2$$

1A

(b) $\frac{x^2}{1+x^3} - \frac{1}{1+x} = \frac{x^2 - (x^2 - x + 1)}{1+x^3}$ 1M

$$= \frac{x-1}{1+x^3} \quad 1A$$

(c) $\int_{\frac{1}{2}}^2 \frac{x-1}{1+x^3} dx = \int_{\frac{1}{2}}^2 \left(\frac{x^2}{1+x^3} - \frac{1}{1+x} \right) dx$ 1M

$$= \ln 2 - \left[\ln |1+x| \right]_{\frac{1}{2}}^2 \quad 1A$$

$$= \ln 2 - \left(\ln 3 - \ln \frac{3}{2} \right)$$

$$= 0$$

1A

$$40. (a) \frac{d}{dx} (x^{n-1} \sqrt{1+x^2})$$

$$= (n-1)x^{n-2} \sqrt{1+x^2} + x^{n-1} \cdot \frac{1}{2} \cdot \frac{2x}{\sqrt{1+x^2}} \quad 1M$$

$$= \frac{(n-1)x^{n-2}(1+x^2)}{\sqrt{1+x^2}} + \frac{x^n}{\sqrt{1+x^2}}$$

$$= \frac{nx^n}{\sqrt{1+x^2}} + \frac{(n-1)x^{n-2}}{\sqrt{1+x^2}}$$

等式兩邊對 x 作積分。

$$\left[x^{n-1} \sqrt{1+x^2} \right]_0^1 = n \int_0^1 \frac{x^n}{\sqrt{1+x^2}} dx + (n-1) \int_0^1 \frac{x^{n-2}}{\sqrt{1+x^2}} dx \quad 1M$$

$$\sqrt{2} = n \int_0^1 \frac{x^n}{\sqrt{1+x^2}} dx + (n-1) \int_0^1 \frac{x^{n-2}}{\sqrt{1+x^2}} dx$$

$$\int_0^1 \frac{x^n}{\sqrt{1+x^2}} dx = \frac{\sqrt{2}}{n} - \frac{n-1}{n} \int_0^1 \frac{x^{n-2}}{\sqrt{1+x^2}} dx \quad 1$$

$$(b) (i) \frac{d}{dx} \left[\ln(\sqrt{1+x^2} + x) \right]$$

$$= \frac{\frac{x}{\sqrt{1+x^2}} + 1}{\sqrt{1+x^2} + x}$$

$$= \frac{x + \sqrt{1+x^2}}{\sqrt{1+x^2}(\sqrt{1+x^2} + x)}$$

$$= \frac{1}{\sqrt{1+x^2}} \quad 1A$$

$$(ii) \int_0^1 \frac{x^2}{\sqrt{1+x^2}} dx$$

$$= \frac{\sqrt{2}}{2} - \frac{1}{2} \int_0^1 \frac{1}{\sqrt{1+x^2}} dx \quad 1M$$

$$= \frac{\sqrt{2}}{2} - \frac{1}{2} \left[\ln(\sqrt{1+x^2} + x) \right]_0^1 \quad 1M$$

$$= \frac{\sqrt{2} - \ln(\sqrt{2} + 1)}{2} \quad 1A$$

$$(c) \text{ 設 } u = \ln[(e-1)\sin x + 1] \text{。則 } du = \frac{(e-1)\cos x}{(e-1)\sin x + 1} dx \text{。} \quad 1M$$

$$\int_0^{\frac{\pi}{2}} \frac{\sin(\frac{\pi}{2} - x) \{\ln[(e-1)\sin x + 1]\}^2}{[(e-1)\sin x + 1] \sqrt{1 + \{\ln[(e-1)\sin x + 1]\}^2}} dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos x \{\ln[(e-1)\sin x + 1]\}^2}{[(e-1)\sin x + 1] \sqrt{1 + \{\ln[(e-1)\sin x + 1]\}^2}} dx$$

$$= \frac{1}{e-1} \int_0^1 \frac{u^2}{\sqrt{1+u^2}} du \quad 1M$$

$$= \frac{\sqrt{2} - \ln(\sqrt{2} + 1)}{2(e-1)} \quad 1A$$

$$(d) \text{ 設 } u = 1 - x \text{。則 } du = -dx \text{。} \quad 1M$$

$$\begin{aligned}
& \int_0^1 \sqrt{x^2 - 2x + 2} \, dx \\
&= \int_0^1 \sqrt{(x-1)^2 + 1} \, dx \\
&= - \int_1^0 \sqrt{(-u)^2 + 1} \, du \\
&= \int_0^1 \frac{1+u^2}{\sqrt{1+u^2}} \, du \\
&= \int_0^1 \frac{1}{1+u^2} \, du + \int_0^1 \frac{u^2}{\sqrt{1+u^2}} \, du \quad 1M \\
&= \ln(\sqrt{2}+1) + \frac{\sqrt{2} - \ln(\sqrt{2}+1)}{2} \\
&= \frac{\sqrt{2} + \ln(\sqrt{2}+1)}{2} \quad 1A
\end{aligned}$$

41. (a) $\frac{d}{d\theta} \ln(\sec \theta + \tan \theta)$

$$\begin{aligned}
&= \frac{1}{\sec \theta + \tan \theta} [-(\cos \theta)^{-2}(-\sin \theta) + \sec^2 \theta] \\
&= \frac{\sec \theta \tan \theta + \sec^2 \theta}{\sec \theta + \tan \theta} \\
&= \frac{\sec \theta(\tan \theta + \sec \theta)}{\sec \theta + \tan \theta} \\
&= \sec \theta \quad 1A
\end{aligned}$$

(b) (i) 設 $u = \tan \theta$ 。則 $du = \sec^2 \theta \, d\theta$ 。 1M

$$\begin{aligned}
\int \frac{1}{\sqrt{u^2+1}} \, du &= \int \frac{1}{\sqrt{\tan^2 \theta + 1}} \sec^2 \theta \, d\theta \\
&= \int \sec \theta \, d\theta \\
&= \ln |\sec \theta + \tan \theta| + \text{常數} \quad 1M \\
&= \ln |\sqrt{u^2+1} + u| + \text{常數} \quad 1A
\end{aligned}$$

(ii) 設 $u = x^2 + 2$ 。則 $du = 2x \, dx$ 。 1M

$$\begin{aligned}
& \int_0^1 \frac{2x}{\sqrt{x^4 + 4x^2 + 5}} dx \\
&= \int_0^1 \frac{2x}{\sqrt{(x^2 + 2)^2 + 1}} dx \\
&= \int_2^3 \frac{1}{\sqrt{u^2 + 1}} du \\
&= \left[\ln |\sqrt{u^2 + 1} + u| \right]_2^3 \\
&= \ln \frac{\sqrt{10} + 3}{\sqrt{5} + 2} \times \frac{\sqrt{5} - 2}{\sqrt{5} - 2} \\
&= \ln \frac{5\sqrt{2} - 2\sqrt{10} + 3\sqrt{5} - 6}{5 - 4} \\
&= \ln(5\sqrt{2} + 3\sqrt{5} - 2\sqrt{10} - 6)
\end{aligned}$$

1M

1A

42. (a) (i) $\sin 2x = 2 \sin x \cos x$

$$\begin{aligned}
&= \frac{2 \sin x \sec x}{\sec^2 x} \\
&= \frac{2 \tan x}{1 + \tan^2 x} \\
&= \frac{2t}{1 + t^2}
\end{aligned}$$

1

$$\begin{aligned}
\cos 2x &= \cos^2 x - \sin^2 x \\
&= \frac{1 - \tan^2 x}{\sec^2 x} \\
&= \frac{1 - t^2}{1 + t^2}
\end{aligned}$$

1

(ii) $3 + 2 \cos 2x + \sin 2x = 3 + 2 \left(\frac{1 - t^2}{1 + t^2} \right) + \frac{2t}{1 + t^2}$

$$\begin{aligned}
&= \frac{3(1 + t^2) + 2(1 - t^2) + 2t}{1 + t^2} \\
&= \frac{5 + 2t + t^2}{1 + t^2}
\end{aligned}$$

1

(b) 設 $t = \tan x$ 。則 $dt = \sec^2 x dx = (1 + t^2) dx$ 。 1M

$$\begin{aligned}
& \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{dx}{3 + 2 \cos 2x + \sin 2x} \\
&= \int_{-1}^1 \frac{1}{\left(\frac{5 + 2t + t^2}{1 + t^2} \right)} \cdot \frac{1}{1 + t^2} dt \\
&= \int_{-1}^1 \frac{dt}{(t + 1)^2 + 4}
\end{aligned}$$

1M

設 $t + 1 = 2 \tan \theta$ 。則 $dt = 2 \sec^2 \theta d\theta$ 。 1M

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{dx}{3 + 2 \cos 2x + \sin 2x} dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{2 \sec^2 \theta}{4 \tan^2 \theta + 4} d\theta$$

1M

$$= \frac{1}{2} \int_0^{\frac{\pi}{4}} d\theta$$

$$= \frac{1}{2} \left[\theta \right]_0^{\frac{\pi}{4}}$$

$$= \frac{\pi}{8}$$

1A

(c) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos^2 x + \sin x \cos x - \sin^2 x}{3 + 2 \cos 2x + \sin 2x} dx$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos 2x - \frac{1}{2} \sin 2x}{3 + 2 \cos 2x + \sin 2x} dx$$

$$= \frac{1}{2} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{2 \cos 2x + \sin 2x}{3 + 2 \cos 2x + \sin 2x} dx$$

$$= \frac{1}{2} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left[1 - \frac{3}{3 + 2 \cos 2x + \sin 2x} \right] dx$$

1M

$$= \frac{1}{2} \left[x \right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} - \frac{3}{2} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{3 + 2 \cos 2x + \sin 2x} dx$$

$$= \frac{\pi}{4} - \frac{3}{2} \cdot \frac{\pi}{8}$$

$$= \frac{\pi}{16}$$

1M

1A

43. 設 $u = e^{3x} + 3x + 1$ 。則 $du = (3e^{3x} + 3) dx$ 。

1M

$$\int_0^3 \frac{x}{e^{3x} + 3x + 1} dx = \frac{1}{3} \int_0^3 \left(1 - \frac{e^{3x} + 1}{e^{3x} + 3x + 1} \right) dx$$

1A

$$= \frac{1}{3} \int_0^3 dx - \frac{1}{9} \int_2^{e^9+10} \frac{du}{u}$$

1A

$$= \frac{1}{3} \left[x \right]_0^3 - \frac{1}{9} \left[\ln |u| \right]_2^{e^9+10}$$

$$= 1 - \frac{1}{9} \ln \frac{e^9 + 10}{2}$$

1A

44. (a) 設 $x = \sin \theta$ 。則 $dx = \cos \theta d\theta$ 。

$$\begin{aligned}\int_0^1 \sqrt{1-x^2} dx &= \int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2 \theta} \cos \theta d\theta & 1M \\ &= \frac{1}{2} \int_0^{\frac{\pi}{2}} (1 + \cos 2\theta) d\theta & 1M \\ &= \frac{1}{2} \left[\theta + \frac{\sin 2\theta}{2} \right]_0^{\frac{\pi}{2}} & 1A \\ &= \frac{1}{2} \left(\frac{\pi}{2} + 0 - 0 - 0 \right) \\ &= \frac{\pi}{4} & 1A\end{aligned}$$

(b) $\frac{d}{dx} [(1-x^2)^{\frac{3}{2}} x] = (1-x^2)^{\frac{3}{2}} + \frac{3}{2} (1-x^2)^{\frac{1}{2}} (-2x)(x)$ 1M

$$\begin{aligned}&= \sqrt{1-x^2} [(1-x^2) - 3x^2] \\ &= \sqrt{1-x^2} - 4x^2 \sqrt{1-x^2}\end{aligned}$$

1

- (c) 利用 (b)， $4x^2 \sqrt{1-x^2} = \sqrt{1-x^2} - \frac{d}{dx} [(1-x^2)^{\frac{3}{2}} x]$ 。

$$\begin{aligned}\int_0^1 x^2 \sqrt{1-x^2} dx &= \frac{1}{4} \int_0^1 \sqrt{1-x^2} dx - \frac{1}{4} \left[(1-x^2)^{\frac{3}{2}} x \right]_0^1 & 1M \\ &= \frac{1}{4} \times \frac{\pi}{4} - \frac{1}{4} (0-0) \\ &= \frac{\pi}{16}\end{aligned}$$

1A

45. 設 $x = \sin^2 \theta$ ，其中 $0 \leq \theta \leq \frac{\pi}{2}$ 。則 $dx = 2 \sin \theta \cos \theta d\theta$ 。 1M

$$\begin{aligned}\int_0^{\frac{3}{4}} \sqrt{\frac{x}{1-x}} dx &= \int_0^{\frac{\pi}{3}} \sqrt{\frac{\sin^2 \theta}{1-\sin^2 \theta}} \cdot 2 \sin \theta \cos \theta d\theta & 1M \\ &= \int_0^{\frac{\pi}{3}} 2 \sin^2 \theta d\theta \\ &= \int_0^{\frac{\pi}{3}} (1 - \cos 2\theta) d\theta & 1M \\ &= \left[\theta - \frac{\sin 2\theta}{2} \right]_0^{\frac{\pi}{3}} & 1A \\ &= \frac{\pi}{3} - \frac{\sqrt{3}}{4} & 1A\end{aligned}$$

46. (a) $a(\sin x + 2 \cos x)b(\cos x - 2 \sin x) = (2a + b) \cos x + (a - 2b) \sin x$

可得 $2a + b = 13$ 及 $a - 2b = -6$ 。

求解後，可得 $a = 4$ 及 $b = 5$ 。

- (b) 設 $u = \sin x + 2 \cos x$ 。設 $du = (\cos x - 2 \sin x) dx$ 。

$$\begin{aligned}
& \int_0^{\frac{\pi}{4}} \frac{13 - 6 \tan x}{\tan x + 2} dx \\
&= \int_0^{\frac{\pi}{4}} \frac{13 \cos x - 6 \sin x}{\sin x + 2 \cos x} dx \\
&= \int_0^{\frac{\pi}{4}} \frac{4(\sin x + 2 \cos x) + 5(\cos x - 2 \sin x)}{\sin x + 2 \cos x} dx \\
&= 4 \int_0^{\frac{\pi}{4}} dx + 5 \int_0^{\frac{\pi}{4}} \frac{\cos x - 2 \sin x}{\sin x + 2 \cos x} dx \quad 1M \\
&= 4 \left[x \right]_0^{\frac{\pi}{4}} + 5 \int_2^{\frac{3\sqrt{2}}{2}} \frac{du}{u} \\
&= \pi + 5 \left[\ln |u| \right]_2^{\frac{3\sqrt{2}}{2}} \quad 1M \\
&= \pi + 5 \ln \frac{3\sqrt{2}}{4} \quad 1A
\end{aligned}$$

47. (a) $\frac{A}{x} + \frac{Bx + C}{1 + x^2} = \frac{A(1 + x^2) + (Bx + C)(x)}{x(1 + x^2)}$
 $= \frac{(A + B)x^2 + Cx + A}{x(1 + x^2)}$

可得 $A + B = 0$ 、 $C = 0$ 及 $A = 2$ 。 1M

求解後，可得 $A = 2$ 、 $B = -2$ 及 $C = 0$ 。 1A

(b) $\int_1^2 \frac{2}{x(1 + x^2)} dx = \int_1^2 \left(\frac{2}{x} + \frac{-2x}{1 + x^2} \right) dx$ 1M
 $= 2 \left[\ln |x| \right]_1^2 - 2 \int_1^2 \frac{x dx}{1 + x^2}$ 1A

設 $u = 1 + x^2$ 。則 $du = 2x dx$ 。 1M

$$\begin{aligned}
\int_1^2 \frac{2}{x(1 + x^2)} dx &= 2(\ln 2 - \ln 1) - \int_2^5 \frac{du}{u} \\
&= 2 \ln 2 - \left[\ln |u| \right]_2^5 \\
&= 2 \ln 2 - (\ln 5 - \ln 2) \\
&= 3 \ln 2 - \ln 5 \quad 1A
\end{aligned}$$

48. (a) $\sin^2 x \tan^2 x$
 $= (1 - \cos^2 x)(\sec^2 x - 1)$
 $= \sec^2 x - 2 + \cos^2 x$
 $= \sec^2 x - 2 + \frac{1}{2}(1 + \cos 2x)$ 1M
 $= \sec^2 x + \frac{1}{2} \cos 2x - \frac{3}{2}$ 1

$$\begin{aligned}
\text{(b)} \quad & \int_0^{\frac{\pi}{4}} \sin^4 x \sec^2 x \, dx \\
&= \int_0^{\frac{\pi}{4}} \sin^2 x \tan^2 x \, dx \\
&= \int_0^{\frac{\pi}{4}} \left(\sec^2 x + \frac{1}{2} \cos 2x - \frac{3}{2} \right) dx && 1M \\
&= \left[\tan x + \frac{\sin 2x}{4} - \frac{3x}{2} \right]_0^{\frac{\pi}{4}} && 1A \\
&= \frac{5}{4} - \frac{3\pi}{8} && 1A
\end{aligned}$$

$$\begin{aligned}
49. \quad & \int_0^3 \left(\frac{x}{e^{2x}} \right)^2 dx = \int_0^3 x^2 e^{-4x} dx \\
&= \left[x^2 \times \frac{e^{-4x}}{-4} \right]_0^3 - \int_0^3 (2x) \left(\frac{e^{-4x}}{-4} \right) dx && 1M+1M \\
&= -\frac{9e^{-12}}{4} + \frac{1}{2} \left[x \left(\frac{e^{-4x}}{-4} \right) \right]_0^3 - \int_0^3 \frac{e^{-4x}}{-4} dx && 1M \\
&= -\frac{9e^{-12}}{4} - \frac{3e^{-12}}{8} + \frac{1}{8} \left[\frac{e^{-4x}}{-4} \right]_0^3 && 1A \\
&= -\frac{21e^{-12}}{8} - \frac{1}{32} (e^{-12} - 1) \\
&= -\frac{85}{32e^{12}} + \frac{1}{32} && 1A
\end{aligned}$$

$$50. \quad \text{設 } u = x + 1 \circ \text{ 則 } du = dx \circ \quad 1M$$

$$\begin{aligned}
& \int_0^1 [\ln(x+1)]^2 dx = \int_1^2 (\ln u)^2 du \\
&= \left[u(\ln u)^2 \right]_1^2 - \int_1^2 u(2) \ln u \times \frac{1}{u} du && 1M+1M \\
&= 2(\ln 2)^2 - 2 \int_1^2 \ln u \, du \\
&= 2(\ln 2)^2 - 2 \left[\left[u \ln u \right]_1^2 - \int_1^2 u \times \frac{1}{u} du \right] && 1M \\
&= 2(\ln 2)^2 - 4 \ln 2 + 2 \left[u \right]_1^2 \\
&= 2(\ln 2)^2 - 4 \ln 2 + 2 && 1A
\end{aligned}$$

$$51. \quad (a) \quad \int_0^{\frac{\pi}{2}} e^{\frac{x}{2}} \sin x \, dx = \left[2e^{\frac{x}{2}} \sin x \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (2e^{\frac{x}{2}})(\cos x) \, dx \quad 1M+1M$$

$$= 2e^{\frac{\pi}{4}} - 2 \left[\left[2e^{\frac{x}{2}} \cos x \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (2e^{\frac{x}{2}})(-\sin x) \, dx \right] \quad 1M$$

$$= 2e^{\frac{\pi}{4}} - 4(0 - 1) - 4 \int_0^{\frac{\pi}{2}} e^{\frac{x}{2}} \sin x \, dx$$

$$5 \int_0^{\frac{\pi}{2}} e^{\frac{x}{2}} \sin x \, dx = 2e^{\frac{\pi}{4}} + 4 \quad 1M$$

$$\int_0^{\frac{\pi}{2}} e^{\frac{x}{2}} \sin x \, dx = \frac{2}{5}(e^{\frac{\pi}{4}} + 2) \quad 1A$$

$$(b) \quad \int_0^{\frac{\pi}{4}} e^x (\sin x + \cos x)^2 \, dx = \int_0^{\frac{\pi}{4}} e^x (1 + \sin 2x) \, dx$$

設 $u = 2x$ 。則 $du = 2 \, dx$ 。

當 $x = 0$ ， $u = 0$ ；當 $x = \frac{\pi}{4}$ ， $u = \frac{\pi}{2}$ 。

$$\int_0^{\frac{\pi}{4}} e^x (\sin x + \cos x)^2 \, dx = \frac{1}{2} \int_0^{\frac{\pi}{2}} e^{\frac{u}{2}} (1 + \sin u) \, du \quad 1M$$

$$= \left[e^{\frac{u}{2}} \right]_0^{\frac{\pi}{2}} + \frac{1}{5}(e^{\frac{\pi}{4}} + 2) \quad 1M$$

$$= e^{\frac{\pi}{4}} - 1 + \frac{1}{5}(e^{\frac{\pi}{4}} + 2)$$

$$= \frac{1}{5}(6e^{\frac{\pi}{4}} - 3) \quad 1A$$

$$52. \quad (a) \quad \int \sin(\ln x) \, dx = x \sin(\ln x) - \int \cos(\ln x) \, dx \quad 1M$$

$$= x \sin(\ln x) - x \cos(\ln x) - \int \sin(\ln x) \, dx \quad 1M$$

$$2 \int \sin(\ln x) \, dx = x \sin(\ln x) - x \cos(\ln x) + \text{常數} \quad 1M$$

$$\int \sin(\ln x) \, dx = \frac{x(\sin(\ln x) - \cos(\ln x))}{2} + \text{常數} \quad 1A$$

$$(b) \quad \text{設 } u = x + 1 \text{。則 } du = dx \text{。} \quad 1M$$

$$\begin{aligned} \int_0^{e^{\pi}-1} \sin[\ln(x+1)] \, dx &= \int_1^{e^{\pi}} \sin(\ln u) \, du \\ &= \left[\frac{u(\sin(\ln u) - \cos(\ln u))}{2} \right]_1^{e^{\pi}} \end{aligned} \quad 1M$$

$$= \frac{e^{\pi} + 1}{2} \quad 1A$$

$$\begin{aligned}
53. \quad \int_0^3 \left(\frac{x}{e^{2x}}\right)^2 dx &= \int_0^3 x^2 e^{-4x} dx \\
&= -\frac{1}{4} \left[x^2 e^{-4x} \right]_0^3 + \frac{1}{2} \int_0^3 x e^{-4x} dx && 1M+1A \\
&= -\frac{9e^{-12}}{4} - \frac{1}{8} \left[x e^{-4x} \right]_0^3 + \frac{1}{8} \int_0^3 e^{-4x} dx \\
&= -\frac{9e^{-12}}{4} - \frac{3e^{-12}}{8} - \frac{1}{32} \left[e^{-4x} \right]_0^3 \\
&= -\frac{85}{32e^{12}} + \frac{1}{32} && 1A
\end{aligned}$$

$$\begin{aligned}
54. \quad (a) \quad \int x \cos kx dx &= x \left(\frac{\sin kx}{k} \right) - \int \frac{\sin kx}{k} dx && 1M+1M \\
&= \frac{x \sin kx}{k} + \frac{\cos kx}{k^2} + C && 1A
\end{aligned}$$

$$\begin{aligned}
(b) \quad \int_0^{\frac{\pi}{2}} x \cos x \cos 2x dx &= \frac{1}{2} \int_0^{\frac{\pi}{2}} x (\cos 3x + \cos x) dx && 1M \\
&= \frac{1}{2} \left[\frac{x \sin 3x}{3} + \frac{\cos 3x}{9} \right]_0^{\frac{\pi}{2}} + \frac{1}{2} \left[x \sin x + \cos x \right]_0^{\frac{\pi}{2}} && 1M \\
&= \frac{1}{2} \left[\left(-\frac{\pi}{6} + 0 \right) - \left(0 + \frac{1}{9} \right) \right] + \frac{1}{2} \left[\left(\frac{\pi}{2} + 0 \right) - (0 + 1) \right] \\
&= \frac{\pi}{6} - \frac{5}{9} && 1A
\end{aligned}$$

$$\begin{aligned}
55. \quad (a) \quad \int_1^6 x \sqrt{x-1} dx &= \left[x \times \frac{2(x-1)^{\frac{3}{2}}}{3} \right]_1^6 - \int_1^6 \frac{2(x-1)^{\frac{3}{2}}}{3} dx && 1M+1M \\
&= (20\sqrt{5} - 0) - \frac{4}{15} \left[(x-1)^{\frac{5}{2}} \right]_1^6 && 1A \\
&= 20\sqrt{5} - \frac{20\sqrt{5}}{3} \\
&= \frac{40\sqrt{5}}{3} && 1A
\end{aligned}$$

(b) 設 $u = x - 1$ 。則 $du = dx$ 。

$$\begin{aligned}
\int_1^6 x \sqrt{x-1} dx &= \int_0^5 (u+1) \sqrt{u} du && 1M \\
&= \int_0^5 (u^{\frac{3}{2}} + u^{\frac{1}{2}}) du && 1M \\
&= \left[\frac{2u^{\frac{5}{2}}}{5} + \frac{2u^{\frac{3}{2}}}{3} \right]_0^5 && 1A \\
&= \frac{40\sqrt{5}}{3} && 1A
\end{aligned}$$

56. (a) 設 $u = \ln x$ 。則 $du = \frac{dx}{x}$ 。 1M

$$\int_1^e \frac{\ln x}{x} dx = \int_0^1 u du$$
 1M

$$= \left[\frac{u^2}{2} \right]_0^1$$
 1A

$$= \frac{1}{2}$$
 1A

(b) $\int_1^e \frac{(1+x^2)\ln x}{x} dx = \int_1^e \frac{\ln x}{x} dx + \int_1^e x \ln x dx$ 1M+1M

$$= \frac{1}{2} + \left[\frac{x^2}{2} \ln x \right]_1^e - \int_1^e \left(\frac{x^2}{2} \right) \left(\frac{1}{x} \right) dx$$

$$= \frac{1}{2} + \frac{e^2}{2} - 0 - \frac{1}{2} \int_1^e x dx$$

$$= \frac{1+e^2}{2} - \frac{1}{2} \left[\frac{x^2}{2} \right]_1^e$$

$$= \frac{1+e^2}{2} - \frac{e^2-1}{4}$$

$$= \frac{3+e^2}{4}$$
 1A

57. (a) $\int_0^{\frac{\pi}{2}} e^{-2x} \cos 2x dx = \left[\left(\frac{e^{-2x}}{-2} \right) \cos 2x \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \left(\frac{e^{-2x}}{-2} \right) (-2 \sin 2x) dx$ 1M+1M

$$= -\frac{1}{2}(-e^{-\pi} - 1) - \int_0^{\frac{\pi}{2}} e^{-2x} \sin 2x dx$$

$$= \frac{e^{-\pi} + 1}{2} - \left[\left[\left(\frac{e^{-2x}}{-2} \right) \sin 2x \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \left(\frac{e^{-2x}}{-2} \right) (2 \cos 2x) dx \right]$$
 1M

$$= \frac{e^{-\pi} + 1}{2} + 0 - \int_0^{\frac{\pi}{2}} e^{-2x} \cos 2x dx$$

$$2 \int_0^{\frac{\pi}{2}} e^{-2x} \cos 2x dx = \frac{e^{-\pi} + 1}{2}$$
 1M

$$\int_0^{\frac{\pi}{2}} e^{-2x} \cos 2x dx = \frac{e^{-\pi} + 1}{4}$$
 1A

(b) $\int_0^{\frac{\pi}{2}} e^{-2x} \cos^2 x dx = \frac{1}{2} \int_0^{\frac{\pi}{2}} e^{-2x} (1 + \cos 2x) dx$ 1M

$$= \frac{1}{2} \left[\frac{e^{-2x}}{-2} \right]_0^{\frac{\pi}{2}} + \frac{1}{2} \times \frac{e^{-\pi} + 1}{4}$$
 1M

$$= -\frac{1}{4}(e^{-\pi} - 1) + \frac{e^{-\pi} + 1}{8}$$

$$= \frac{3 - e^{-\pi}}{8}$$
 1A

58. (a) $\int x(x+2)e^x dx = x(x+2)e^x - \int (2x+2)e^x dx$ 1M+1M

$$= (x^2 + 2x)e^x - 2 \left[(x+1)e^x - \int e^x dx \right]$$

1M

$$= (x^2 + 2x)e^x - 2(x+1)e^x + 2e^x + C$$

$$= x^2e^x + C$$

1A

(b) $\int_1^e x(x+2)e^x \ln x dx = \left[x^2e^x \ln x \right]_1^e - \int_1^e (x^2e^x) \left(\frac{1}{x} \right) dx$ 1M+1M

$$= (e^{2+e} - 0) - \int_1^e xe^x dx$$

$$= e^{2+e} - \left[\left[xe^x \right]_1^e - \int_1^e e^x dx \right]$$

1M

$$= e^{2+e} - (e^{1+e} - e) + \left[e^x \right]_1^e$$

1A

$$= e^{2+e} - e^{1+e} + e + (e^e - e)$$

$$= e^{2+e} - e^{1+e} + e^e$$

1A

59. $\int_0^{\frac{\pi}{4}} e^x \sin 2x dx = \left[e^x \sin 2x \right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} e^x (2 \cos 2x) dx$ 1M+1M

$$= (e^{\frac{\pi}{4}} - 0) - 2 \left[\left[e^x \cos 2x \right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} e^x (-2 \sin 2x) dx \right]$$

1M

$$5 \int_0^{\frac{\pi}{4}} e^x \sin 2x dx = e^{\frac{\pi}{4}} - 2(0 - 1)$$

1M

$$\int_0^{\frac{\pi}{4}} e^x \sin 2x dx = \frac{1}{5} (e^{\frac{\pi}{4}} + 2)$$

1A

60. (a) $\int e^x \cos(\pi x) dx = e^x \cos(\pi x) + \pi \int e^x \sin(\pi x) dx$ 1M

$$= e^x \cos(\pi x) + \pi e^x \sin(\pi x) - \pi^2 \int e^x \cos(\pi x) dx$$

1M

$$(1 + \pi^2) \int e^x \cos(\pi x) dx = e^x \cos(\pi x) + \pi e^x \sin(\pi x) + \text{常數}$$

1M

$$\int e^x \cos(\pi x) dx = \frac{1}{1 + \pi^2} [e^x \cos(\pi x) + \pi e^x \sin(\pi x)] + \text{常數}$$

1A

(b) 設 $u = 2 - x$ 。則 $du = -dx$ 。 1M

$$\begin{aligned} & \int_0^2 e^{2-x} \cos(\pi x) dx \\ &= - \int_2^0 e^u \cos(2\pi - u) du \\ &= \int_0^2 e^u \cos u du \\ &= \frac{1}{1 + \pi^2} \left[e^u \cos(\pi u) + \pi e^u \sin(\pi u) \right]_0^2 \\ &= \frac{e^2 - 1}{1 + \pi^2} \end{aligned}$$

1M
1A

61. (a) 設 $x = a \tan \theta$ 。則 $dx = a \sec^2 \theta d\theta$ 。 1M

$$\begin{aligned} \int \frac{dx}{x^2 + a^2} &= \int \frac{a \sec^2 \theta d\theta}{a^2 \tan^2 \theta + a^2} \\ &= \frac{1}{a} \int d\theta \\ &= \frac{1}{a} \tan^{-1} \frac{x}{a} + C \end{aligned}$$

1M
1A

(b) $\int_0^{3\sqrt{3}} \tan^{-1} \frac{x}{3} dx = \left[x \tan^{-1} \frac{x}{3} \right]_0^{3\sqrt{3}} - \int_0^{3\sqrt{3}} x \times 3 \times \frac{1}{x^2 + 3^2} dx$ 1M+1M

$$= 3\sqrt{3} \times \frac{\pi}{3} - 0 - 3 \int_0^{3\sqrt{3}} \frac{x}{x^2 + 9} dx$$

設 $u = x^2 + 9$ 。則 $du = 2x dx$ 。 1M

$$\begin{aligned} \int_0^{3\sqrt{3}} \tan^{-1} \frac{x}{3} dx &= \sqrt{3}\pi - \frac{3}{2} \int_9^{36} \frac{du}{u} \\ &= \sqrt{3}\pi - \frac{3}{2} \left[\ln |u| \right]_9^{36} \\ &= \sqrt{3}\pi - 3 \ln 2 \end{aligned}$$

1A
1A

62. (a) $\int \ln x dx = x \ln x - \int x \left(\frac{1}{x} \right) dx$ 1M

$$= x \ln x - x + \text{常數}$$

1A

(b) 設 $u = \sqrt{x} + 1$ 。則 $du = \frac{1}{2\sqrt{x}} dx$ 。 1M

$$\begin{aligned} \int_0^{e^2} \frac{\ln(\sqrt{x} + 1)}{\sqrt{x}} dx &= 2 \int_1^{e+1} \ln u du \\ &= 2 \left[u \ln u - u \right]_1^{e+1} \\ &= 2(e+1) \ln(e+1) - 2e \end{aligned}$$

1A
1A

$$\begin{aligned}
63. \quad (a) \quad \int_1^e \ln x \, dx &= \left[x \ln x \right]_1^e - \int_1^e x \times \frac{1}{x} \, dx && 1M+1M \\
&= (e-0) - \left[x \right]_1^e \\
&= e - (e-1) \\
&= 1 && 1A
\end{aligned}$$

$$\begin{aligned}
(b) \quad \int_1^e (\ln x)^2 \, dx &= \left[x(\ln x)^2 \right]_1^e - \int_1^e x \cdot 2 \ln x \times \frac{1}{x} \, dx && 1M \\
&= (e-0) - 2 \int_1^e \ln x \, dx \\
&= e - 2 && 1A
\end{aligned}$$

$$\begin{aligned}
64. \quad (a) \quad \int_0^{\frac{\pi}{8}} x \cos 2x \, dx &= \left[x \times \frac{\sin 2x}{2} \right]_0^{\frac{\pi}{8}} - \int_0^{\frac{\pi}{8}} \frac{\sin 2x}{2} \, dx && 1M \\
&= \left(\frac{\pi}{8} \times \frac{\sqrt{2}}{4} - 0 \right) + \left[\frac{\cos 2x}{4} \right]_0^{\frac{\pi}{8}} && 1A \\
&= \frac{\pi\sqrt{2}}{32} + \frac{\sqrt{2}}{8} - \frac{1}{4} && 1A
\end{aligned}$$

$$\begin{aligned}
(b) \quad \int_0^{\frac{\pi}{8}} x^2 \sin 2x \, dx &= \left[x^2 \left(-\frac{\cos 2x}{2} \right) \right]_0^{\frac{\pi}{8}} - \int_0^{\frac{\pi}{8}} (2x) \left(-\frac{\cos 2x}{2} \right) \, dx && 1M+1M \\
&= \left[\frac{\pi^2}{64} \times \left(-\frac{\sqrt{2}}{4} \right) - 0 \right] + \int_0^{\frac{\pi}{8}} x \cos 2x \, dx \\
&= -\frac{\pi^2\sqrt{2}}{256} + \frac{\pi\sqrt{2}}{32} + \frac{\sqrt{2}}{8} - \frac{1}{4} && 1A
\end{aligned}$$

$$\begin{aligned}
65. \quad \int_0^{\frac{\pi}{6}} x^2 \cos 3x \, dx &= \left[x^2 \left(\frac{\sin 3x}{3} \right) \right]_0^{\frac{\pi}{6}} - \int_0^{\frac{\pi}{6}} (2x) \left(\frac{\sin 3x}{3} \right) \, dx && 1M+1M \\
&= \frac{\pi^2}{108} - 0 - \frac{2}{3} \left[\left[x \left(-\frac{\cos 3x}{3} \right) \right]_0^{\frac{\pi}{6}} - \int_0^{\frac{\pi}{6}} \left(-\frac{\cos 3x}{3} \right) \, dx \right] && 1M \\
&= \frac{\pi^2}{108} + \frac{2}{9}(0-0) - \frac{2}{9} \left[\frac{\sin 3x}{3} \right]_0^{\frac{\pi}{6}} && 1A \\
&= \frac{\pi^2}{108} - \frac{2}{27} && 1A
\end{aligned}$$

$$\begin{aligned}
66. \quad (a) \quad \int \sin(k \ln x) \, dx &= x \sin(k \ln x) - k \int \cos(k \ln x) \, dx && 1M \\
&= x \sin(k \ln x) - kx \cos(k \ln x) - k^2 \int \sin(k \ln x) \, dx && 1M \\
(1+k^2) \int \sin(k \ln x) \, dx &= x[\sin(k \ln x) - k \cos(k \ln x)] + \text{常數} \\
\int \sin(k \ln x) \, dx &= \frac{x[\sin(k \ln x) - k \cos(k \ln x)]}{1+k^2} + \text{常數} && 1
\end{aligned}$$

$$\begin{aligned}
\text{(b)} \quad \int_1^e \sin(\pi \ln x) \cos(\pi \ln x) dx &= \frac{1}{2} \int_1^e \sin(2\pi \ln x) dx & 1M \\
&= \frac{1}{2} \left[\frac{x[\sin(2\pi \ln x) - 2\pi \cos(2\pi \ln x)]}{1 + 4\pi^2} \right]_1^e & 1M \\
&= \frac{(1-e)\pi}{1 + 4\pi^2} & 1A
\end{aligned}$$

$$\begin{aligned}
67. \quad \text{(a)} \quad \int x^2 e^x dx &= x^2 e^x - \int (2x) e^x dx & 1M+1M \\
&= x^2 e^x - 2 \left[x e^x - \int e^x dx \right] & 1M \\
&= x^2 e^x - 2x e^x + 2e^x + C & 1A
\end{aligned}$$

$$\text{(b)} \quad \text{設 } u = \sqrt[3]{x} \circ \text{ 則 } du = \frac{1}{3} x^{-\frac{2}{3}} dx \circ \quad 1M$$

$$\begin{aligned}
\int_1^8 e^{\sqrt[3]{x}} dx &= 3 \int_1^8 x^{\frac{2}{3}} e^{\sqrt[3]{x}} \cdot \frac{1}{3} x^{-\frac{2}{3}} dx \\
&= 3 \int_1^2 u^2 e^u du & 1M \\
&= 3 \left[u^2 e^u - 2u e^u + 2e^u \right]_1^2 & 1M \\
&= 3[(4e^2 - 4e^2 + 2e^2) - (e - 2e + 2e)] \\
&= 6e^2 - 3e & 1A
\end{aligned}$$

$$\begin{aligned}
\text{(c)} \quad \int_1^8 2^{\sqrt[3]{x}} dx &= \int_1^8 e^{\sqrt[3]{x} \ln 2} dx & 1M \\
\text{設 } u &= \sqrt[3]{x} \ln 2 \circ \text{ 則 } du = \frac{\ln 2}{3} x^{-\frac{2}{3}} dx \circ & 1M
\end{aligned}$$

$$\begin{aligned}
\int_1^8 2^{\sqrt[3]{x}} dx &= \frac{3}{\ln 2} \int_1^8 x^{\frac{2}{3}} e^{\sqrt[3]{x}} \cdot \frac{\ln 2}{3} x^{-\frac{2}{3}} dx \\
&= \frac{3}{\ln 2} \int_{\ln 2}^{2 \ln 2} \left(\frac{u}{\ln 2} \right)^2 e^u du & 1M \\
&= \frac{3}{(\ln 2)^3} \left[u^2 e^u - 2u e^u + 2e^u \right]_{\ln 2}^{2 \ln 2} & 1M \\
&= \frac{42}{\ln 2} - \frac{36}{(\ln 2)^2} + \frac{12}{(\ln 2)^3} & 1A
\end{aligned}$$

$$\begin{aligned}
68. \quad \int_0^{\frac{\pi}{6}} x^2 \cos 3x dx &= \left[\frac{x^2 \sin 3x}{3} \right]_0^{\frac{\pi}{6}} - \frac{2}{3} \int_0^{\frac{\pi}{6}} x \sin 3x dx & 1M+1A \\
&= \frac{\pi^2}{108} + \frac{2}{9} \left[x \cos 3x \right]_0^{\frac{\pi}{6}} - \frac{2}{9} \int_0^{\frac{\pi}{6}} \cos 3x dx \\
&= \frac{\pi^2}{108} + 0 - \frac{2}{27} \left[\sin 3x \right]_0^{\frac{\pi}{6}} \\
&= \frac{\pi^2}{108} - \frac{2}{27} & 1A
\end{aligned}$$

69. (a) 設 $u = \frac{\pi}{4} - x$ 。則 $du = -dx$ 。 1M

$$\begin{aligned} \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln \left[\cos \left(\frac{\pi}{4} - x \right) \right] dx &= - \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln(\cos u) du \\ &= \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln(\cos u) du & 1M \\ &= \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln(\cos x) dx & 1 \end{aligned}$$

(b) $\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln \left[\cos \left(\frac{\pi}{4} - x \right) \right] dx - \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln(\cos x) dx = 0$

$$\begin{aligned} \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln \frac{\cos \frac{\pi}{4} \cos x + \sin \frac{\pi}{4} \sin x}{\cos x} dx &= 0 & 1M \\ \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \left[-\frac{1}{2} \ln 2 + \ln(1 + \tan x) \right] dx &= 0 & 1M \\ -\frac{\ln 2}{2} \left[x \right]_{\frac{\pi}{12}}^{\frac{\pi}{6}} + \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln(1 + \tan x) dx &= 0 \\ \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln(1 + \tan x) dx &= \frac{\pi \ln 2}{24} & 1 \end{aligned}$$

(c) $\tan \frac{\pi}{12} = \tan \left(\frac{\pi}{4} - \frac{\pi}{6} \right)$

$$\begin{aligned} &= \frac{\tan \frac{\pi}{4} - \tan \frac{\pi}{6}}{1 + \tan \frac{\pi}{4} \tan \frac{\pi}{6}} & 1M \\ &= \frac{1 - \frac{\sqrt{3}}{3}}{1 + \frac{\sqrt{3}}{3}} \times \frac{\sqrt{3} - 1}{\sqrt{3} - 1} \\ &= \frac{4 - 2\sqrt{3}}{2} \\ &= 2 - \sqrt{3} & 1A \end{aligned}$$

(d) 留意 $\frac{d}{dx} \ln(1 + \tan x) = \frac{\sec^2 x}{1 + \tan x}$ 。

$$\begin{aligned}
& \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \frac{(x + \pi) \sec^2 x}{1 + \tan x} dx \\
&= \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \frac{x \sec^2 x}{1 + \tan x} dx + \pi \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \frac{\sec^2 x}{1 + \tan x} dx \\
&= \left[x \ln(1 + \tan x) \right]_{\frac{\pi}{12}}^{\frac{\pi}{6}} - \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln(1 + \tan x) dx + \pi \left[\ln(1 + \tan x) \right]_{\frac{\pi}{12}}^{\frac{\pi}{6}} \quad 1M \\
&= \frac{\pi}{6} \ln \left(1 + \tan \frac{\pi}{6} \right) - \frac{\pi}{12} \ln \left(1 + \tan \frac{\pi}{12} \right) - \frac{\pi \ln 2}{24} + \pi \ln \left(1 + \tan \frac{\pi}{6} \right) - \pi \ln \left(1 + \tan \frac{\pi}{12} \right) \quad 1M \\
&= \frac{7\pi}{6} \ln \left(1 + \frac{1}{\sqrt{3}} \right) - \frac{13\pi}{12} \ln(3 - \sqrt{3}) - \frac{\pi \ln 2}{24} \quad 1M \\
&= \frac{7\pi}{6} [\ln 2 - \ln(3 - \sqrt{3})] - \frac{13\pi}{12} \ln(3 - \sqrt{3}) - \frac{\pi \ln 2}{24} \quad 1M \\
&= \frac{9\pi \ln 2}{8} - \frac{9\pi}{4} \ln(3 - \sqrt{3}) \\
&= -\frac{9\pi}{8} \left[-\ln 2 + \ln(3 - \sqrt{3})^2 \right] \\
&= -\frac{9\pi}{8} \left[-\ln 2 + \ln(12 - 6\sqrt{3}) \right] \\
&= -\frac{9\pi}{8} \ln(6 - 3\sqrt{3}) \quad 1
\end{aligned}$$

70. (a) $\frac{A}{u^2 + 1} + \frac{B}{u^2 + 4} = \frac{A(u^2 + 4) + B(u^2 + 1)}{(u^2 + 1)(u^2 + 4)}$
 $= \frac{(A + B)u^2 + (4A + B)}{(u^2 + 1)(u^2 + 4)}$

比較同類項，可得 $A + B = 0$ 及 $4A + B = 1$ 。 1M

求解後，可得 $A = \frac{1}{3}$ 及 $B = -\frac{1}{3}$ 。

因此， $\frac{1}{(u^2 + 1)(u^2 + 4)} = \frac{1}{3(u^2 + 1)} - \frac{1}{3(u^2 + 4)}$ 。 1A

(b) 設 $u = -x$ 。則 $du = -dx$ 。 1M

$$\begin{aligned}
\int_{-a}^a f(x) dx &= \int_{-a}^0 f(x) dx + \int_0^a f(x) dx \\
&= -\int_a^0 f(-u) du + \int_0^a f(x) dx \quad 1M \\
&= \int_0^a f(u) du + \int_0^a f(x) dx \\
&= 2 \int_0^a f(x) dx \quad 1
\end{aligned}$$

(c) (i) 利用 (b) 的結果。

$$\int_{-\sqrt{3}}^{\sqrt{3}} \frac{du}{(u^2 + 1)(u^2 + 4)} = \int_{-\sqrt{3}}^{\sqrt{3}} \left(\frac{1}{3(u^2 + 1)} - \frac{1}{3(u^2 + 4)} \right) du \quad 1M$$

設 $u = \tan \theta$ 。則 $du = \sec^2 \theta d\theta$ 。 1M

設 $u = 2 \tan \phi$ 。則 $du = 2 \sec^2 \phi \, d\phi$ 。

$$\int_{-\sqrt{3}}^{\sqrt{3}} \frac{du}{(u^2+1)(u^2+4)} = \frac{1}{3} \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{\sec^2 \theta}{\tan^2 \theta + 1} d\theta - \frac{1}{3} \int_{-\tan^{-1} \frac{\sqrt{3}}{2}}^{\tan^{-1} \frac{\sqrt{3}}{2}} \frac{2 \sec^2 \phi}{4 \tan^2 \phi + 4} d\phi \quad 1A$$

$$= \frac{1}{3} \left[\theta \right]_{-\frac{\pi}{3}}^{\frac{\pi}{3}} - \frac{1}{6} \left[\phi \right]_{-\tan^{-1} \frac{\sqrt{3}}{2}}^{\tan^{-1} \frac{\sqrt{3}}{2}}$$

$$= \frac{2\pi}{9} - \frac{1}{3} \tan^{-1} \frac{\sqrt{3}}{2} \quad 1A$$

(ii) 留意 $\frac{1}{\tan^2(-x)+4} = \frac{1}{\tan^2 x + 4}$ 。

設 $u = \tan x$ 。則 $du = \sec^2 x \, dx = (u^2 + 1) \, dx$ 。

$$\int_0^{\frac{\pi}{3}} \frac{dx}{\tan^2 x + 4} = \frac{1}{2} \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{dx}{\tan^2 x + 4} \quad 1M$$

$$= \frac{1}{2} \int_{-\sqrt{3}}^{\sqrt{3}} \frac{du}{(u^2+1)(u^2+4)} \quad 1M$$

$$= \frac{\pi}{9} - \frac{1}{6} \tan^{-1} \frac{\sqrt{3}}{2} \quad 1A$$

71. (a) 設 $x = \sqrt{3} \tan \theta$ 。則 $dx = \sqrt{3} \sec^2 \theta \, d\theta$ 。 1M

$$\int \frac{1}{x^2+3} dx = \int \frac{\sqrt{3} \sec^2 \theta}{3 \tan^2 \theta + 3} d\theta$$

$$= \frac{\sqrt{3}}{3} \int d\theta \quad 1M$$

$$= \frac{\sqrt{3}}{3} \tan^{-1} \frac{\sqrt{3}x}{3} + \text{常數} \quad 1A$$

(b) 設 $t = \tan x$ 。則 $dt = \sec^2 x \, dx$ 。

$$\int \frac{1}{1+2\cos^2 x} dx = \int \frac{\sec^2 x}{\sec^2 x + 2} dx \quad 1M$$

$$= \int \frac{dt}{t^2+3} \quad 1A$$

$$= \frac{\sqrt{3}}{3} \tan^{-1} \frac{\sqrt{3}t}{3} + \text{常數}$$

$$= \frac{\sqrt{3}}{3} \tan^{-1} \frac{\sqrt{3} \tan x}{3} + \text{常數} \quad 1A$$

(c) 設 $y = 2\pi - x$ 。則 $dy = -dx$ 。

$$\int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} f(x) \ln(1 + e^{\sin x}) dx = - \int_{\frac{5\pi}{4}}^{\frac{3\pi}{4}} f(2\pi - y) \ln(1 + e^{\sin(2\pi - y)}) dy \quad 1M$$

$$= \int_{\frac{5\pi}{4}}^{\frac{3\pi}{4}} f(y) \ln(1 + e^{-\sin y}) dy \quad 1A$$

$$= - \int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} f(y) \ln\left(\frac{1 + e^{\sin y}}{e^{\sin y}}\right) dy$$

$$= - \int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} f(y) \ln(1 + e^{\sin y}) dy + \int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} f(y) \sin y dy$$

$$= - \int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} f(x) \ln(1 + e^{\sin x}) dx + \int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} f(x) \sin x dx$$

$$\int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} f(x) \ln(1 + e^{\sin x}) dx = \frac{1}{2} \int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} f(x) \sin x dx \quad 1$$

(d) 代 $f(x) = \frac{\sin x}{1 + 2\cos^2 x}$ 。

$$f(2\pi - x) = \frac{\sin(2\pi - x)}{1 + 2\cos^2(2\pi - x)} = \frac{-\sin x}{1 + 2\cos^2 x} = -f(x) \quad 1M$$

$$\int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} \frac{\sin x (1 + e^{\sin x})}{1 + 2\cos^2 x} dx$$

$$= \frac{1}{2} \int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} \frac{\sin^2 x}{1 + 2\cos^2 x} dx$$

$$= \frac{1}{2} \int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} \frac{1 - \cos^2 x}{1 + 2\cos^2 x} dx$$

$$= \frac{1}{2} \int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} \frac{1 - \frac{1}{2}(1 + 2\cos^2 x) + \frac{1}{2}}{1 + 2\cos^2 x} dx \quad 1M$$

$$= \frac{3}{4} \int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} \frac{dx}{1 + 2\cos^2 x} - \frac{1}{4} \int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} dx$$

$$= \frac{3}{4} \left[\frac{\sqrt{3}}{3} \tan^{-1} \frac{\sqrt{3} \tan x}{3} \right]_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} - \frac{1}{4} \left[x \right]_{\frac{3\pi}{4}}^{\frac{5\pi}{4}}$$

$$= \frac{\sqrt{3}\pi}{12} - \frac{\pi}{8} \quad 1A$$

$$72. (a) \frac{1 + \sin 2t}{1 + \cos 2t} = \frac{1 + 2 \sin t \cos t}{1 + (2 \cos^2 t - 1)} \quad 1M$$

$$= \frac{1 + 2 \sin t \cos t}{2 \cos^2 t}$$

$$= \frac{1}{2} \sec^2 t + \tan t \quad 1$$

$$(b) \int e^x f'(x) dx = e^x f(x) - \int e^x f(x) dx \quad 1M$$

$$\int e^x [f(x) + f'(x)] dx = e^x f(x) + \text{常數} \quad 1$$

(c) 設 $f(x) = \tan \frac{x}{2}$ 。則 $f'(x) = \frac{1}{2} \sec^2 \frac{x}{2}$ 。 1M

$$\int_0^{\frac{\pi}{2}} \frac{e^x(1 + \sin x)}{1 + \cos x} dx = \int_0^{\frac{\pi}{2}} e^x \left(\frac{1}{2} \sec^2 \frac{x}{2} + \tan \frac{x}{2} \right) dx \quad 1M$$

$$= \left[e^x \tan \frac{x}{2} \right]_0^{\frac{\pi}{2}} \quad 1M$$

$$= e^{\frac{\pi}{2}} \quad 1A$$

(d) 設 $u = \frac{\pi}{2} - x$ 。則 $du = -dx$ 。 1M

$$\int_0^{\frac{\pi}{2}} \frac{1 + \cos x}{e^x(1 + \sin x)} dx = - \int_{\frac{\pi}{2}}^0 \frac{1 + \cos(\frac{\pi}{2} - x)}{e^{\frac{\pi}{2}-u}(1 + \sin(\frac{\pi}{2} - u))} du$$

$$= e^{-\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{e^u(1 + \sin u)}{1 + \cos u} du$$

$$= e^{-\frac{\pi}{2}} \cdot e^{\frac{\pi}{2}} \quad 1M$$

$$= 1 \quad 1A$$

73. (a) (i) $\frac{d}{d\theta} \ln(\csc \theta + \cot \theta) = \frac{-(\sin \theta)^{-2} \cos \theta - (\tan \theta)^{-2} \sec^2 \theta}{\csc \theta + \cot \theta}$ 1M

$$= \frac{-\csc \theta \cot \theta - \csc^2 \theta}{\csc \theta + \cot \theta}$$

$$= -\csc \theta$$

因此， $\int \csc \theta d\theta = -\ln(\csc \theta + \cot \theta) + \text{常數}$ 。 1

(ii) $\int \frac{1}{\sin x \cos x} dx = 2 \int \csc 2x dx$ 1M

$$= -\ln(\csc 2x + \cot 2x) + \text{常數} \quad 1A$$

(b) 設 $u = \frac{\pi}{2} - x$ 。則 $du = -dx$ 。 1M

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} f(x) \ln(1 + e^{\sin x - \cos x}) dx = - \int_{\frac{\pi}{3}}^{\frac{\pi}{6}} f\left(\frac{\pi}{2} - u\right) \ln\left[1 + e^{\sin(\frac{\pi}{2}-u) - \cos(\frac{\pi}{2}-u)}\right] du$$

$$= - \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} f(u) \ln\left(\frac{e^{\sin u - \cos u} + 1}{e^{\sin u - \cos u}}\right) du \quad 1M$$

$$= - \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} f(u) \ln(1 + e^{\sin x - \cos x}) dx$$

$$+ \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} f(x)(\sin x - \cos x) dx \quad 1M$$

$$2 \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} f(x) \ln(1 + e^{\sin x - \cos x}) dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} f(x)(\sin x - \cos x) dx \quad 1M$$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} f(x) \ln(1 + e^{\sin x - \cos x}) dx = \frac{1}{2} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} f(x)(\sin x - \cos x) dx \quad 1$$

(c) 代 $f(x) = \frac{\sin x - \cos x}{\sin x \cos x}$ 。 1M

$$\begin{aligned}
f\left(\frac{\pi}{2}-x\right) &= \frac{\sin\left(\frac{\pi}{2}-x\right)-\cos\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)\cos\left(\frac{\pi}{2}-x\right)} \\
&= \frac{\cos x-\sin x}{\cos x \sin x} \\
&= -f(x) \\
\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{(\sin x-\cos x) \ln(1+e^{\sin x-\cos x})}{\sin x \cos x} dx & \\
&= \frac{1}{2} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{(\sin x-\cos x)^2}{\sin x \cos x} dx & 1M \\
&= \frac{1}{2} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1-2 \sin x \cos x}{\sin x \cos x} dx \\
&= \frac{1}{2} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{\sin x \cos x} dx - \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} dx \\
&= \frac{1}{2} \left[-\ln(\csc 2x + \cot 2x) \right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} - \left[x \right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} & 1M \\
&= \frac{1}{2} \left(\ln 3 - \frac{\pi}{3} \right) & 1A
\end{aligned}$$

74. (a) (i) 設 $u = x - \frac{\pi}{2}$ 。則 $du = dx$ 。

$$\begin{aligned}
\int_{\frac{\pi}{2}}^{\pi} \frac{\sin^4 x}{\sin^4 x + \cos^4 x} dx &= \int_0^{\frac{\pi}{2}} \frac{\sin^4\left(u + \frac{\pi}{2}\right)}{\sin^4\left(u + \frac{\pi}{2}\right) + \cos^4\left(u + \frac{\pi}{2}\right)} du & 1M \\
&= \int_0^{\frac{\pi}{2}} \frac{\cos^4 x}{\sin^4 x + \cos^4 x} dx & 1
\end{aligned}$$

$$\begin{aligned}
(ii) \int_0^{\pi} \frac{\sin^4 x}{\sin^4 x + \cos^4 x} dx & \\
&= \int_0^{\frac{\pi}{2}} \frac{\sin^4 x}{\sin^4 x + \cos^4 x} dx + \int_{\frac{\pi}{2}}^{\pi} \frac{\sin^4 x}{\sin^4 x + \cos^4 x} dx & 1M \\
&= \int_0^{\frac{\pi}{2}} \frac{\sin^4 x}{\sin^4 x + \cos^4 x} dx + \int_0^{\frac{\pi}{2}} \frac{\cos^4 x}{\sin^4 x + \cos^4 x} dx & 1M \\
&= \int_0^{\frac{\pi}{2}} dx & 1M \\
&= \left[x \right]_0^{\frac{\pi}{2}} \\
&= \frac{\pi}{2} & 1A
\end{aligned}$$

(b) 設 $u = \lambda - x$ 。則 $du = -dx$ 。 1M

$$\begin{aligned}\int_0^\lambda x f(x) dx &= - \int_\lambda^0 (\lambda - u) f(\lambda - u) du \\ &= \int_0^\lambda (\lambda - u) f(u) du & 1M \\ &= \lambda \int_0^\lambda f(x) dx - \int_0^\lambda x f(x) dx \\ &= \frac{\lambda}{2} \int_0^\lambda f(x) dx & 1\end{aligned}$$

(c) 代 $f(x) = \frac{4 \sin^2 x - \sin^2 2x}{\sin^4 x + \cos^4 x}$ 。

可得 $f(\pi - x) = \frac{4 \sin^2(\pi - x) - \sin^2(2\pi - 2x)}{\sin^4(\pi - x) + \cos^4(\pi - x)} = \frac{4 \sin^2 x - \sin^2 2x}{\sin^4 x + \cos^4 x} = f(x)$ 。

$$\begin{aligned}\int_0^\pi \frac{4x \sin^2 x - x \sin^2 2x}{\sin^4 x + \cos^4 x} dx &= \int_0^\pi x \left(\frac{4 \sin^2 x - \sin^2 2x}{\sin^4 x + \cos^4 x} \right) dx \\ &= \frac{\pi}{2} \int_0^\pi \frac{4 \sin^2 x - \sin^2 2x}{\sin^4 x + \cos^4 x} dx & 1M \\ &= \frac{\pi}{2} \int_0^\pi \frac{4 \sin^2 x - (2 \sin x \cos x)^2}{\sin^4 x + \cos^4 x} dx & 1M \\ &= \frac{\pi}{2} \int_0^\pi \frac{4 \sin^2 x (1 - \cos^2 x)}{\sin^4 x + \cos^4 x} dx \\ &= 2\pi \int_0^\pi \frac{\sin^4 x}{\sin^4 x + \cos^4 x} dx \\ &= 2\pi \left(\frac{\pi}{2} \right) & 1M \\ &= \pi^2 & 1A\end{aligned}$$

75. (a) 設 $u = \sqrt{3}k \tan \theta$ 。則 $du = \sqrt{3}k \sec^2 \theta d\theta$ 。 1M

$$\begin{aligned}\int_0^k \frac{k}{u^2 + 3u^2} du &= \int_0^{\frac{\pi}{6}} \frac{k \cdot \sqrt{3}k \sec^2 \theta}{3k^2 \tan^2 \theta + 3k^2} d\theta & 1M \\ &= \frac{\sqrt{3}}{3} \int_0^{\frac{\pi}{6}} d\theta \\ &= \frac{\sqrt{3}}{3} \left[\theta \right]_0^{\frac{\pi}{6}} \\ &= \frac{\sqrt{3}\pi}{18} & 1A\end{aligned}$$

(b) 設 $u = a - x$ 。則 $du = -dx$ 。 1M

$$\begin{aligned}\int_0^a x f(x) dx &= - \int_a^0 (a - u) f(a - u) du \\ &= \int_0^a (a - u) f(u) du \\ &= a \int_0^a f(x) dx - \int_0^a x f(x) dx \\ &= \frac{a}{2} \int_0^a f(x) dx\end{aligned}$$

1M
1

(c) 代 $f(x) = \frac{\sin x}{\cos 2x + 7}$ 。
則 $f(\pi - x) = \frac{\sin(\pi - x)}{\cos(2\pi - 2x) + 7} = \frac{\sin x}{\cos 2x + 7} = f(x)$ 。 1M

$$\begin{aligned}\int_0^\pi \frac{x \sin x}{\cos 2x + 7} dx &= \frac{\pi}{2} \int_0^\pi \frac{\sin x}{\cos 2x + 7} dx \\ &= \frac{\pi}{2} \int_0^\pi \frac{\sin x}{2 \cos^2 x - 1 + 7} \\ &= \frac{\pi}{4} \int_0^\pi \frac{\sin x}{\cos^2 x + 3} dx\end{aligned}$$

1M

設 $u = \cos x$ 。則 $du = -\sin x dx$ 。 1M

$$\begin{aligned}\int_0^\pi \frac{x \sin x}{\cos 2x + 7} dx &= -\frac{\pi}{4} \int_1^{-1} \frac{du}{u^2 + 3} \\ &= \frac{\pi}{4} \int_{-1}^1 \frac{du}{u^2 + 3} \\ &= \frac{\pi}{2} \int_0^1 \frac{du}{u^2 + 3} \\ &= \frac{\pi}{2} \left(\frac{\sqrt{3}\pi}{18} \right) \\ &= \frac{\sqrt{3}\pi^2}{36}\end{aligned}$$

1M
1A

76. (a) 設 $t = \pi - x$ 。則 $dt = -dx$ 。 1M

$$\begin{aligned}\int_0^\pi x f(\sin x) dx &= - \int_\pi^0 (\pi - t) f[\sin(\pi - t)] dt \\ &= \int_0^\pi (\pi - t) f(\sin t) dt \\ &= \pi \int_0^\pi f(\sin x) dx - \int_0^\pi x f(\sin x) dx \\ &= \frac{\pi}{2} \int_0^\pi f(\sin x) dx\end{aligned}$$

1M
1

(b) 設 $u = \pi - x$ 。則 $du = -dx$ 。 1M

$$\begin{aligned}\int_0^\pi x f(\sin x) dx &= \frac{\pi}{2} \int_0^\pi f(\sin x) dx \\ &= \frac{\pi}{2} \left[\int_0^{\frac{\pi}{2}} f(\sin x) dx + \int_{\frac{\pi}{2}}^\pi f(\sin x) dx \right] \\ &= \frac{\pi}{2} \int_0^{\frac{\pi}{2}} f(\sin x) dx - \frac{\pi}{2} \int_{\frac{\pi}{2}}^0 f[\sin(\pi - u)] du \\ &= \frac{\pi}{2} \int_0^{\frac{\pi}{2}} f(\sin x) dx + \frac{\pi}{2} \int_0^{\frac{\pi}{2}} f(\sin x) dx \\ &= \pi \int_0^{\frac{\pi}{2}} f(\sin x) dx\end{aligned}$$

1M
1

(c) 留意 $\frac{\sin x}{1 + \cos^2 x} = \frac{\sin x}{1 + (1 - \sin^2 x)} = \frac{\sin x}{2 - \sin^2 x}$ 。
代 $f(\sin x) = \frac{\sin x}{2 - \sin^2 x}$ 。

$$\int_0^\pi \frac{x \sin x}{1 + \cos^2 x} dx = \pi \int_0^{\frac{\pi}{2}} \frac{\sin x}{1 + \cos^2 x} dx$$

1M

設 $u = \cos x$ 。則 $du = -\sin x dx$ 。 1M

$$\begin{aligned}\int_0^\pi \frac{x \sin x}{1 + \cos^2 x} dx &= -\pi \int_1^0 \frac{du}{1 + u^2} \\ &= \pi \int_0^1 \frac{du}{1 + u^2}\end{aligned}$$

設 $u = \tan \theta$ 。則 $du = \sec^2 \theta d\theta$ 。 1M

$$\begin{aligned}\int_0^\pi \frac{x \sin x}{1 + \cos^2 x} dx &= \pi \int_0^{\frac{\pi}{4}} \frac{\sec^2 \theta}{1 + \tan^2 \theta} d\theta \\ &= \pi \int_0^{\frac{\pi}{4}} d\theta \\ &= \pi \left[\theta \right]_0^{\frac{\pi}{4}} \\ &= \frac{\pi^2}{4}\end{aligned}$$

1A

(d) 代 $f(x) = \frac{x}{2 - x^2}$ 。 1M

由於 $f(-x) = \frac{-x}{2 - (-x)^2} = \frac{-x}{2 - x^2} = -f(x)$ ， $f(x)$ 為奇函數。

留意 $(-x)f(\sin(-x)) = -x(-f(\sin x)) = xf(\sin x) = \frac{x \sin x}{1 + \cos^2 x}$ 。

$$\begin{aligned}\int_{-\pi}^{\pi} xf(\sin x) dx &= 2 \int_0^{\pi} xf(\sin x) dx \\ &= 2 \left(\frac{\pi^2}{4} \right) \\ &= \frac{\pi^2}{2} \\ &\neq 0\end{aligned}$$

不同意該宣稱。

1A

77. (a) 設 $u = \pi - x$ 。則 $du = -dx$ 。

1M

$$\begin{aligned}\int_0^{\pi} xf(\sin x) dx &= - \int_{\pi}^0 (\pi - u)f(\sin(\pi - u)) du \\ &= \int_0^{\pi} (\pi - u)f(\sin u) du \\ &= \pi \int_0^{\pi} f(\sin u) du - \int_0^{\pi} uf(\sin u) du \\ &= \pi \int_0^{\pi} f(\sin x) dx - \int_0^{\pi} xf(\sin x) dx \\ 2 \int_0^{\pi} xf(\sin x) dx &= \pi \int_0^{\pi} f(\sin x) dx \\ \int_0^{\pi} xf(\sin x) dx &= \frac{\pi}{2} \int_0^{\pi} f(\sin x) dx\end{aligned}$$

1M

1

(b) (i) 設 $v = -x$ 。則 $dv = -dx$ 。

1M

$$\begin{aligned}\int_{-\pi}^0 \frac{g(x)}{1 + e^x} dx &= - \int_{\pi}^0 \frac{g(-v)}{1 + e^{-v}} dv \\ &= \int_0^{\pi} \frac{e^v g(v)}{e^v(1 + e^{-v})} dv \\ &= \int_0^{\pi} \frac{e^v g(v)}{1 + e^v} dv \\ &= \int_0^{\pi} \frac{e^x g(x)}{1 + e^x} dx\end{aligned}$$

1

$$\begin{aligned}\text{(ii)} \quad \int_{-\pi}^{\pi} \frac{g(x)}{1 + e^x} dx &= \int_{-\pi}^0 \frac{g(x)}{1 + e^x} dx + \int_0^{\pi} \frac{g(x)}{1 + e^x} dx \\ &= \int_0^{\pi} \frac{e^x g(x)}{1 + e^x} dx + \int_0^{\pi} \frac{g(x)}{1 + e^x} dx \\ &= \int_0^{\pi} \frac{(1 + e^x)g(x)}{1 + e^x} dx \\ &= \int_0^{\pi} g(x) dx\end{aligned}$$

1

$$\text{(c)} \quad \text{代 } g(x) = \frac{\sqrt{2 + \cos x}}{\sqrt{2 + \cos x} + \sqrt{2 - \cos x}}。$$

$$g(-x) = \frac{\sqrt{2 + \cos(-x)}}{\sqrt{2 + \cos(-x)} + \sqrt{2 - \cos(-x)}} = \frac{\sqrt{2 + \cos x}}{\sqrt{2 + \cos x} + \sqrt{2 - \cos x}} = g(x) \quad 1M$$

$g(x)$ 為在 $[-\pi, \pi]$ 上的連續偶函數。

$$\begin{aligned} & \int_{-\pi}^{\pi} \frac{\sqrt{2 + \cos x}}{(1 + e^x)(\sqrt{2 + \cos x} + \sqrt{2 - \cos x})} dx \\ &= \int_0^{\pi} g(x) dx \\ &= \left[xg(x) \right]_0^{\pi} - \int_0^{\pi} xg'(x) dx \end{aligned} \quad 1M$$

$$g'(x) = \frac{\frac{-\sin x}{2\sqrt{2+\cos x}}(\sqrt{2+\cos x} + \sqrt{2-\cos x}) - (\sqrt{2+\cos x})\left(\frac{-\sin x}{2\sqrt{2+\cos x}} + \frac{\sin x}{2\sqrt{2-\cos x}}\right)}{(\sqrt{2+\cos x} + \sqrt{2-\cos x})^2} \quad 1M$$

$$= \frac{-\sin x}{2\sqrt{3 + \sin^2 x} + (3 + \sin^2 x)} \quad 1A$$

取 $f(x) = \frac{-x}{2\sqrt{3+x^2} + 3+x^2}$ 使得它在 $[0, 1]$ 連續及 $g'(x) = f(\sin x)$ 。

$$\begin{aligned} & \int_{-\pi}^{\pi} \frac{\sqrt{2 + \cos x}}{(1 + e^x)(\sqrt{2 + \cos x} + \sqrt{2 - \cos x})} dx \\ &= \left[xg(x) \right]_0^{\pi} - \int_0^{\pi} xf(\sin x) dx \\ &= \pi g(\pi) - \frac{\pi}{2} \int_0^{\pi} f(\sin x) dx \\ &= \pi g(\pi) - \frac{\pi}{2} \int_0^{\pi} g'(x) dx \\ &= \pi g(\pi) - \frac{\pi}{2} \left[g(x) \right]_0^{\pi} \\ &= \frac{\pi}{2} \left(\frac{\sqrt{2-1}}{\sqrt{2-1} + \sqrt{2-1}} + \frac{\sqrt{2+1}}{\sqrt{2+1} + \sqrt{2-1}} \right) \\ &= \frac{\pi}{2} \end{aligned} \quad 1A$$

$$\begin{aligned} 78. \quad (a) \quad \frac{2}{3 \cos 2x + 5} &= \frac{2}{3(\cos^2 x - 1) + 5} \\ &= \frac{2}{6 \cos^2 x + 2} \\ &= \frac{\sec^2 x}{3 + \sec^2 x} \end{aligned}$$

1

(b) 設 $u = \tan x$ 。則 $du = \sec^2 x dx$ 。

1M

$$\begin{aligned} \int_0^{\frac{\pi}{4}} \frac{2}{3 \cos 2x + 5} dx &= \int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{3 + \sec^2 x} dx \\ &= \int_0^1 \frac{1}{4 + u^2} du \end{aligned}$$

設 $u = 2 \tan \theta$ 。則 $du = 2 \sec^2 \theta d\theta$ 。 1M

$$\begin{aligned} \int_0^{\frac{\pi}{4}} \frac{2}{3 \cos 2x + 5} dx &= \int_0^{\tan^{-1} \frac{1}{2}} \frac{2 \sec^2 \theta}{4 + 4 \tan^2 \theta} d\theta \\ &= \frac{1}{2} \left[\theta \right]_0^{\tan^{-1} \frac{1}{2}} \\ &= \frac{1}{2} \tan^{-1} \frac{1}{2} \end{aligned} \quad 1A$$

(c) 設 $u = -x$ 。則 $du = -dx$ 。 1M

$$\begin{aligned} &\int_{-\alpha}^{\alpha} f(x) \ln(e^x + 1) dx \\ &= \int_{-\alpha}^0 f(x) \ln(e^x + 1) dx + \int_0^{\alpha} f(x) \ln(e^x + 1) dx \quad 1M \\ &= - \int_{\alpha}^0 f(-u) \ln(e^{-u} + 1) du + \int_0^{\alpha} f(x) \ln(e^x + 1) dx \\ &= - \int_0^{\alpha} f(x) \ln\left(\frac{1 + e^x}{e^x}\right) dx + \int_0^{\alpha} f(x) \ln(e^x + 1) dx \\ &= - \int_0^{\alpha} f(x) \ln(1 + e^x) dx + \int_0^{\alpha} f(x) \ln e^x dx + \int_0^{\alpha} f(x) \ln(e^x + 1) dx \\ &= \int_0^{\alpha} x f(x) dx \end{aligned} \quad 1$$

(d) 代 $f(x) = \frac{6 \sin 2x}{(3 \cos 2x + 5)^2}$ 。 1M

$$\begin{aligned} \text{則 } f(-x) &= \frac{6 \sin(-2x)}{(3 \cos(-2x) + 5)^2} = \frac{-6 \sin 2x}{(3 \cos 2x + 5)^2} = -f(x) \text{。} \\ \text{留意 } \frac{d}{dx} \left(\frac{2}{3 \cos 2x + 5} \right) &= \frac{6 \sin 2x}{(3 \cos 2x + 5)^2} \text{。} \\ \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{6 \sin 2x}{(3 \cos 2x + 5)^2} \ln(e^x + 1) dx \\ &= \int_0^{\frac{\pi}{4}} \frac{6x \sin 2x}{(3 \cos 2x + 5)^2} dx \quad 1M \\ &= \left[\frac{x}{3 \cos 2x + 5} \right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \frac{1}{3 \cos 2x + 5} dx \quad 1M \\ &= \frac{\frac{\pi}{4}}{3(0) + 5} - \frac{1}{2} \left(\frac{1}{2} \tan^{-1} \frac{1}{2} \right) \quad 1M \\ &= \frac{\pi}{20} - \frac{1}{4} \tan^{-1} \frac{1}{2} \quad 1A \end{aligned}$$

$$\begin{aligned} 79. \quad (a) \quad (i) \quad \frac{\sin 3x}{\sin x} - (2 \cos 2x + 1) &= \frac{1}{\sin x} [\sin 3x - 2 \sin x \cos 2x - \sin x] \\ &= \frac{1}{\sin x} [\sin 3x - (\sin 3x - \sin x) - \sin x] \quad 1M \\ &= 0 \end{aligned}$$

因此， $\frac{\sin 3x}{\sin x} = 2 \cos 2x + 1$ 。 1

(ii) 代 $x = \frac{\pi}{4} + y$ 。

$$\frac{\sin 3\left(\frac{\pi}{4} + y\right)}{\sin\left(\frac{\pi}{4} + y\right)} = 2 \cos 2\left(\frac{\pi}{4} + y\right) + 1$$

$$\frac{\sin \frac{3\pi}{4} \cos 3y + \cos \frac{3\pi}{4} \sin 3y}{\sin \frac{\pi}{4} \cos y + \cos \frac{\pi}{4} \sin y} = -2 \sin 2y + 1 \quad 1A$$

$$\frac{\cos 3y - \sin 3y}{\cos y + \sin y} = -2 \sin 2y + 1$$

$$\frac{\sin 3y - \cos 3y}{\sin x + \cos y} = 2 \sin 2y - 1 \quad 1A$$

(b) (i) 設 $u = a - x$ 。則 $du = -dx$ 。

$$\begin{aligned} \int_0^a f(x) dx &= - \int_a^0 f(a-u) du \\ &= \int_0^a f(a-u) du \\ &= \int_0^a f(a-x) dx \end{aligned} \quad \begin{array}{l} 1M \\ 1A \end{array}$$

(ii) 代 $f(x) = \frac{\sin 3x}{\sin x + \cos x}$ 。

$$f\left(\frac{\pi}{2} - x\right) = \frac{\sin 3\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)} = \frac{-\cos 3x}{\cos x + \sin x} \quad 1$$

$$\text{可得 } \int_0^{\frac{\pi}{2}} \frac{\sin 3x}{\sin x + \cos x} dx = \int_0^{\frac{\pi}{2}} \frac{-\cos 3x}{\sin x + \cos x} dx。 \quad 1A$$

$$2 \int_0^{\frac{\pi}{2}} \frac{\sin 3x}{\sin x + \cos x} dx = \int_0^{\frac{\pi}{2}} \frac{\sin 3x}{\sin x + \cos x} dx + \int_0^{\frac{\pi}{2}} \frac{-\cos 3x}{\sin x + \cos x} dx \quad 1M$$

$$\int_0^{\frac{\pi}{2}} \frac{\sin 3x}{\sin x + \cos x} dx = \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{\sin 3x - \cos 3x}{\sin x + \cos x} dx \quad 1A$$

(c) $\int_0^{\frac{\pi}{2}} \frac{\sin 3x}{\sin x + \cos x} dx = \frac{1}{2} \int_0^{\frac{\pi}{2}} (2 \sin 2x - 1) dx \quad 1M$

$$= \frac{1}{2} \left[-\cos 2x - x \right]_0^{\frac{\pi}{2}} \quad 1A$$

$$= 1 - \frac{\pi}{4} \quad 1A$$

80. (a) $\frac{1 - \cos 2x}{1 + \cos 2x} = \frac{1 - (1 - 2 \sin^2 x)}{1 + (2 \cos^2 x - 1)} \quad 1M$

$$= \frac{2 \sin^2 x}{2 \cos^2 x}$$

$$= \tan^2 x \quad 1$$

(b) 設 $u = a - x$ 。則 $du = -dx$ 。

$$\begin{aligned}\int_0^a f(a-x) dx &= -\int_a^0 f(u) du \\ &= \int_0^a f(u) du \\ &= \int_0^a f(x) dx\end{aligned}$$

1

(c) 設 $u = \cos x$ 。則 $du = -\sin x dx$ 。

$$\begin{aligned}\int \tan x dx &= \int \frac{\sin x}{\cos x} dx \\ &= -\int \frac{du}{u} \\ &= -\ln|u| + \text{常數} \\ &= -\ln|\cos x| + \text{常數}\end{aligned}$$

1M

1

(d) $\int x \sec^2 x dx = x \tan x - \int \tan x dx$

1M

$$= x \tan x - (-\ln|\cos x|) + \text{常數}$$

$$= x \tan x + \ln|\cos x| + \text{常數}$$

1A

(e) $\int_0^{\frac{\pi}{4}} \frac{(\cos x - \sin x + \sqrt{x})(\cos x - \sin x - \sqrt{x})}{1 + \sin 2x} dx$

$$= \int_0^{\frac{\pi}{4}} \frac{(\cos x - \sin x)^2 - x}{1 + \sin 2x} dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{\sin^2 x - 2 \sin x \cos x + \cos^2 x - x}{1 + \sin 2x} dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{1 - \sin 2x - x}{1 + \sin 2x} dx$$

1M

$$= \int_0^{\frac{\pi}{4}} \frac{1 - \sin(\frac{\pi}{2} - 2x) - (\frac{\pi}{4} - x)}{1 + \sin(\frac{\pi}{2} - 2x)} dx$$

1M

$$= \int_0^{\frac{\pi}{4}} \frac{1 - \cos 2x}{1 + \cos 2x} dx + \int_0^{\frac{\pi}{4}} \frac{x - \frac{\pi}{4}}{1 + \cos 2x} dx$$

$$= \int_0^{\frac{\pi}{4}} \tan^2 x dx + \int_0^{\frac{\pi}{4}} \frac{x - \frac{\pi}{4}}{2 \cos^2 x} dx$$

1M

$$= \int_0^{\frac{\pi}{4}} (\sec^2 x - 1) dx + \frac{1}{2} \int_0^{\frac{\pi}{4}} x \sec^2 x dx - \frac{\pi}{8} \int_0^{\frac{\pi}{4}} \sec^2 x dx$$

1M

$$= \left[\tan x - x \right]_0^{\frac{\pi}{4}} + \frac{1}{2} \left[x \tan x + \ln(\cos x) \right]_0^{\frac{\pi}{4}} - \frac{\pi}{8} \left[\tan x \right]_0^{\frac{\pi}{4}}$$

1M

$$= 1 - \frac{\pi}{4} - \frac{1}{4} \ln 2$$

1A

81. (a) $\int_0^\pi \cos^4 x \, dx$

$$= \int_0^\pi \left(\frac{1 + \cos 2x}{2} \right)^2 dx \quad 1M$$

$$= \int_0^\pi \left(\frac{1}{4} + \frac{\cos 2x}{2} + \frac{\cos^2 2x}{8} \right) dx$$

$$= \int_0^\pi \left(\frac{1}{4} + \frac{\cos 2x}{2} + \frac{1 + \cos 4x}{16} \right) dx \quad 1M$$

$$= \left[\frac{3x}{8} + \frac{\sin 2x}{4} + \frac{\sin 4x}{32} \right]_0^\pi$$

$$= \frac{3\pi}{8} \quad 1A$$

(b) 設 $u = k - x$ 。則 $du = -dx$ 。 1M

$$\begin{aligned} \int_0^k f(x)g(x) \, dx &= - \int_k^0 f(k-u)g(k-u) \, du \\ &= \int_0^k f(u)g(k-u) \, du \\ &= \int_0^k f(x)g(k-x) \, dx \\ &= \frac{1}{2} \left[\int_0^k f(x)g(x) \, dx + \int_0^k f(x)g(k-x) \, dx \right] \quad 1M \\ &= \frac{1}{2} \int_0^k f(x)[g(x) + g(k-x)] \, dx \\ &= \frac{a}{2} \int_0^k f(x) \, dx \quad 1 \end{aligned}$$

(c) (i) 代 $f(x) = \cos^4 x$ 及 $g(x) = \frac{1}{1 + e^{\sin x}}$ 。 1M

$$f(2\pi - x) = \cos^4(2\pi - x) = \cos^4 x = f(x)$$

$$g(x) + g(2\pi - x) = \frac{1}{1 + e^{\sin x}} + \frac{1}{1 + e^{\sin(2\pi - x)}}$$

$$= \frac{1}{1 + e^{\sin x}} + \frac{1}{1 + e^{-\sin x}} \times \frac{e^{\sin x}}{e^{\sin x}}$$

$$= \frac{1}{1 + e^{\sin x}} + \frac{e^{\sin x}}{1 + e^{\sin x}}$$

$$= 1 \quad 1M$$

設 $u = 2\pi - x$ 。則 $du = -dx$ 。

$$\begin{aligned}\int_0^{2\pi} \frac{\cos^4 x}{1 + e^{\sin x}} dx &= \frac{1}{2} \int_0^{2\pi} \cos^4 x dx \\ &= \frac{1}{2} \left(\int_0^{\pi} \cos^4 x dx + \int_{\pi}^{2\pi} \cos^4 x dx \right) \\ &= \frac{1}{2} \int_0^{\pi} \cos^4 x dx - \frac{1}{2} \int_{\pi}^0 \cos^4(2\pi - u) du \\ &= \frac{1}{2} \int_0^{\pi} \cos^4 x dx + \frac{1}{2} \int_0^{\pi} \cos^4 u du \\ &= \frac{1}{2} \left(\frac{3\pi}{8} \right) + \frac{1}{2} \left(\frac{3\pi}{8} \right) \\ &= \frac{3\pi}{8}\end{aligned}$$

1M

1A

$$\begin{aligned}\text{(ii)} \quad \int_0^{2\pi} \frac{e^{\sin x} \cos^4 x}{1 + e^{\sin x}} dx &= \int_0^{2\pi} \left[\frac{(1 + e^{\sin x}) \cos^4 x}{1 + e^{\sin x}} - \frac{\cos^4 x}{1 + e^{\sin x}} \right] dx \\ &= \int_0^{2\pi} \cos^4 x dx - \int_0^{2\pi} \frac{\cos^4 x}{1 + e^{\sin x}} dx \\ &= 2 \left(\frac{3\pi}{8} \right) - \frac{3\pi}{8} \\ &= \frac{3\pi}{8}\end{aligned}$$

1M

1A

82. (a) 設 $u = k - x$ 。則 $du = -dx$ 。

1M

$$\begin{aligned}\int_0^k x f(x) dx &= - \int_k^0 (k - u) f(\pi - u) du \\ &= \int_0^k (k - u) f(u) du \\ &= \int_0^k (k - x) f(x) dx\end{aligned}$$

1

代 $k = \pi$ 。

$$\begin{aligned}\int_0^{\pi} x f(x) dx &= \int_0^{\pi} (\pi - x) f(x) dx \\ &= \frac{1}{2} \left[\int_0^{\pi} (\pi - x) f(x) dx + \int_0^{\pi} x f(x) dx \right] \\ &= \frac{\pi}{2} \int_0^{\pi} f(x) dx\end{aligned}$$

1M

1

(b) 設 $u = \cos x$ 。則 $du = -\sin x dx$ 。

1M

$$\begin{aligned}\int_0^{\pi} \frac{\sin x}{1 + \cos^2 x} dx &= - \int_1^{-1} \frac{du}{1 + u^2} \\ &= \int_{-1}^1 \frac{du}{1 + u^2}\end{aligned}$$

1A

設 $u = \tan \theta$ 。則 $du = \sec^2 \theta d\theta$ 。 1M

$$\begin{aligned} \int_0^{\pi} \frac{\sin x}{1 + \cos^2 x} dx &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sec^2 \theta}{1 + \tan^2 \theta} d\theta & 1A \\ &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} d\theta \\ &= \left[\theta \right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \\ &= \frac{\pi}{2} & 1A \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad & \frac{\cos(\frac{\pi}{6} - x) - \cos(\frac{\pi}{6} + x)}{3 + \cos 2x} \\ &= \frac{-2 \sin \frac{\pi}{6} \sin(-x)}{3 + (2 \cos^2 x - 1)} & 1M \end{aligned}$$

$$\begin{aligned} &= \frac{1}{2} \cdot \frac{\sin x}{1 + \cos^2 x} \\ \text{代 } f(x) &= \frac{\sin x}{1 + \cos^2 x}。 & 1M \end{aligned}$$

$$\text{則 } f(\pi - x) = \frac{\sin(\pi - x)}{1 + \cos^2(\pi - x)} = \frac{\sin x}{1 + \cos^2 x} = f(x)。$$

$$\begin{aligned} & \int_0^{\pi} x \frac{[\cos(\frac{\pi}{6} - x) - \cos(\frac{\pi}{6} + x)]}{3 + \cos 2x} dx \\ &= \frac{1}{2} \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx \\ &= \frac{1}{2} \cdot \frac{\pi}{2} \int_0^{\pi} \frac{\sin x}{1 + \cos^2 x} dx \\ &= \frac{\pi}{4} \cdot \frac{\pi}{2} \\ &= \frac{\pi^2}{8} & 1A \end{aligned}$$

83. (a) 設 $u = \pi - x$ 。則 $du = -dx$ 。 1M

$$\begin{aligned} \int_0^{\pi} f(\pi - x) dx &= - \int_{\pi}^0 f(u) du \\ &= \int_0^{\pi} f(u) du & 1M \\ &= \int_0^{\pi} f(x) dx & 1 \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad & \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx = \int_0^{\pi} \frac{(\pi - x) \sin(\pi - x)}{1 + \cos^2(\pi - x)} dx & 1M \\ &= \int_0^{\pi} \frac{(\pi - x) \sin x}{1 + \cos^2 x} dx \\ &= \frac{1}{2} \left[\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx + \int_0^{\pi} \frac{(\pi - x) \sin x}{1 + \cos^2 x} dx \right] & 1M \\ &= \frac{\pi}{2} \int_0^{\pi} \frac{\sin x}{1 + \cos^2 x} dx & 1A \end{aligned}$$

(c) (i) 設 $x = \tan \theta$ 。則 $dx = \sec^2 \theta d\theta$ 。 1M

$$\begin{aligned}\int \frac{1}{1+x^2} dx &= \int \frac{\sec^2 \theta}{1+\tan^2 \theta} d\theta \\ &= \int d\theta \\ &= \theta + \text{常數} \\ &= \tan^{-1} x + \text{常數}\end{aligned}$$

1A

(ii)
$$\begin{aligned}\int_0^\pi \frac{(x + \cos^5 x) \sin x}{1 + \cos^2 x} dx \\ &= \int_0^\pi \frac{x \sin x}{1 + \cos^2 x} dx + \int_0^\pi \frac{\cos^5 x}{1 + \cos^2 x} dx \\ &= \frac{\pi}{2} \int_0^\pi \frac{\sin x}{1 + \cos^2 x} dx + \int_0^\pi \frac{\cos^5 x \sin x}{1 + \cos^2 x} dx\end{aligned}$$
 1M

由於 $\frac{\cos^5(-x) \sin(-x)}{1 + \cos^2(-x)} = \frac{-\cos^5 x \sin x}{1 + \cos^2 x}$ ，它是奇函數。

設 $u = \cos x$ 。則 $du = -\sin x dx$ 。 1M

$$\begin{aligned}\int_0^\pi \frac{(x + \cos^5 x) \sin x}{1 + \cos^2 x} dx \\ &= -\frac{\pi}{2} \int_1^{-1} \frac{du}{1+u^2} + 0 \\ &= \frac{\pi}{2} \int_{-1}^1 \frac{du}{1+u^2} \\ &= \frac{\pi}{2} \left[\tan^{-1} u \right]_{-1}^1 \\ &= \frac{\pi^2}{4}\end{aligned}$$

1A

84. (a)
$$\begin{aligned}\int g(x) dx &= \sin^2 x \left(\frac{\sin 2x}{2} \right) - \frac{1}{2} \int (2 \sin x \cos x) \sin 2x dx \\ &= \frac{1}{2} \sin^2 x \sin 2x - \frac{1}{2} \int \sin^2 2x dx\end{aligned}$$
 1M

1

(b)
$$\begin{aligned}\int_0^\pi g(x) dx &= \frac{1}{2} \left[\sin^2 x \sin 2x \right]_0^\pi - \frac{1}{2} \int_0^\pi \sin^2 2x dx \\ &= 0 - \frac{1}{4} \int_0^\pi (1 - \cos 4x) dx \\ &= -\frac{1}{4} \left[x - \frac{\sin 4x}{4} \right]_0^\pi \\ &= -\frac{\pi}{4}\end{aligned}$$
 1M

1A

(c) 設 $u = \pi - x$ 。則 $du = -dx$ 。 1M

$$\begin{aligned}
 \int_0^{\pi} xg(x) dx &= - \int_{\pi}^0 (\pi - u)g(\pi - u) du \\
 &= \int_0^{\pi} (\pi - u) \sin^2(\pi - u) \cos(2\pi - 2u) du & 1M \\
 &= \int_0^{\pi} (\pi - x) \sin^2 x \cos 2x dx \\
 &= \pi \int_0^{\pi} g(x) dx - \int_0^{\pi} xg(x) dx \\
 &= \frac{\pi}{2} \int_0^{\pi} g(x) dx & 1M \\
 &= \frac{\pi}{2} \left(-\frac{\pi}{4} \right) \\
 &= -\frac{\pi^2}{8} & 1A
 \end{aligned}$$

(d) 留意 $(-x)g(-x) = -x \sin^2(-x) \cos(-2x) = -x \sin^2 \cos 2x = -xg(x)$ 。 1M

因此， $xg(x)$ 為奇函數。

設 $u = x - \pi$ 。則 $du = dx$ 。 1M

$$\begin{aligned}
 \int_{-\pi}^{2\pi} xg(x) dx &= \int_{-\pi}^{\pi} xg(x) dx + \int_{\pi}^{2\pi} xg(x) dx \\
 &= 0 + \int_{\pi}^{2\pi} xg(x) dx \\
 &= \int_0^{\pi} (u + \pi) \sin^2(u + \pi) \cos(2u + 2\pi) du \\
 &= \int_0^{\pi} (\pi + x)g(x) dx & 1M \\
 &= \pi \int_0^{\pi} g(x) dx + \int_0^{\pi} xg(x) dx \\
 &= \pi \left(-\frac{\pi}{4} \right) + \left(-\frac{\pi^2}{8} \right) \\
 &= -\frac{3\pi^2}{8} & 1A
 \end{aligned}$$

$$\begin{aligned}
 85. (a) \quad \sin x + \sin x \tan^2 \frac{x}{2} &= 2 \sin \frac{x}{2} \cos \frac{x}{2} \left(1 + \frac{\sin^2 \frac{x}{2}}{\cos^2 \frac{x}{2}} \right) \\
 &= 2 \tan \frac{x}{2} \left(\cos^2 \frac{x}{2} + \sin^2 \frac{x}{2} \right) \\
 &= 2 \tan \frac{x}{2}
 \end{aligned}$$

1

$$(b) \quad \text{留意 } x^2 + ax + a^2 = \left(x + \frac{a}{2} \right)^2 + \frac{3a^2}{4}。$$

$$\text{設 } x + \frac{a}{2} = \frac{\sqrt{3}a}{2} \tan \theta \circ \text{ 則 } dx = \frac{\sqrt{3}a}{2} \sec^2 \theta d\theta \circ \quad 1M$$

$$\begin{aligned} \int_0^a \frac{dx}{x^2 + ax + a^2} &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\frac{\sqrt{3}a}{2} \sec^2 \theta}{\frac{3a^2}{4} \tan^2 \theta + \frac{3a^2}{4}} d\theta \\ &= \frac{2}{\sqrt{3}a} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} d\theta \\ &= \frac{2}{\sqrt{3}a} \left[\theta \right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} \\ &= \frac{\pi}{3\sqrt{3}a} \\ &= \frac{\sqrt{3}\pi}{9a} \end{aligned} \quad 1M \quad 1A$$

(c) 利用 (a) 的結果。

$$\begin{aligned} \sin x + \sin x \tan^2 \frac{x}{2} &= 2 \tan \frac{x}{2} \\ \sin x &= \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} \end{aligned}$$

$$\text{設 } t = \tan \frac{x}{2} \circ \text{ 則 } dt = \frac{1}{2} \sec^2 \frac{x}{2} dx = \frac{1+t^2}{2} dx \circ \quad 1M$$

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \frac{dx}{\sin x + 2} &= \int_0^1 \frac{1}{\frac{2t}{1+t^2} + 2} \cdot \frac{2}{1+t^2} dt \\ &= \int_0^1 \frac{dt}{t^2 + t + 1} \\ &= \frac{\sqrt{3}\pi}{9(1)} \\ &= \frac{\sqrt{3}\pi}{9} \end{aligned} \quad 1M \quad 1A$$

$$\begin{aligned} \text{(d)} \quad \frac{d}{dx} \left(\frac{\cos x}{\sin x + 2} \right) &= \frac{(-\sin x)(\sin x + 2) - \cos x(\cos x)}{(\sin x + 2)^2} \\ &= \frac{-2 \sin x - 1}{(\sin x + 2)^2} \\ &= \frac{-2(\sin x + 2) + 3}{(\sin x + 2)^2} \\ &= -\frac{2}{\sin x + 2} + \frac{3}{(\sin x + 2)^2} \end{aligned} \quad 1M$$

兩邊對 x 積分。

$$\begin{aligned} \left[\frac{\cos x}{\sin x + 2} \right]_0^{\frac{\pi}{2}} &= -2 \int_0^{\frac{\pi}{2}} \frac{dx}{\sin x + 2} + 3 \int_0^{\frac{\pi}{2}} \frac{dx}{(\sin x + 2)^2} \\ \frac{1}{2} - 0 &= -2 \left(\frac{\sqrt{3}\pi}{9} \right) + 3 \int_0^{\frac{\pi}{2}} \frac{dx}{(\sin x + 2)^2} \\ \int_0^{\frac{\pi}{2}} \frac{dx}{(\sin x + 2)^2} &= \frac{2\sqrt{3}\pi}{27} - \frac{1}{6} \end{aligned} \quad 1M$$

$$\text{留意 } \frac{1}{[\cos(-x) + 2]^2} = \frac{1}{(\cos x + 2)^2} \circ$$

因此， $\frac{1}{(\cos x + 2)^2}$ 為偶函數。

設 $u = \frac{\pi}{2} - x$ 。則 $du = -dx$ 。

1M

$$\begin{aligned} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{dx}{(\cos x + 2)^2} &= 2 \int_0^{\frac{\pi}{2}} \frac{dx}{(\cos x + 2)^2} \\ &= -2 \int_{-\frac{\pi}{2}}^0 \frac{du}{[\cos(\frac{\pi}{2} - u) + 2]^2} \\ &= 2 \int_0^{\frac{\pi}{2}} \frac{du}{(\sin u + 2)^2} \\ &= 2 \left(\frac{2\sqrt{3}\pi}{27} - \frac{1}{6} \right) \\ &= \frac{4\sqrt{3}\pi}{27} - \frac{1}{3} \end{aligned}$$

1M

1M

1A

$$\begin{aligned} 86. \quad (a) \quad \int_0^{\pi} x^2 \cos^2 x \, dx &= \frac{1}{2} \int_0^{\pi} x^2 (1 + \cos 2x) \, dx \\ &= \left[\frac{x^3}{6} \right]_0^{\pi} + \frac{1}{4} \left[x^2 \sin 2x \right]_0^{\pi} - \frac{1}{2} \int_0^{\pi} x \sin 2x \, dx \\ &= \frac{\pi^3}{6} + 0 + \frac{1}{4} \left[x \cos 2x \right]_0^{\pi} - \frac{1}{4} \int_0^{\pi} \cos 2x \, dx \\ &= \frac{\pi^3}{6} + \frac{\pi}{4} - \frac{1}{8} \left[\sin 2x \right]_0^{\pi} \\ &= \frac{\pi^3}{6} + \frac{\pi}{4} \end{aligned}$$

1M

1M

1M

1A

(b) (i) 設 $u = -x$ 。則 $du = -dx$ 。

1M

$$\begin{aligned} \int_{-a}^a f(x) \, dx &= \int_{-a}^0 f(x) \, dx + \int_0^a f(x) \, dx \\ &= - \int_a^0 f(-u) \, du + \int_0^a f(x) \, dx \\ &= \int_0^a f(-u) \, du + \int_0^a f(x) \, dx \\ &= \int_0^a [f(x) + f(-x)] \, dx \end{aligned}$$

1

(ii) 設 $u = a - x$ 。則 $du = -dx$ 。

1M

$$\begin{aligned} \int_0^a xg(x) \, dx &= - \int_a^0 (a - u)g(a - u) \, du \\ &= \int_0^a (a - u)g(u) \, du \\ &= a \int_0^a g(x) \, dx - \int_0^a xg(x) \, dx \\ &= \frac{a}{2} \int_0^a g(x) \, dx \end{aligned}$$

1

(c) 代 $f(x) = \ln(1 + e^x)(\sin^3 x + x) \cos^2 x$ 。

$$\begin{aligned} f(x) + f(-x) &= \ln(1 + e^x)(\sin^3 x + x) \cos^2 x + \ln(1 + e^{-x})(\sin^3(-x) - x) \cos^2(-x) \\ &= \ln(1 + e^x)(\sin^3 x + x) \cos^2 x - \ln\left(\frac{e^x + 1}{e^x}\right)(\sin^3 x + x) \cos^2 x \\ &= \ln(e^x)(\sin^3 x + x) \cos^2 x \\ &= x(\sin^3 x + x) \cos^2 x \end{aligned}$$

利用 (b)(i) 的結果。

$$\begin{aligned} \int_{-\pi}^{\pi} \ln(1 + e^x)(\sin^3 x + x) \cos^2 x \, dx &= \int_0^{\pi} x(\sin^3 x + x) \cos^2 x \, dx & 1M \\ &= \int_0^{\pi} x \sin^3 x \cos^2 x \, dx + \int_0^{\pi} x^2 \cos^2 x \, dx \\ &= \int_0^{\pi} x \sin^3 x \cos^2 x \, dx + \frac{\pi^3}{6} + \frac{\pi}{4} \end{aligned}$$

代 $g(x) = \sin^3 x \cos^2 x$ 。則 $g(\pi - x) = \sin^3(\pi - x) \cos^2(\pi - x) = \sin^3 x \cos^2 x = g(x)$ 。

設 $u = \cos x$ 。則 $du = -\sin x \, dx$ 。 & 1M

$$\begin{aligned} \int_0^{\pi} x \sin^3 x \cos^2 x \, dx &= \frac{\pi}{2} \int_0^{\pi} \sin^3 x \cos^2 x \, dx & 1M \\ &= -\frac{\pi}{2} \int_1^{-1} (1 - u^2)u^2 \, du \\ &= \frac{\pi}{2} \int_{-1}^1 (u^2 - u^4) \, du \\ &= \frac{\pi}{2} \left[\frac{u^3}{3} - \frac{u^5}{5} \right]_{-1}^1 \\ &= \frac{2\pi}{15} \end{aligned}$$

結合各結果。

$$\begin{aligned} \int_{-\pi}^{\pi} \ln(1 + e^x)(\sin^3 x + x) \cos^2 x \, dx &= \frac{2\pi}{15} + \left(\frac{\pi^3}{6} + \frac{\pi}{4} \right) \\ &= \frac{\pi^3}{6} + \frac{23\pi}{60} & 1A \end{aligned}$$

$$\begin{aligned} 87. \quad (a) \quad \frac{\sin 3\theta}{\sin \theta} &= \frac{\sin 2\theta \cos \theta + \cos 2\theta \sin \theta}{\sin \theta} & 1M \\ &= \frac{2 \sin \theta \cos^2 \theta + \cos 2\theta \sin \theta}{\sin \theta} \\ &= (1 + \cos 2\theta) + \cos 2\theta \\ &= 2 \cos 2\theta + 1 & 1 \end{aligned}$$

$$(b) \quad \frac{\sin \left[3 \left(x + \frac{\pi}{4} \right) \right]}{\sin \left(x + \frac{\pi}{4} \right)} = \frac{\sin 3x \cos \frac{3\pi}{4} + \cos 3x \sin \frac{3\pi}{4}}{\sin x \cos \frac{\pi}{4} + \cos x \sin \frac{\pi}{4}} \quad 1M$$

$$= \frac{-\sin 3x \cdot \frac{\sqrt{2}}{2} + \cos 3x \cdot \frac{\sqrt{2}}{2}}{\sin x \cdot \frac{\sqrt{2}}{2} + \cos x \cdot \frac{\sqrt{2}}{2}}$$

$$= \frac{\cos 3x - \sin 3x}{\cos x + \sin x} \quad 1$$

(c) (i) 設 $u = \frac{\pi}{2} - x$ 。則 $du = -dx$ 。

$$\int_0^{\frac{\pi}{2}} \frac{\cos 3x}{\cos x + \sin x} dx = - \int_{\frac{\pi}{2}}^0 \frac{\cos \left(\frac{3\pi}{2} - 3u \right)}{\cos \left(\frac{\pi}{2} - u \right) + \sin \left(\frac{\pi}{2} - u \right)} du$$

$$= \int_{\frac{\pi}{2}}^0 \frac{\sin 3u}{\sin u + \cos u} du$$

$$= \int_{\frac{\pi}{2}}^0 \frac{\sin 3x}{\cos x + \sin x} dx \quad 1$$

(ii) $\int_0^{\frac{\pi}{2}} \frac{\cos 3x}{\cos x + \sin x} dx$

$$= \frac{1}{2} \left[\int_0^{\frac{\pi}{2}} \frac{\cos 3x}{\cos x + \sin x} dx + \int_{\frac{\pi}{2}}^0 \frac{\sin 3x}{\cos x + \sin x} dx \right] \quad 1M$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{\cos 3x}{\cos x + \sin x} dx - \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{\sin 3x}{\cos x + \sin x} dx$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{\cos 3x - \sin 3x}{\cos x + \sin x} dx$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{\sin \left(3x + \frac{3\pi}{4} \right)}{\sin \left(x + \frac{\pi}{4} \right)} dx$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} \left[2 \cos \left(2x + \frac{\pi}{2} \right) + 1 \right] dx \quad 1M$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} (1 - 2 \sin 2x) dx$$

$$= \frac{1}{2} \left[x + \cos 2x \right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{4} - 1 \quad 1A$$

88. (a) 設 $u = a - x$ 。則 $du = -dx$ 。

$$\int_0^a f(a-x) dx = - \int_a^0 f(u) du$$

$$= \int_0^a f(u) du \quad 1M$$

$$= \int_0^a f(x) dx \quad 1$$

$$(b) \quad (i) \quad I = \int_0^{\pi} \frac{(\pi-x) \sin(\pi-x)}{1+\cos^2(\pi-x)} dx \quad 1M$$

$$= \int_0^{\pi} \frac{(\pi-x) \sin x}{1+\cos^2 x} dx \quad 1$$

(ii) 利用 (b)(i) 的結果。

$$I = \frac{1}{2} \left[\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx + \int_0^{\pi} \frac{(\pi-x) \sin x}{1+\cos^2 x} dx \right] \quad 1M$$

$$= \frac{\pi}{2} \int_0^{\pi} \frac{\sin x}{1+\cos^2 x} dx$$

設 $u = \pi - x$ 。則 $du = -dx$ 。 1M

$$I = \frac{\pi}{2} \left[\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x} dx + \int_{\frac{\pi}{2}}^{\pi} \frac{\sin x}{1+\cos^2 x} dx \right]$$

$$= \frac{\pi}{2} \left[\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x} dx - \int_{\frac{\pi}{2}}^0 \frac{\sin(\pi-u)}{1+\cos^2(\pi-u)} du \right]$$

$$= \frac{\pi}{2} \left[\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x} dx + \int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x} dx \right]$$

$$= \pi \int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x} dx \quad 1$$

(iii) 設 $u = \cos x$ 。則 $du = -\sin x dx$ 。 1M

$$I = \pi \int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x} dx$$

$$= -\pi \int_1^{-1} \frac{du}{1+u^2}$$

$$= \pi \int_{-1}^1 \frac{du}{1+u^2}$$

設 $u = \tan \theta$ 。則 $du = \sec^2 \theta d\theta$ 。 1M

$$I = \pi \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sec^2 \theta}{1+\tan^2 \theta} d\theta$$

$$= \pi \left[\theta \right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$= \frac{\pi^2}{4} \quad 1$$

89. (a) 設 $u = \frac{\pi}{4} - x$ 。則 $du = -dx$ 。 1M

$$\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln \left[\cos \left(\frac{\pi}{4} - x \right) \right] dx = - \int_{\frac{\pi}{6}}^{\frac{\pi}{12}} \ln \cos(u) du$$

$$= \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln \cos(u) du \quad 1M$$

$$= \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln(\cos x) dx \quad 1$$

$$(b) \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln \left[\cos \left(\frac{\pi}{4} - x \right) \right] dx - \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln(\cos x) dx = 0$$

$$\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln \frac{\cos \frac{\pi}{4} \cos x + \sin \frac{\pi}{4} \sin x}{\cos x} dx = 0 \quad 1M$$

$$\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln \frac{1 + \tan x}{\sqrt{2}} dx = 0$$

$$\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln(\tan x + 1) dx = \left(\frac{1}{2} \ln 2 \right) \left[x \right]_{\frac{\pi}{12}}^{\frac{\pi}{6}} \quad 1M$$

$$\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln(\tan x + 1) dx = \frac{\pi}{24} \ln 2 \quad 1A$$

$$(c) \quad (i) \quad \tan \frac{\pi}{12} = \tan \left(\frac{\pi}{3} - \frac{\pi}{4} \right)$$

$$= \frac{\tan \frac{\pi}{3} - \tan \frac{\pi}{4}}{1 + \tan \frac{\pi}{3} \tan \frac{\pi}{4}} \quad 1M$$

$$= \frac{\sqrt{3} - 1}{1 + \sqrt{3}} \times \frac{\sqrt{3} - 1}{\sqrt{3} - 1}$$

$$= 2 - \sqrt{3} \quad 1$$

$$(ii) \quad \text{留意 } \frac{d}{dx} \ln(\tan x + 1) = \frac{\sec^2 x}{\tan x + 1} \quad 1M$$

$$\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \frac{x \sec^2 x}{\tan x + 1} dx$$

$$= \left[x \ln(\tan x + 1) \right]_{\frac{\pi}{12}}^{\frac{\pi}{6}} - \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln(\tan x + 1) dx \quad 1M$$

$$= \frac{\pi}{6} \ln \left(\tan \frac{\pi}{6} + 1 \right) - \frac{\pi}{12} \ln \left(\tan \frac{\pi}{12} + 1 \right) - \frac{\pi}{24} \ln 2 \quad 1M$$

$$= \frac{\pi}{24} \ln \frac{(\sqrt{3} + 3)^4}{3^4} - \frac{\pi}{24} \ln (3 - \sqrt{3})^2 - \frac{\pi}{24} \ln 2$$

$$= \frac{\pi}{24} \ln \left[\frac{(3 + \sqrt{3})^4}{3^4(2)(3 - \sqrt{3})^2} \times \frac{(3 + \sqrt{3})^2}{(3 + \sqrt{3})^2} \right] \quad 1M$$

$$= \frac{\pi}{24} \ln \left[\frac{(3 + \sqrt{3})^6}{(3^6)(2^3)} \right]$$

$$= \frac{\pi}{4} \ln \frac{3 + \sqrt{3}}{3\sqrt{2}}$$

$$= \frac{\pi}{4} \ln \frac{\sqrt{6} + 3\sqrt{2}}{6} \quad 1$$

90. (a) 設 $u = \cos \pi x$ 。則 $du = -\pi \sin \pi x \, dx$ 。 1M

$$\begin{aligned} \int \cos^4 \pi x \sin^3 \pi x \, dx &= -\frac{1}{\pi} \int u^4 (1 - u^2) \, du & 1M \\ &= -\frac{1}{\pi} \int (u^4 - u^6) \, du \\ &= -\frac{u^5}{5\pi} + \frac{u^7}{7\pi} + \text{常數} \\ &= -\frac{\cos^5 \pi x}{5\pi} + \frac{\cos^7 \pi x}{7\pi} + \text{常數} & 1A \end{aligned}$$

(b) 設 $u = 3 - x$ 。則 $du = -dx$ 。 1M

$$\begin{aligned} \int_0^3 x \cos^4 \pi x \sin^3 \pi x \, dx &= -\int_3^0 (3 - u) \cos^4 (3\pi - \pi u) \sin^3 (3\pi - \pi u) \, du \\ &= \int_0^3 (3 - x) \cos^4 \pi x \sin^3 \pi x \, dx \\ 2 \int_0^3 x \cos^4 \pi x \sin^3 \pi x \, dx &= 3 \int_0^3 \cos^4 \pi x \sin^3 \pi x \, dx & 1M \\ \int_0^3 x \cos^4 \pi x \sin^3 \pi x \, dx &= \frac{3}{2} \left[-\frac{\cos^5 \pi x}{5\pi} + \frac{\cos^7 \pi x}{7\pi} \right]_0^3 & 1M \\ &= \frac{6}{35\pi} & 1A \end{aligned}$$

91. (a) 設 $u = k - x$ 。則 $du = -dx$ 。

$$\begin{aligned} \int_0^k \ln(1 + \tan k \tan x) \, dx &= -\int_k^0 \ln[1 + \tan k \tan(k - u)] \, du \\ &= \int_0^k \ln \left[1 + \tan k \cdot \frac{\tan k - \tan u}{1 + \tan k \tan u} \right] \, du & 1M \\ &= \int_0^k \ln \left[\frac{1 + \tan k \tan u + \tan k (\tan k - \tan u)}{1 + \tan k \tan u} \right] \, du \\ &= \int_0^k \ln \frac{\sec^2 k}{1 + \tan k \tan u} \, du & 1M \\ &= \int_0^k \ln(\sec^2 k) \, dx - \int_0^k \ln(1 + \tan k \tan x) \, dx \\ 2 \int_0^k \ln(1 + \tan k \tan x) \, dx &= 2 \ln(\sec k) \left[x \right]_0^k & 1M \\ \int_0^k \ln(1 + \tan k \tan x) \, dx &= k \ln(\sec k) & 1 \end{aligned}$$

(b) 代 $x = \frac{\pi}{4}$ 。

$$\begin{aligned} \int_0^{\frac{\pi}{4}} \ln \left(1 + \tan \frac{\pi}{4} \tan x \right) \, dx &= \frac{\pi}{4} \ln \left(\sec \frac{\pi}{4} \right) & 1M \\ \int_0^{\frac{\pi}{4}} \ln(1 + \tan x) \, dx &= \frac{\pi}{4} \ln \sqrt{2} \\ &= \frac{\pi \ln 2}{8} & 1A \end{aligned}$$

$$(c) \quad \frac{d}{dx} [\ln(1 + \tan x)] = \frac{\sec^2 x}{1 + \tan x} \quad 1A$$

$$(d) \quad \text{設 } x = \tan \theta \circ \text{則 } dx = \sec^2 \theta d\theta \circ \quad 1M$$

$$\int_0^1 \frac{\tan^{-1} x}{1+x} dx = \int_0^{\frac{\pi}{4}} \frac{\theta}{1+\tan \theta} \sec^2 \theta d\theta \quad 1M$$

$$= \left[\theta \ln(1 + \tan \theta) \right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \ln(1 + \tan \theta) d\theta \quad 1M$$

$$= \frac{\pi}{4} \ln 2 - \frac{\pi \ln 2}{8}$$

$$= \frac{\pi \ln 2}{8} \quad 1A$$

$$92. (a) \quad \int x e^x dx = x e^x - \int e^x dx \quad 1M$$

$$= x e^x - e^x + \text{常數} \quad 1A$$

$$(b) \quad \text{設 } u = x^2 \circ \text{則 } du = 2x dx \circ \quad 1M$$

$$\int_0^1 x^3 e^{x^2} dx = \frac{1}{2} \int_0^1 u e^u du$$

$$= \frac{1}{2} \left[u e^u - e^u \right]_0^1 \quad 1M$$

$$= \frac{1}{2} \quad 1A$$

$$(c) \quad \text{設 } f(x) = x^5 e^{x^4} \circ$$

$$f(-x) = (-x)^5 e^{(-x)^4} = -x^5 e^{x^4} = -f(x) \quad 1M$$

$f(x)$ 為奇函數。

$$\text{因此, } \int_{-9}^9 x^5 e^{x^4} dx = 0 \circ \quad 1A$$

$$93. (a) \quad \text{設 } x = 3 \tan \theta \circ \text{則 } dx = 3 \sec^2 \theta d\theta \circ \quad 1M$$

$$\int_0^1 \frac{1}{x^2 + 9} dx = \int_0^{\tan^{-1} \frac{1}{3}} \frac{3 \sec^2 \theta}{9 \tan^2 \theta + 9} d\theta$$

$$= \frac{1}{3} \left[\theta \right]_0^{\tan^{-1} \frac{1}{3}} \quad 1M$$

$$= \frac{1}{3} \tan^{-1} \frac{1}{3} \quad 1A$$

$$(b) (i) \quad \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} = \frac{1 - \frac{\sin^2 \theta}{\cos^2 \theta}}{1 + \frac{\sin^2 \theta}{\cos^2 \theta}} \times \frac{\cos^2 \theta}{\cos^2 \theta}$$

$$= \frac{\cos^2 \theta - \sin^2 \theta}{\cos^2 \theta + \sin^2 \theta}$$

$$= \cos 2\theta$$

1

(ii) 設 $t = \tan \theta$ 。則 $dt = \sec^2 \theta d\theta = (1 + t^2) d\theta$ 。 1M

$$\begin{aligned} \int_0^{\frac{\pi}{4}} \frac{1}{4 \cos 2\theta + 5} d\theta &= \int_0^1 \frac{1}{4 \left(\frac{1-t^2}{1+t^2} \right) + 5} \cdot \frac{1}{1+t^2} dt \\ &= \int_0^1 \frac{1}{t^2 + 9} dt \\ &= \frac{1}{3} \tan^{-1} \frac{1}{3} \end{aligned} \quad \begin{array}{l} 1M \\ 1A \end{array}$$

(c) 設 $u = \frac{\pi}{4} - \theta$ 。則 $du = -d\theta$ 。 1M

$$\begin{aligned} \int_0^{\frac{\pi}{4}} \frac{1}{4 \cos 2\theta + 5} d\theta &= - \int_{\frac{\pi}{4}}^0 \frac{1}{4 \cos \left(\frac{\pi}{4} - 2u \right) + 5} du \\ &= \int_0^{\frac{\pi}{4}} \frac{1}{4 \sin 2u + 5} du \\ &= \int_0^{\frac{\pi}{4}} \frac{1}{4 \sin 2\theta + 5} d\theta \end{aligned} \quad \begin{array}{l} 1 \end{array}$$

$$\begin{aligned} \text{(d)} \quad \int_0^{\frac{\pi}{4}} \frac{4 \sin 2\theta + 4 \cos 2\theta + 10}{(4 \sin 2\theta + 5)(4 \cos 2\theta + 5)} d\theta \\ &= \int_0^{\frac{\pi}{4}} \frac{1}{\cos 2\theta + 5} d\theta + \int_0^{\frac{\pi}{4}} \frac{1}{4 \sin 2\theta + 5} d\theta \\ &= 2 \int_0^{\frac{\pi}{4}} \frac{1}{4 \cos 2\theta + 5} d\theta \\ &= 2 \left(\frac{1}{3} \tan^{-1} \frac{1}{3} \right) \\ &= \frac{2}{3} \tan^{-1} \frac{1}{3} \end{aligned} \quad \begin{array}{l} 1M \\ 1M \\ 1A \end{array}$$

94. (a) 設 $u = a \tan \theta$ 。則 $du = a \sec^2 \theta d\theta$ 。 1M

$$\begin{aligned} \int \frac{u^2}{a^2 + u^2} du &= \int \frac{a^2 \tan^2 \theta}{a^2 + a^2 \tan^2 \theta} \cdot a \sec^2 \theta d\theta \\ &= a \int \tan^2 \theta d\theta \\ &= a \int (\sec^2 \theta - 1) d\theta \\ &= a \tan \theta - a\theta + \text{常數} \\ &= u - a \tan^{-1} \frac{u}{a} + \text{常數} \end{aligned} \quad \begin{array}{l} 1M \\ 1A \end{array}$$

(b) 設 $u = -x$ 。則 $du = -dx$ 。 1M

$$\begin{aligned} & \int_{-b}^b g(x) \log_{25} (5^{h(x)} + 1) dx \\ &= \int_{-b}^0 g(x) \log_{25} (5^{h(x)} + 1) dx + \int_0^b g(x) \log_{25} (5^{h(x)} + 1) dx \end{aligned} \quad 1M$$

$$\begin{aligned} &= - \int_b^0 g(-u) \log_{25} (5^{h(-u)} + 1) du + \int_0^b g(x) \log_{25} (5^{h(x)} + 1) dx \\ &= - \int_0^b g(u) \log_{25} (5^{-h(u)} + 1) du + \int_0^b g(x) \log_{25} (5^{h(x)} + 1) dx \end{aligned} \quad 1M$$

$$= - \int_0^b g(x) \log_{25} (5^{-h(x)} + 1) dx + \int_0^b g(x) \log_{25} (5^{h(x)} + 1) dx$$

$$= \int_0^b g(x) \log_{25} \left[(5^{h(x)} + 1) \div \left(\frac{1 + 5^{h(x)}}{5^{h(x)}} \right) \right] dx$$

$$= \int_0^b g(x) \log_{25} (5^{h(x)}) dx$$

$$= \frac{1}{2} \int_0^b g(x) h(x) dx \quad 1$$

(c) 代 $g(x) = \frac{16^x - 16^{-x}}{16^x + 16^{-x} - 1}$ 及 $h(x) = 4^x - 4^{-x}$ 。

$$g(-x) = \frac{16^{-x} + 16^x - 1}{16^{-x} - 16^x} = -g(x)$$

$$h(-x) = 4^{-x} - 4^x = -h(x)$$

$g(x)$ 及 $h(x)$ 均為奇函數。 1M

設 $u = 4^x - 4^{-x}$ 。則 $du = (4^x + 4^{-x})(\ln 4) dx$ 。

$$\begin{aligned} & \int_{-1}^1 \frac{(16^x - 16^{-x}) \log_{25} (5^{4^x - 4^{-x}} + 1)}{16^x + 16^{-x} - 1} dx \\ &= \frac{1}{2} \int_0^1 \frac{(16^x - 16^{-x})(4^x - 4^{-x})}{16^x + 16^{-x} - 1} dx \end{aligned} \quad 1M$$

$$= \frac{1}{2} \int_0^1 \frac{(4^x + 4^{-x})(4^x - 4^{-x})^2}{(4^x + 4^{-x})^2 + 1} dx$$

$$= \frac{1}{2 \ln 4} \int_0^{\frac{15}{4}} \frac{u^2}{1 + u^2} du \quad 1M$$

$$= \frac{1}{4 \ln 2} \left[x - \tan^{-1} x \right]_0^{\frac{15}{4}} \quad 1M$$

$$= \frac{15}{16 \ln 2} - \frac{1}{4 \ln 2} \tan^{-1} \frac{15}{4} \quad 1A$$

95. (a) 設 $u = -x$ 。則 $du = -dx$ 。 1M

$$\int_{-a}^a f(x) dx = \int_{-a}^0 f(x) dx + \int_0^a f(x) dx$$

$$= - \int_a^0 f(-u) du + \int_0^a f(x) dx$$

$$= \int_0^a f(-u) du + \int_0^a f(x) dx \quad 1M$$

$$= \int_0^a [f(x) + f(-x)] dx \quad 1$$

$$(b) \quad \tan\left(\frac{\pi}{8} + \frac{\pi}{8}\right) = \frac{2 \tan \frac{\pi}{8}}{1 - \tan^2 \frac{\pi}{8}} \quad 1M$$

$$1 = \frac{2 \tan \frac{\pi}{8}}{1 - \tan^2 \frac{\pi}{8}}$$

$$1 - \tan^2 \frac{\pi}{8} = 2 \tan \frac{\pi}{8}$$

$$1 - 2 \tan \frac{\pi}{8} = \tan^2 \frac{\pi}{8} \quad 1A$$

$$(c) \quad \int_{-1}^1 \frac{\tan \frac{\pi}{8}}{(1 + e^x)(x^2 + 1 - 2 \tan \frac{\pi}{8})} dx$$

$$= \int_{-1}^1 \frac{\tan \frac{\pi}{8}}{(1 + e^x)(x^2 + \tan^2 \frac{\pi}{8})} dx \quad 1M$$

$$= \int_0^1 \left[\frac{\tan \frac{\pi}{8}}{(1 + e^x)(x^2 + \tan^2 \frac{\pi}{8})} + \frac{\tan \frac{\pi}{8}}{(1 + e^{-x})((-x)^2 + \tan^2 \frac{\pi}{8})} \right] dx \quad 1M+1M$$

$$= \int_0^1 \left[\frac{\tan \frac{\pi}{8}}{(1 + e^x)(x^2 + \tan^2 \frac{\pi}{8})} + \frac{e^x \tan \frac{\pi}{8}}{(e^x + 1)((-x)^2 + \tan^2 \frac{\pi}{8})} \right] dx$$

$$= \int_0^1 \frac{\tan \frac{\pi}{8}}{x^2 + \tan^2 \frac{\pi}{8}} dx \quad 1A$$

設 $x = \tan \frac{\pi}{8} \tan \theta$ 。則 $dx = \tan \frac{\pi}{8} \sec^2 \theta d\theta$ 。 $1M$

$$\int_{-1}^1 \frac{\tan \frac{\pi}{8}}{(1 + e^x)(x^2 + 1 - 2 \tan \frac{\pi}{8})} dx$$

$$= \int_0^{\frac{3\pi}{8}} \frac{\tan \frac{\pi}{8} \cdot \tan \frac{\pi}{8} \sec^2 \theta}{\tan^2 \frac{\pi}{8} \tan^2 \theta + \tan^2 \frac{\pi}{8}} d\theta \quad 1A+1A$$

$$= \int_0^{\frac{3\pi}{8}} d\theta$$

$$= \left[\theta \right]_0^{\frac{3\pi}{8}}$$

$$= \frac{3\pi}{8} \quad 1A$$

96. (a) (i) 可得 $p + q = 0$ 及 $p(4 + \sqrt{7}) + q(4 - \sqrt{7}) = 1$ 。

$$p(4 + \sqrt{7}) + (-p)(4 - \sqrt{7}) = 1$$

$$p = \frac{1}{2\sqrt{7}}$$

因此， $p = \frac{1}{2\sqrt{7}}$ 及 $q = -\frac{1}{2\sqrt{7}}$ 。 $1A$

$$\begin{aligned}
\text{(ii)} \quad \int_0^1 \frac{1}{t^2 + 8t + 9} dt &= \int_0^1 \frac{1}{(t+4)^2 - 7} dy \\
&= \int_0^1 \frac{p(t+4+\sqrt{7}) + q(t+4-\sqrt{7})}{(t+4+\sqrt{7})(t+4-\sqrt{7})} dt & 1M \\
&= \frac{1}{2\sqrt{7}} \int_0^1 \left(\frac{1}{t+4-\sqrt{7}} - \frac{1}{t+4+\sqrt{7}} \right) dt \\
&= \frac{1}{2\sqrt{7}} \left[\ln|t+4-\sqrt{7}| - \ln|t+4+\sqrt{7}| \right]_0^1 & 1M \\
&= \frac{1}{2\sqrt{7}} \ln \frac{(4+\sqrt{7})(5-\sqrt{7})}{(4-\sqrt{7})(5+\sqrt{7})} \\
&= \frac{1}{2\sqrt{7}} \ln \frac{13+\sqrt{7}}{13-\sqrt{7}} & 1A
\end{aligned}$$

$$\begin{aligned}
\text{(b) (i)} \quad \sin 2x &= 2 \sin x \cos x \\
&= \frac{2 \sin x \cos x \sec^2 x}{\sec^2 x} \\
&= \frac{2 \tan x}{1 + \tan^2 x} & 1 \\
\cos 2x &= \cos^2 x - \sin^2 x \\
&= \frac{(\cos^2 x - \sin^2 x) \sec^2 x}{\sec^2 x} \\
&= \frac{1 - \tan^2 x}{1 + \tan^2 x} & 1
\end{aligned}$$

$$\begin{aligned}
\text{(ii)} \quad \text{設 } t = \tan x \circ \text{ 則 } dt &= \sec^2 x \, dx \circ & 1M \\
\int_0^{\frac{\pi}{4}} \frac{1}{4 \sin 2x + 4 \cos 2x + 5} dx \\
&= \int_0^1 \frac{1}{4 \left(\frac{2t}{1+t^2} \right) + 4 \left(\frac{1-t^2}{1+t^2} \right) + 5} \cdot \frac{1}{t^2 + 1} dt \\
&= \int_0^1 \frac{1}{t^2 + 8t + 9} dt \\
&= \frac{1}{2\sqrt{7}} \ln \frac{13 + \sqrt{7}}{13 - \sqrt{7}} & 1A
\end{aligned}$$

$$\begin{aligned}
\text{(c)} \quad \text{設 } u = \frac{\pi}{4} - x \circ \text{ 則 } du &= -dx \circ \\
\int_0^{\frac{\pi}{4}} \frac{8 \sin 2x + 5}{4 \sin 2x + 4 \cos 2x + 5} dx \\
&= - \int_{\frac{\pi}{4}}^0 \frac{8 \sin \left(\frac{\pi}{2} - 2u \right) + 5}{4 \sin \left(\frac{\pi}{2} - 2u \right) + 4 \cos \left(\frac{\pi}{2} - 2u \right) + 5} du & 1M \\
&= \int_0^{\frac{\pi}{4}} \frac{8 \cos 2u + 5}{4 \cos 2u + 4 \sin 2u + 5} du \\
&= \int_0^{\frac{\pi}{4}} \frac{8 \cos 2x + 5}{4 \sin 2x + 4 \cos 2x + 5} dx & 1
\end{aligned}$$

$$\begin{aligned}
\text{(d)} \quad & \int_0^{\frac{\pi}{4}} \frac{16 \sin 2x + 11}{4 \sin 2x + 4 \cos 2x + 5} dx \\
&= 2 \int_0^{\frac{\pi}{4}} \frac{8 \sin 2x + 5}{4 \sin 2x + 4 \cos 2x + 5} dx + \int_0^{\frac{\pi}{4}} \frac{1}{4 \sin 2x + 4 \cos 2x + 5} dx & 1M \\
&= \int_0^{\frac{\pi}{4}} \frac{8 \sin 2x + 5}{4 \sin 2x + 4 \cos 2x + 5} dx + \int_0^{\frac{\pi}{4}} \frac{8 \cos 2x + 5}{4 \sin 2x + 4 \cos 2x + 5} dx + \int_0^{\frac{\pi}{4}} \frac{1}{4 \sin 2x + 4 \cos 2x + 5} dx \\
&= \int_0^{\frac{\pi}{4}} \frac{8 \sin 2x + 8 \cos 2x + 10}{4 \sin 2x + 4 \cos 2x + 5} dx + \int_0^{\frac{\pi}{4}} \frac{1}{4 \sin 2x + 4 \cos 2x + 5} dx \\
&= \int_0^{\frac{\pi}{4}} 2 dx + \frac{1}{2\sqrt{7}} \ln \frac{13 + \sqrt{7}}{13 - \sqrt{7}} & 1M \\
&= \left[2x \right]_0^{\frac{\pi}{4}} + \frac{1}{2\sqrt{7}} \ln \frac{13 + \sqrt{7}}{13 - \sqrt{7}} \\
&= \frac{\pi}{2} + \frac{1}{2\sqrt{7}} \ln \frac{13 + \sqrt{7}}{13 - \sqrt{7}} & 1A
\end{aligned}$$

97. (a) 設 $u = T - x$ 。則 $du = -dx$ 。 1M

$$\begin{aligned}
& \int_0^T x f(x) dx = - \int_T^0 (T - u) f(T - u) du \\
&= \int_0^T (T - u) f(u) du & 1M \\
&= T \int_0^T f(x) dx - \int_0^T x f(x) dx \\
2 \int_0^T x f(x) dx &= T \int_0^T f(x) dx \\
\int_0^T x f(x) dx &= \frac{T}{2} \int_0^T f(x) dx & 1
\end{aligned}$$

(b) (i) 設 $u = \cos x$ 。則 $du = -\sin x dx$ 。

$$\begin{aligned}
& \int_0^{\pi} \frac{\sin^3 x}{2 - \sin^2 x} dx = - \int_1^{-1} \frac{1 - u^2}{2 - (1 - u^2)} du & 1M \\
&= \int_{-1}^1 \frac{1 - u^2}{1 + u^2} du \\
&= \int_{-1}^1 \left(\frac{2}{1 + u^2} - 1 \right) du & 1M \\
&= 2 \int_{-1}^1 \frac{1}{1 + u^2} du - \left[u \right]_{-1}^1 \\
&= 2 \int_{-1}^1 \frac{1}{1 + u^2} du - 2 & 1
\end{aligned}$$

(ii) 設 $u = \tan \theta$ 。則 $du = \sec^2 \theta d\theta$ 。 1M

$$\begin{aligned} \int_0^\pi \frac{\sin^3 x}{2 - \sin^2 x} dx &= 2 \int_{-1}^1 \frac{1}{1 + u^2} du - 2 \\ &= 2 \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sec^2 \theta}{1 + \tan^2 \theta} d\theta - 2 \\ &= 2 \left[\theta \right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} - 2 \\ &= \pi - 2 \end{aligned} \quad \begin{array}{l} 1A \\ 1A \end{array}$$

(c) 代 $f(x) = \frac{\sin^3 x}{2 - \sin^2 x}$ 。

$$f(\pi - x) = \frac{\sin^3(\pi - x)}{2 - \sin^2(\pi - x)} = \frac{\sin^3 x}{2 - \sin^2 x} = f(x) \quad 1M$$

$$\begin{aligned} \int_0^\pi \frac{x \sin^3 x}{2 - \sin^2 x} dx &= \frac{\pi}{2} \int_0^\pi \frac{\sin^3 x}{2 - \sin^2 x} dx \\ &= \frac{\pi}{2} (\pi - 2) \end{aligned} \quad \begin{array}{l} 1M \\ 1A \end{array}$$

98. (a) (i) $\frac{1}{1 + 2 \sin^2 x} = \frac{1}{1 + 2(1 - \cos^2 x)} \div \frac{\cos^2 x}{\cos^2 x}$

$$= \frac{\sec^2 x}{3 \sec^2 x - 2} \quad 1$$

(ii) 設 $u = \tan x$ 。則 $du = \sec^2 x dx$ 。 1M

$$\begin{aligned} \int_0^{\frac{\pi}{4}} \frac{1}{1 + 2 \sin^2 x} dx &= \int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{3 \sec^2 x - 2} dx \\ &= \int_0^1 \frac{du}{3(u^2 + 1) - 2} \\ &= \int_0^1 \frac{du}{1 + 3u^2} \end{aligned}$$

設 $\sqrt{3}u = \tan \theta$ 。則 $\sqrt{3} du = \sec^2 \theta d\theta$ 。 1M

$$\begin{aligned} \int_0^{\frac{\pi}{4}} \frac{1}{1 + 2 \sin^2 x} dx &= \frac{\sqrt{3}}{3} \int_0^{\frac{\pi}{3}} \frac{\sec^2 \theta}{1 + \tan^2 \theta} d\theta \\ &= \frac{\sqrt{3}}{3} \left[\theta \right]_0^{\frac{\pi}{3}} \\ &= \frac{\sqrt{3}\pi}{9} \end{aligned} \quad 1A$$

(b) 設 $u = -x$ 。則 $du = -dx$ 。 1M

$$\begin{aligned}
 \int_{-m}^m g(x)h(x) dx &= \int_{-m}^0 g(x)h(x) dx + \int_0^m g(x)h(x) dx & 1M \\
 &= -\int_m^0 g(-u)h(-u) du + \int_0^m g(x)h(x) dx \\
 &= -\int_0^m g(-u)h(u) du + \int_0^m g(x)h(x) dx \\
 &= -\int_0^m g(-x)h(x) dx + \int_0^m g(x)h(x) dx \\
 &= \int_0^m [g(x) - g(-x)]h(x) dx & 1M \\
 &= \int_0^m xh(x) dx & 1
 \end{aligned}$$

(c) 代 $g(x) = \ln(e^x + 1)$ 及 $h(x) = \frac{\sin 2x}{(1 + 2 \sin^2 x)^2}$ 。 1M

$$h(-x) = \frac{\sin(-2x)}{(1 + 2 \sin^2(-2x))^2} = \frac{-\sin 2x}{(1 + 2 \sin^2 2x)^2} = -h(x)。$$

$$g(x) - g(-x) = \ln(e^x + 1) - \ln(e^{-x} + 1)$$

$$= \ln(e^x + 1) - \ln\left(\frac{1 + e^x}{e^x}\right)$$

$$= \ln e^x$$

$$= x$$

$$\begin{aligned}
 &\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\ln(e^x + 1) \sin 2x}{(1 + 2 \sin^2 x)^2} dx \\
 &= \int_0^{\frac{\pi}{4}} \frac{x \sin 2x}{(1 + 2 \sin^2 x)^2} dx & 1M
 \end{aligned}$$

$$= \left[\frac{-x}{2(1 + \sin^2 x)} \right]_0^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}} \frac{1}{2(1 + 2 \sin^2 x)} dx & 1M$$

$$= \frac{-\frac{\pi}{4}}{2\left(1 + 2 \sin^2 \frac{\pi}{4}\right)} + \frac{1}{2} \left(\frac{\sqrt{3}\pi}{9} \right) & 1M$$

$$= \frac{(8\sqrt{3} - 9)\pi}{144} & 1A$$