M2REG-DEFINT-2425-ASM-SET 4-MATH
建議題解

1. 設 $u = e^x - e^{-x}$。則 $\mathrm{d}u = (e^x + e^{-x})\,\mathrm{d}x$。 1M

$$\int_{\ln 2}^{\ln 4} \frac{e^x + e^{-x}}{\sqrt{e^x - e^{-x}}}\,\mathrm{d}x = \int_{\frac{3}{2}}^{\frac{15}{4}} \frac{\mathrm{d}u}{\sqrt{u}} \qquad \text{1A}$$

$$= \left[2u^{\frac{1}{2}}\right]_{\frac{3}{2}}^{\frac{15}{4}}$$

$$= \sqrt{15} - \sqrt{6} \qquad \text{1A}$$

2. $$\int_{\frac{\pi}{9}}^{\frac{5\pi}{36}} \frac{\sqrt{3} + \tan 3x}{1 - \sqrt{3}\tan 3x}\,\mathrm{d}x = \int_{\frac{\pi}{9}}^{\frac{5\pi}{36}} \frac{\tan\frac{\pi}{3} + \tan 3x}{1 - \tan\frac{\pi}{3}\tan 3x}\,\mathrm{d}x$$

$$= \int_{\frac{\pi}{9}}^{\frac{5\pi}{36}} \tan\left(3x + \frac{\pi}{3}\right)\mathrm{d}x \qquad \text{1M}$$

$$= \int_{\frac{\pi}{9}}^{\frac{5\pi}{36}} \frac{\sin\left(3x + \frac{\pi}{3}\right)}{\cos\left(3x + \frac{\pi}{3}\right)}\,\mathrm{d}x$$

設 $u = \cos\left(3x + \frac{\pi}{3}\right)$。則 $\mathrm{d}u = -3\sin\left(3x + \frac{\pi}{3}\right)\mathrm{d}x$。 1M

$$\int_{\frac{\pi}{9}}^{\frac{5\pi}{36}} \frac{\sqrt{3} + \tan 3x}{1 - \sqrt{3}\tan 3x}\,\mathrm{d}x = -\frac{1}{3}\int_{-\frac{1}{2}}^{-\frac{\sqrt{2}}{2}} \frac{\mathrm{d}u}{u} \qquad \text{1A}$$

$$= -\frac{1}{3}\Big[\ln|u|\Big]_{-\frac{1}{2}}^{-\frac{\sqrt{2}}{2}}$$

$$= -\frac{1}{6}\ln 2 \qquad \text{1A}$$

3. (a) $$f(-x) = \sum_{i=0}^{77} \sin^{2i+1}(-x)$$

$$= -\sum_{i=0}^{77} \sin^{2i+1} x$$

$$= -f(x)$$

因此，$f(x)$ 為奇函數。 1

(b) $\displaystyle\int_{-\frac{\pi}{9}}^{\frac{\pi}{9}} f(x)\,\mathrm{d}x = 0$ 1A

4. $$\frac{\tan(-x)^3 + \sin(-x)^3}{(-x)^2 + \cos(-x)} = \frac{-\tan x^3 - \sin x^3}{x^2 + \cos x}$$

$\dfrac{\tan x^3 + \sin x^3}{x^2 + \cos x}$ 為奇函數。 1M

因此，$\displaystyle\int_{-\frac{2\pi}{9}}^{\frac{2\pi}{9}} \frac{\tan x^3 + \sin x^3}{x^2 + \cos x}\,\mathrm{d}x = 0$。 1A

5. $(-x)^3 \sin\left[(-x)^2 + e^{(-x)^4}\right] = -x^3 \sin\left(x^2 + e^{x^4}\right)$

$x^3 \sin\left(x^2 + e^{x^4}\right)$ 為奇函數。 1M

因此，$\int_{-1}^{1} x^3 \sin\left(x^2 + e^{x^4}\right) \mathrm{d}x = 0$。 1A

6. $\ln \dfrac{4 + \sin(-x)}{4 - \sin(-x)} = \ln \dfrac{4 - \sin x}{4 + \sin x} = -\ln \dfrac{4 + \sin x}{4 - \sin x}$

$\ln \dfrac{4 + \sin x}{4 - \sin x}$ 為奇函數。 1M

因此，$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \ln \dfrac{4 + \sin x}{4 - \sin x} \mathrm{d}x = 0$。 1A

7. 設 $u = \tan x$。則 $\mathrm{d}u = \sec^2 x \, \mathrm{d}x$。 1M

$$\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{\mathrm{d}x}{\cos^2 x\sqrt{4 - \tan^2 x}} = \int_{-\sqrt{3}}^{\sqrt{3}} \frac{\mathrm{d}u}{\sqrt{4 - u^2}}$$

設 $u = 2\sin\theta$。則 $\mathrm{d}u = 2\cos\theta \, \mathrm{d}\theta$。 1M

$$\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{\mathrm{d}x}{\cos^2 x\sqrt{4 - \tan^2 x}} = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{2\cos\theta}{\sqrt{4 - 4\sin^2\theta}} \mathrm{d}\theta$$

$$= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \mathrm{d}\theta \quad \text{1A}$$

$$= \left[\theta\right]_{-\frac{\pi}{3}}^{\frac{\pi}{3}}$$

$$= \frac{2\pi}{3} \quad \text{1A}$$

8. $\dfrac{-2(-x)^5}{1 + (-x)^6} = -\dfrac{-2x^5}{1 + x^6}$

$\dfrac{-2x^5}{1 + x^6}$ 為奇函數。 1M

$$\int_{-1}^{1} \frac{x^2 - 2x^5 + x^8}{1 + x^6} \mathrm{d}x = \int_{-1}^{1} \frac{-2x^5}{1 + x^6} \mathrm{d}x + \int_{-1}^{1} \frac{x^2(1 + x^6)}{1 + x^6} \mathrm{d}x$$

$$= 0 + \int_{-1}^{1} x^2 \, \mathrm{d}x \quad \text{1M}$$

$$= \left[\frac{x^3}{3}\right]_{-1}^{1}$$

$$= \frac{2}{3} \quad \text{1A}$$

9. $\sin(\sin(-x)) = -\sin(\sin x)$

$\sin(\sin x)$ 為奇函數。 1M

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} [\tan^2 x + \sin(\sin x)]\,dx = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan^2 x\,dx + \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin(\sin x)\,dx$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\sec^2 x - 1)\,dx + 0$$ 1M

$$= \Big[\tan x - x\Big]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$= 2 - \frac{\pi}{2}$$ 1A

10. (a) $f(-x) = (-x)^4 \sin(-x)$

$= -x^4 \sin x$

$= -f(x)$

因此，$f(x)$ 為奇函數。 1

(b) $$\int_{-a}^{a} (1 - x^4 \sin x)\,dx = \frac{\pi}{2}$$

$$\int_{-a}^{a} dx - \int_{-a}^{a} x^2 \sin x\,dx = \frac{\pi}{2}$$

$$\Big[x\Big]_{-a}^{a} - 0 = \frac{\pi}{2}$$ 1M

$$2a = \frac{\pi}{2}$$

$$a = \frac{\pi}{4}$$ 1A

11. (a) $$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan^2 x\,dx = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\sec^2 x - 1)\,dx$$ 1M

$$= \Big[\tan x - x\Big]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$= 2 - \frac{\pi}{2}$$ 1A

(b) $$\frac{(-x)\tan^2(-x)}{1 + (-x)^2} = -\frac{x\tan^2 x}{1 + x^2}$$

$\dfrac{x\tan^2 x}{1 + x^2}$ 為奇函數。 1M

因此，$\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{x\tan^2 x}{1 + x^2}\,dx = 0$。 1A

12. $\tan^5(-x) = -\tan^5 x$

$\tan^5 x$ 為奇函數。

$$\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} (\tan^5 x + \sin^2 x)\,\mathrm{d}x = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \tan^5 x\,\mathrm{d}x + \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \sin^2 x\,\mathrm{d}x$$

$$= 0 + \frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} (1 - \cos 2x)\,\mathrm{d}x$$ 1M+1M

$$= \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right]_{-\frac{\pi}{6}}^{\frac{\pi}{6}}$$

$$= \frac{\pi}{6} - \frac{\sqrt{3}}{4}$$ 1A

13. (a) $$\frac{1}{4\cos 2x+5} = \frac{1}{4(2\cos^2 x-1)+5}$$
$$= \frac{1}{8\cos^2 x+1}$$
$$= \frac{\sec^2 x}{8+\sec^2 x}$$ 1

(b) 設 $u = \tan x$。則 $\mathrm{d}u = \sec^2 x\,\mathrm{d}x$。 1M

$$\int_0^{\frac{\pi}{4}} \frac{1}{4\cos 2x+5}\,\mathrm{d}x = \int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{8+\sec^2 x}\,\mathrm{d}x$$ 1M
$$= \int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{9+\tan^2 x}\,\mathrm{d}x$$
$$= \int_0^{1} \frac{\mathrm{d}u}{u^2+9}$$

設 $u = 3\tan\theta$。則 $\mathrm{d}u = 3\sec^2\theta\,\mathrm{d}\theta$。

$$\int_0^{\frac{\pi}{4}} \frac{1}{4\cos 2x+5}\,\mathrm{d}x = \int_0^{\tan^{-1}\frac{1}{3}} \frac{3\sec^2\theta}{9\tan^2\theta+9}\,\mathrm{d}\theta$$
$$= \frac{1}{3}\int_0^{\tan^{-1}\frac{1}{3}} \mathrm{d}\theta$$
$$= \frac{1}{3}\Big[\theta\Big]_0^{\tan^{-1}\frac{1}{3}}$$
$$= \frac{1}{3}\tan^{-1}\frac{1}{3}$$ 1A

(c) 設 $u = -x$。則 $\mathrm{d}u = -\mathrm{d}x$。

$$\int_{-a}^{a} f(x)\ln(1+e^x)\,\mathrm{d}x$$
$$= \int_{-a}^{0} f(x)\ln(1+e^x)\,\mathrm{d}x + \int_0^a f(x)\ln(1+e^x)\,\mathrm{d}x$$ 1M
$$= -\int_a^0 f(-u)\ln(1+e^{-u})\,\mathrm{d}u + \int_0^a f(x)\ln(1+e^x)\,\mathrm{d}x$$ 1M
$$= -\int_a^0 [-f(x)]\ln(1+e^{-x})\,\mathrm{d}x + \int_0^a f(x)\ln(1+e^x)\,\mathrm{d}x$$ 1M
$$= -\int_0^a f(x)\ln\left(\frac{e^x+1}{e^x}\right)\mathrm{d}x + \int_0^a f(x)\ln(1+e^x)\,\mathrm{d}x$$
$$= \int_0^a f(x)[\ln e^x - \ln(1+e^x) + \ln(1+e^x)]\,\mathrm{d}x$$
$$= \int_0^a xf(x)\,\mathrm{d}x$$ 1

(d) 設 $g(x) = \dfrac{4\sin 2x}{(4\cos 2x+5)^2}$。

$$g(-x) = \frac{4\sin[2(-x)]}{(4\cos[2(-x)]+5)^2} = -\frac{4\sin 2x}{(4\cos 2x+5)^2} = -g(x)$$ 1M

因此，$g(x)$ 為奇函數。。

留意 $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\dfrac{1}{2(4\cos 2x+5)}\right] = \dfrac{4\sin 2x}{(4\cos 2x+5)^2}$。

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{4\sin 2x}{(4\cos 2x+5)^2}\,\mathrm{d}x$$

$$= \int_0^{\frac{\pi}{4}} \frac{4x\sin 2x}{(4\cos 2x+5)^2}\,\mathrm{d}x \qquad \text{1M}$$

$$= \left[\frac{x}{2(4\cos 2x+5)}\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \frac{\mathrm{d}x}{2(4\cos 2x+5)} \qquad \text{1M}$$

$$= \frac{1}{2}\left(\frac{\pi}{4}\right)\left(\frac{1}{0+5}\right) - \frac{1}{2}\left(\frac{1}{3}\tan^{-1}\frac{1}{3}\right) \qquad \text{1M}$$

$$= \frac{\pi}{40} - \frac{1}{6}\tan^{-1}\frac{1}{3} \qquad \text{1A}$$