

1. (a) $\frac{d}{dx} \left[(x^6 + 1) \ln(x^2 + 1) \right]$
 $= (x^6 + 1) \frac{2x}{x^2 + 1} + 6x^5 \ln(x^2 + 1)$ 1M+1A
 $= (x^2 + 1)(x^4 - x^2 + 1) \frac{2x}{x^2 + 1} + 6x^5 \ln(x^2 + 1)$ 1M
 $= 2x(x^4 - x^2 + 1) + 6x^5 \ln(x^2 + 1)$ 1A
- (b) 利用 (a) , $\int [2x(x^4 - x^2 + 1) + 6x^5 \ln(x^2 + 1)] dx = (x^6 + 1) \ln(x^2 + 1) + \text{常數}$ 1M
- $6 \int x^5 \ln(x^2 + 1) dx = (x^6 + 1) \ln(x^2 + 1) - 2 \int (x^5 - x^3 + x) dx$
 $= (x^6 + 1) \ln(x^2 + 1) - 2 \left(\frac{x^6}{6} - \frac{x^4}{4} + \frac{x^2}{2} \right) + \text{常數}$ 1A
- $\int x^5 \ln(x^2 + 1) dx = \frac{x^6 + 1}{6} \ln(x^2 + 1) - \frac{x^6}{18} + \frac{x^4}{12} - \frac{x^2}{6} + \text{常數}$ 1A
2. (a) $\int \frac{x+1}{x} dx = \int \left(1 + \frac{1}{x} \right) dx$ 1M
 $= x + \ln|x| + \text{常數}$ 1A
- (b) 設 $u = x^2 - 1$ 。則 $du = 2x dx$ 。
- $\int \frac{x^3}{x^2 - 1} dx = \frac{1}{2} \int \frac{u+1}{u} du$ 1A
 $= \frac{u}{2} + \frac{\ln|u|}{2} + \text{常數}$ 1M
 $= \frac{x^2 - 1}{2} + \frac{\ln|x^2 - 1|}{2} + \text{常數}$ 1A

3. (a) 設 $u = 9 - x$ 。則 $du = -dx$ 。 1M

$$\int \frac{dx}{\sqrt{9-x}} = \int \frac{du}{\sqrt{u}} \quad 1A$$

$$= -2u^{\frac{1}{2}} + \text{常數}$$

$$= -2\sqrt{9-x} + \text{常數} \quad 1A$$

- (b) 設 $x = 3 \sin \theta$ 。則 $dx = 3 \cos \theta d\theta$ 。 1M

$$\int \frac{dx}{\sqrt{9-x^2}} = \int \frac{3 \cos \theta d\theta}{\sqrt{9-9 \sin^2 \theta}} \quad 1A$$

$$= \int d\theta$$

$$= \theta + \text{常數}$$

$$= \sin^{-1} \frac{x}{3} + \text{常數} \quad 1A$$

4. $\int \frac{x^4+1}{x^2} dx = \int \left(x^2 + \frac{1}{x^2}\right) dx \quad 1M$

$$= \frac{x^3}{3} - \frac{1}{x} + \text{常數} \quad 1A$$

5. $\int \frac{2}{\cot \frac{x}{2} + \tan \frac{x}{2}} dx = \int \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} dx \quad 1M$

$$= \int 2 \tan \frac{x}{2} \cos^2 \frac{x}{2} dx$$

$$= \int 2 \sin \frac{x}{2} \cos \frac{x}{2} dx$$

$$= \int \sin x dx \quad 1M$$

$$= -\cos x + \text{常數} \quad 1A$$

$$6. \frac{dy}{d\theta} = \tan^2 \theta \sec^2 \theta - \sec^2 \theta \quad 1M$$

$$= (\tan^2 \theta - 1)(\tan^2 \theta + 1)$$

$$= \tan^4 \theta - 1 \quad 1A$$

$$\int \tan^4 \theta \, d\theta = \int (\tan^4 \theta - 1 + 1) \, d\theta \quad 1M$$

$$= \frac{\tan^3 \theta}{3} - \tan \theta + \theta + \text{常數} \quad 1A$$

$$7. \int (1 + \cos \theta)^2 \, d\theta = \int (1 + 2 \cos \theta + \cos^2 \theta) \, d\theta$$

$$= \theta + 2 \sin \theta + \frac{1}{2} \int (1 + \cos 2\theta) \, d\theta \quad 1M+1A$$

$$= \frac{3\theta}{2} + 2 \sin \theta + \frac{\sin 2\theta}{4} + \text{常數} \quad 1A$$

$$8. \quad (a) \int \cos^2 2x \, dx = \frac{1}{2} \int (1 + \cos 4x) \, dx \quad 1M$$

$$= \frac{x}{2} + \frac{\sin 4x}{8} + \text{常數} \quad 1A$$

$$(b) \int \sin^2 2x \, dx = \int (1 - \cos^2 2x) \, dx \quad 1M$$

$$= x - \left(\frac{x}{2} + \frac{\sin 4x}{8} \right) + \text{常數} \quad 1M$$

$$= \frac{x}{2} - \frac{\sin 4x}{8} + \text{常數} \quad 1A$$

$$\begin{aligned}
9. \quad \int (\sin x - \cos x)^2 dx &= \int (\sin^2 x + \cos^2 x - 2 \sin x \cos x) dx && 1A \\
&= \int (1 - \sin 2x) dx && 1M+1M \\
&= x + \frac{\cos 2x}{2} + \text{常數} && 1A
\end{aligned}$$

$$\begin{aligned}
10. \quad \int \cos^2 \theta d\theta &= \frac{1}{2} \int (1 + \cos 2\theta) d\theta && 1M+1A \\
&= \frac{\theta}{2} + \frac{\sin 2\theta}{4} + \text{常數}
\end{aligned}$$

$$\begin{aligned}
11. \quad (a) \quad \int (4x+1)^2 dx &= \frac{(4x+1)^3}{3(4)} + \text{常數} \\
&= \frac{(4x+1)^3}{12} + \text{常數} && 1M+1A
\end{aligned}$$

$$\begin{aligned}
(b) \quad \int \sin 3\theta \cos \theta d\theta &= \frac{1}{2} \int (\sin 4\theta + \sin 2\theta) d\theta && 1M \\
&= -\frac{\cos 4\theta}{8} - \frac{\cos 2\theta}{4} + \text{常數} && 1M+1A
\end{aligned}$$

$$12. \quad \int \sqrt{2x+1} dx = \frac{(2x+1)^{\frac{3}{2}}}{\frac{3}{2}} + \text{常數} \quad 1M+1A$$

$$13. \quad \text{設 } u = x - 1 \text{。則 } du = dx \text{。} \quad 1M$$

$$\begin{aligned}
\int (x+2)\sqrt{x-1} dx &= \int (u+3)\sqrt{u} du \\
&= \int (u^{\frac{3}{2}} + 3u^{\frac{1}{2}}) du && 1M \\
&= \frac{2u^{\frac{5}{2}}}{5} + 2u^{\frac{3}{2}} + \text{常數} && 1A \\
&= \frac{2}{5}(x-1)^{\frac{5}{2}} + 2(x-1)^{\frac{3}{2}} + \text{常數} && 1A
\end{aligned}$$

14. 設 $u = x - 1$ 。則 $du = dx$ 。 1M

$$\int x\sqrt{x-1} \, dx = \int (u+1)\sqrt{u} \, du \quad 1A$$

$$= \int (u^{\frac{3}{2}} + u^{\frac{1}{2}}) \, du$$

$$= \frac{2u^{\frac{5}{2}}}{5} + \frac{2u^{\frac{3}{2}}}{3} + \text{常數} \quad 1A$$

$$= \frac{2(x-1)^{\frac{5}{2}}}{5} + \frac{2(x-1)^{\frac{3}{2}}}{3} + \text{常數} \quad 1A$$

15. 設 $u = x + 2$ 。則 $du = dx$ 。 1M

$$\int x(x+2)^{99} \, dx = \int (u-2)u^{99} \, du \quad 1A$$

$$= \int (u^{100} - 2u^{99}) \, du$$

$$= \frac{u^{101}}{101} - \frac{2u^{100}}{100} + \text{常數} \quad 1A$$

$$= \frac{(x+2)^{101}}{101} - \frac{(x+2)^{100}}{50} + \text{常數} \quad 1A$$

16. 設 $u = 3x^2 + 1$ 。則 $du = 6x \, dx$ 。 1M

$$\int \frac{x}{\sqrt{3x^2+1}} \, dx = \frac{1}{6} \int \frac{6x}{\sqrt{3x^2+1}} \, dx$$

$$= \frac{1}{6} \int \frac{du}{\sqrt{u}} \quad 1A$$

$$= \frac{1}{3} u^{\frac{1}{2}} + \text{常數} \quad 1$$

$$= \frac{1}{3} (3x^2+1)^{\frac{1}{2}} + \text{常數} \quad 1A$$

17. (a) $\int \cos(3x+1) \, dx = \frac{\sin(3x+1)}{3} + \text{常數}$ 1M+1A

(b) 設 $u = 2 - x$ 。則 $du = -dx$ 。

$$\int (2-x)^{2004} \, dx = - \int u^{2004} \, du \quad 1M$$

$$= -\frac{u^{2005}}{2005} + \text{常數}$$

$$= -\frac{(2-x)^{2005}}{2005} + \text{常數} \quad 1A$$

18. $\int (8x+5)^{250} \, dx = \frac{(8x+5)^{251}}{251(8)} + \text{常數}$ 1M

$$= \frac{(8x+5)^{251}}{2008} + \text{常數} \quad 1A$$

19. 設 $u = \sqrt{x+9}$ 。則 $du = \frac{1}{2}(x+9)^{-\frac{1}{2}} \, dx$ 。 1M

$$\int \frac{x}{\sqrt{x+9}} \, dx = 2 \int \frac{x}{2(x+9)^{\frac{1}{2}}} \, dx$$

$$= 2 \int (u^2 - 9) \, du \quad 1A$$

$$= \frac{2u^3}{3} - 18u + \text{常數} \quad 1A$$

$$= \frac{2}{3}(x+9)^{\frac{3}{2}} - 18(x+9)^{\frac{1}{2}} + \text{常數} \quad 1A$$

20. 設 $u = \sin^2 x$ 。則 $du = 2 \sin x \cos x \, dx$ 。 1M

$$\begin{aligned} \int \frac{\sin x \cos x}{\sqrt{9 \sin^2 x + 4 \cos^2 x}} \, dx &= \frac{1}{2} \int \frac{2 \sin x \cos x}{\sqrt{5 \sin^2 x + 4}} \, dx \\ &= \frac{1}{2} \int \frac{du}{\sqrt{5u + 4}} & 1A \\ &= \frac{1}{5} \sqrt{5u + 4} + \text{常數} & 1A \\ &= \frac{1}{5} \sqrt{5 \sin^2 x + 4} + \text{常數} & 1A \end{aligned}$$

21. 設 $u = x^2$ 。則 $du = 2x \, dx$ 。 1M

$$\begin{aligned} \int x \sin^2(x^2) \, dx &= \frac{1}{2} \int \sin^2 u \, du \\ &= \frac{1}{4} \int (1 - \cos 2u) \, du & 1M \\ &= \frac{u}{4} - \frac{\sin 2u}{8} + \text{常數} & 1A \\ &= \frac{x^2}{4} - \frac{\sin 2x^2}{8} + \text{常數} & 1A \end{aligned}$$

22. (a) $\int \tan^3 x \, dx = \int \tan x (\sec^2 x - 1) \, dx$

$$\begin{aligned} &= \int \tan x \, d(\tan x) + \int \frac{d(\cos x)}{\cos x} & 1M \\ &= \frac{\tan^2 x}{2} + \ln |\cos x| + \text{常數} & 1A \end{aligned}$$

(b) (i) $\frac{A}{x} + \frac{B}{x-2} + \frac{C}{(x-2)^2} = \frac{A(x-2)^2 + Bx(x-2) + Cx}{x(x-2)^2}$

$$= \frac{(A+B)x^2 + (-4A-2B+C)x + 4A}{x(x-2)^2}$$

故此， $A+B=1$ 、 $-4A-2B+C=-1$ 及 $4A=2$ 。 1M

求解後，可得 $A = \frac{1}{2}$ 、 $B = \frac{1}{2}$ 及 $C = 2$ 。 1A+1A

(ii) $\int \frac{x^2 - x + 2}{x(x-2)^2} \, dx = \int \left[\frac{1}{2x} + \frac{1}{2(x-2)} + \frac{2}{(x-2)^2} \right] \, dx$ 1M

$$= \frac{1}{2} \ln |x| + \frac{1}{2} \ln |x-2| - \frac{2}{x-2} + \text{常數} \quad 1A$$

$$\begin{aligned}
 23. \quad (a) \quad 3I + 4J &= \int \frac{3 \sin x + 4 \cos x}{3 \sin x + 4 \cos x} dx \\
 &= \int dx \\
 &= x + \text{常數}
 \end{aligned}$$

1A

$$\begin{aligned}
 (b) \quad 4I - 3J &= \int \frac{4 \sin x - 3 \cos x}{3 \sin x + 4 \cos x} dx. \\
 \text{設 } u &= 3 \sin x + 4 \cos x. \text{ 則 } du = (3 \cos x - 4 \sin x) dx. \\
 4I - 3J &= - \int \frac{du}{u} \\
 &= -\ln |u| + \text{常數} \\
 &= -\ln |3 \sin x + 4 \cos x| + \text{常數}
 \end{aligned}$$

1M

1A

(c) 利用 (a) 及 (b) ,

$$\begin{aligned}
 3(3I + 4J) + 4(4I - 3J) &= 3x - 4 \ln |3 \sin x + 4 \cos x| + \text{常數} \\
 25I &= 3x - 4 \ln |3 \sin x + 4 \cos x| + \text{常數} \\
 I &= \frac{3x}{25} - \frac{4}{25} \ln |3 \sin x + 4 \cos x| + \text{常數}
 \end{aligned}$$

1M

1A

24. (a) 設 $x = \sin^2 \theta$. 則 $dx = 2 \sin \theta \cos \theta d\theta$.

$$\begin{aligned}
 \int \frac{f(x)}{\sqrt{x(1-x)}} dx &= \int \frac{f(\sin^2 \theta) \cdot 2 \sin \theta \cos \theta}{\sqrt{\sin^2 \theta (1 - \sin^2 \theta)}} d\theta \\
 &= 2 \int f(\sin^2 \theta) d\theta
 \end{aligned}$$

1

$$\begin{aligned}
 (b) \quad \int \frac{dx}{\sqrt{x(1-x)}} &= 2 \int d\theta \\
 &= 2\theta + \text{常數} \\
 &= 2 \sin^{-1} \sqrt{x} + \text{常數} \\
 \int \sqrt{\frac{x}{1-x}} dx &= \int \frac{x}{\sqrt{x(1-x)}} dx \\
 &= 2 \int \sin^2 \theta d\theta \\
 &= \int (1 - \cos 2\theta) d\theta \\
 &= \theta - \frac{1}{2} \sin 2\theta + \text{常數} \\
 &= \sin^{-1} \sqrt{x} - \sqrt{x(1-x)} + \text{常數}
 \end{aligned}$$

1A

1A

1A

$$25. \int e^{2x} (\sin x + \cos x)^2 dx = \int e^{2x} (1 + 2 \sin x \cos x) dx \quad 1M$$

$$= \frac{e^{2x}}{2} + \int e^{2x} \sin 2x dx \quad 1M$$

考慮積分 $\int e^{2x} \sin 2x dx$,

$$\int e^{2x} \sin 2x dx = \frac{e^{2x}}{2} \sin 2x - \int \frac{e^{2x}}{2} (2 \cos 2x) dx \quad 1M$$

$$= \frac{e^{2x} \sin 2x}{2} - \frac{e^{2x}}{2} \cos 2x + \int \frac{e^{2x}}{2} (-2 \sin 2x) dx \quad 1M$$

$$2 \int e^{2x} \sin 2x dx = \frac{e^{2x} \sin 2x}{2} - \frac{e^{2x} \cos 2x}{2} + \text{常數} \quad 1M$$

$$\int e^{2x} \sin 2x dx = \frac{e^{2x}}{4} (\sin 2x - \cos 2x) + \text{常數} \quad 1A$$

$$\text{故此, } \int e^{2x} (\sin x + \cos x)^2 dx = \frac{e^{2x}}{2} + \frac{e^{2x}}{4} (\sin 2x - \cos 2x) + \text{常數} \quad 1A$$

26. (a) 設 $u = 1 + x^2$ 。則 $du = 2x dx$ 。

$$\int \frac{x^3}{1+x^2} dx = \frac{1}{2} \int \frac{u-1}{u} du \quad 1M$$

$$= \frac{1}{2} \int (1 - u^{-1}) du$$

$$= \frac{u}{2} - \frac{1}{2} \ln |u| + \text{常數}$$

$$= \frac{x^2}{2} - \frac{1}{2} \ln(1+x^2) + \text{常數} \quad 1A$$

$$(b) \int x^2 \tan^{-1} x dx = \frac{x^3}{3} \tan^{-1} x - \frac{1}{3} \int \frac{x^3}{1+x^2} dx \quad 1M+1A$$

$$= \frac{x^3}{3} \tan^{-1} x - \frac{x^2}{6} + \frac{1}{6} \ln(1+x^2) + \text{常數} \quad 1A$$

27. (a) 設 $2x = \tan \theta$ 。則 $2 \, dx = \sec^2 \theta \, d\theta$ 。 1M

$$\int \frac{1}{1+4x^2} \, dx = \frac{1}{2} \int \frac{\sec^2 \theta}{1+\tan^2 \theta} \, d\theta \quad 1M$$

$$= \frac{\theta}{2} + \text{常數}$$

$$= \frac{1}{2} \tan^{-1} 2x + \text{常數} \quad 1A$$

$$(b) \int \ln(1+4x^2) \, dx = x \ln(1+4x^2) - \int x \cdot \frac{8x}{1+4x^2} \, dx \quad 1M+1M$$

$$= x \ln(1+4x^2) - 2 \int \left(1 - \frac{1}{1+4x^2}\right) \, dx \quad 1M$$

$$= x \ln(1+4x^2) - 2x + \tan^{-1} 2x + \text{常數} \quad 1A$$