Solution	Marks
REG-2425-MOCK-SET 13-MATH-EP(M2)	
Suggested solutions	
1. (a) $(1+4x+4x^2)^n = (1+2x)^{2n}$	1M
$= 1 + C_1^{2n}(2x) + C_2^{2n}(2x)^2 + \ldots$	
$= 1 + 4nx + 4n(2n-1)x^2 + \ldots$	
$(1-x)^9 = 1 - 9x + 36x^2 + \ldots$	
$4n(2n-1) + 4n(-9) + 36 = 4$	1M
$8n^2 - 40n + 32 = 0$	
$n = 4$ or 1 (rejected)	1A
(b) $(1+4x+4x^2)^4(1-x)^9$	
$= (1+2x)^8(1-x)^9$	
$= (1+16x+112x^2+448x^3+\ldots)(1-9x+36x^2-84x^3+\ldots)$	
Required coefficient $= (448)(1) + (112)(-9) + (16)(36) + (1)(-84)$	1M
$= -68$	1A
2. $f(e+h) - f(e)$	
$= (e+h)\ln(e+h) - e\ln e$	
$= (e+h)\ln(e+h) - (e+h)\ln e + h$	1M
$= (e+h)\ln\left(1+\frac{h}{e}\right) + h$	1
$f'(e) = \lim_{h\to 0}\frac{f(e+h)-f(e)}{h}$	1M
$= \lim_{h\to 0}\left[\left(\frac{e+h}{h}\right)\ln\left(1+\frac{h}{e}\right)+1\right]$	
$= \lim_{h\to 0}\left[\ln\left(1+\frac{h}{e}\right)^{\frac{e}{h}\times\frac{e+h}{e}}+1\right]$	1M
$= \ln e^1 + 1$	
$= 2$	1A

Solution	Marks
3. (a) $\tan A = \dfrac{2 - \tan B}{1 - 2\tan B}$	
$\dfrac{\sin A}{\cos A} = \dfrac{2\cos B - \sin B}{\cos B - 2\sin B}$	1M
$\sin A\cos B - 2\sin A\cos B = 2\cos A\cos B - \cos A\sin B$	
$\sin A\cos B + \cos A\sin B = 2(\cos A\cos B + \sin A\sin B)$	
$\sin(A+B) = 2\cos(A-B)$	1
(b) $\tan\left(x + \dfrac{5\pi}{18}\right) = \dfrac{2 - \tan\left(x - \frac{\pi}{18}\right)}{1 - 2\tan\left(x - \frac{\pi}{18}\right)}$	
$\sin\left(x + \dfrac{5\pi}{18} + x - \dfrac{\pi}{18}\right) = 2\cos\left[x + \dfrac{5\pi}{18} - \left(x - \dfrac{\pi}{18}\right)\right]$	1M
$\sin\left(2x + \dfrac{2\pi}{9}\right) = 2\cos\dfrac{\pi}{3}$	
$2\sin\left(2x + \dfrac{2\pi}{9}\right) = 1$	1M
$2x + \dfrac{2\pi}{9} = \dfrac{\pi}{2}$	
$x = \dfrac{5\pi}{36}$	1A
4. (a) $\dfrac{A(x-1)}{x^2-2x+2} + \dfrac{B(x+1)}{x^2+2x+2}$	
$= \dfrac{A(x-1)(x^2+2x+2) + B(x+1)(x^2-2x+2)}{(x^2-2x+2)(x^2+2x+2)}$	
$= \dfrac{(A+B)x^2 + (A-B)x + (-2A+2B)}{x^4+4}$	
We have $A + B = 0$, $A - B = 1$ and $-2A + 2B = -2$.	1M
Solving, we have $A = \dfrac{1}{2}$ and $B = -\dfrac{1}{2}$.	1A
(b) Let $u = x^2 - 2x + 2$. Then $\mathrm{d}u = (2x-2)\,\mathrm{d}x$.	1M
Let $w = x^2 + 2x + 2$. Then $\mathrm{d}w = (2x+2)\,\mathrm{d}x$.	
$\displaystyle\int_0^1 \frac{x^2-2}{x^4+4}\,\mathrm{d}x = \int_0^1 \left(\frac{x-1}{2(x^2-2x+2)} - \frac{x+1}{2(x^2+2x+2)}\right)\mathrm{d}x$	1M
$\displaystyle= \frac{1}{4}\int_2^1 \frac{\mathrm{d}u}{u} - \frac{1}{4}\int_2^5 \frac{\mathrm{d}w}{w}$	
$\displaystyle= \frac{1}{4}\Big[\ln\lvert u\rvert\Big]_2^1 - \frac{1}{4}\Big[\ln\lvert w\rvert\Big]_2^5$	1M
$= -\dfrac{1}{4}\ln 5$	1A

Solution	Marks
5. (a) $V = \pi \int_0^b [16 - (y-4)^2]\, \mathrm{d}y$	1M
$= \pi \left[16y - \frac{(y-4)^3}{3} \right]_0^h$	1A
$= \frac{\pi h^2 (12 - h)}{3}$	1A
(b) $V = \frac{\pi}{3}(12h^2 - h^3)$	
$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\pi}{3}(24h - 3h^2)\frac{\mathrm{d}h}{\mathrm{d}t}$	1M
$= \pi h (8 - h) \frac{\mathrm{d}h}{\mathrm{d}t}$	1A
When $h = 3$,	
$\pi = \pi(3)(8-3)\frac{\mathrm{d}h}{\mathrm{d}t}$	
$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{1}{15}$	1A
Required rate is $\frac{1}{15}$ cm/s.	
6. (a) $\det P = 2$	
$P^{-1} = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$	1A
$P^{-1}AP = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 4 & 0 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$	1A
(b) $(P^{-1}AP)^n = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}^n$	
$P^{-1}A^nP = \begin{pmatrix} 4^n & 0 \\ 0 & 1 \end{pmatrix}$	1M
$A^n = P\begin{pmatrix} 4^n & 0 \\ 0 & 1 \end{pmatrix}P^{-1}$	
$= \frac{1}{2}\begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 4^n & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$	
$= \begin{pmatrix} 4^n & 0 \\ 1 - 4^n & 1 \end{pmatrix}$	1A
(c) $A^n - \alpha I = \begin{pmatrix} 4^n - \alpha & 0 \\ 1 - 4^n & 1 - \alpha \end{pmatrix}$	
Suppose $A^n - \alpha I$ is a singular matrix.	
$(\alpha - 4^n)(\alpha - 1) = 0$	1M
$\alpha = 4^n$ or 1 (rejected)	1A

Solution	Marks
7. (a) $f(x) = \int (2x - 12)\,\mathrm{d}x$	1M
$= x^2 - 12x + C$ where C is a constant.	
$13 = (3)^2 - 12(3) + C$	1M
$C = 40$	
Required equation is $y = x^2 - 12x + 40$.	1A
(b) (i) Let $(a,\ b)$ be the coordinates of P.	
$2a - 12 = \dfrac{b+4}{a-5}$	1M
$(2a - 12)(a - 5) = (a^2 - 12a + 40) + 4$	
$a^2 - 10a + 16 = 0$	1M
$a = 2$ (rejected) or 8	
When $a = 8$, $b = 8^2 - 12(8) + 40 = 8$.	
Required coordinates are $(8,\ 8)$.	1A
(ii) Slope $= 2(8) - 12 = 4$	
Required equation is	
$\dfrac{y-8}{x-8} = 4$	1M
$4x - y - 24 = 0$	1A

Solution	Marks

8. (a) When $n = 1$,

L.H.S. $= 2\cos\theta\,[-\cos 2\theta + \cos 4\theta]$

$= -(\cos 3\theta \cos\theta) + (\cos 5\theta + \cos 3\theta)$

$= \cos 5\theta - \cos\theta$

R.H.S. $= \cos 5\theta - \cos\theta =$ L.H.S.

It is true for $n = 1$. — 1

Assume $2\cos\theta \sum_{k=1}^{2p}(-1)^k \cos 2k\theta = \cos(4p+1)\theta - \cos\theta$, where $p \in \mathbb{Z}^+$. — 1

$$2\cos\theta \sum_{k=1}^{2p+2}(-1)^k \cos 2k\theta$$

$$= 2\cos\theta \sum_{k=1}^{2p}(-1)^k \cos 2k\theta + 2\cos\theta\left[(-1)^{2p+1}\cos(4p+2)\theta + (-1)^{2p+2}\cos(4p+4)\theta\right]$$

$$= \cos(4p+1)\theta - \cos\theta + 2\cos\theta[-\cos(4p+2)\theta + \cos(4p+4)\theta]$$ — 1M

$$= \cos(4p+1)\theta - \cos\theta - [\cos(4p+3)\theta + \cos(4p+1)\theta] + [\cos(4p+5)\theta + \cos(4p+3)\theta]$$ — 1M

$$= \cos(4p+5)\theta - \cos\theta$$

It is true for $n = p + 1$.

By M.I., it is true $\forall n \in \mathbb{Z}^+$. — 1

(b) Put $\theta = \frac{\pi}{14}$ and $n = 64$.

$$2\cos\frac{\pi}{14}\sum_{k=1}^{128}(-1)^k \cos\frac{2k\pi}{14} = \cos\frac{257\pi}{14} - \cos\frac{\pi}{14}$$ — 1M

$$2\cos\frac{\pi}{14}\sum_{k=1}^{128}(-1)^k\left(2\cos^2\frac{k\pi}{14} - 1\right) = \cos\left(18\pi + \frac{\pi}{2} - \frac{\pi}{7}\right) - \cos\frac{\pi}{14}$$ — 1M

$$2\cos\frac{\pi}{14}\left[2\sum_{k=1}^{128}(-1)^k\cos^2\frac{k\pi}{14} - \sum_{k=1}^{128}(-1)^k\right] = 2\sin\frac{\pi}{14}\cos\frac{\pi}{14} - \cos\frac{\pi}{14}$$

$$4\sum_{k=1}^{128}(-1)^k\cos^2\frac{k\pi}{14} - 0 = 2\sin\frac{\pi}{14} - 1$$

$$\sum_{k=1}^{128}(-1)^k\cos^2\frac{k\pi}{14} = \frac{1}{2}\sin\frac{\pi}{14} - \frac{1}{4}$$

Thus, $a = \frac{1}{2}$, $b = \frac{1}{14}$ and $c = -\frac{1}{4}$. — 1A

Solution	Marks
9. (a) (i) (1) $\begin{vmatrix} 1 & a+2 & 1 \\ 1 & -a & 6b-1 \\ 2 & a+3 & 2b+1 \end{vmatrix} = \begin{vmatrix} 1 & -a \\ 2 & a+3 \end{vmatrix} - (6b-1)\begin{vmatrix} 1 & a+2 \\ 2 & a+3 \end{vmatrix} + (2b+1)\begin{vmatrix} 1 & a+2 \\ 1 & -a \end{vmatrix}$	1M
$= (3a+3) - (6b-1)(-a-1) + (2b+1)(-2a-2)$	
$= 2b(a+1)$	
Since (E) has unique solution, we have $2b(a+1) \neq 0$.	1M
Thus, $a \neq -1$ and $b \neq 0$.	1
(2) $x = \dfrac{\begin{vmatrix} 2 & a+2 & 1 \\ 1 & -a & 6b-1 \\ 3 & a+3 & 2b+1 \end{vmatrix}}{2b(a+1)}$	1M
$= \dfrac{-4b+1}{2b(a+1)}$	1A
$y = \dfrac{\begin{vmatrix} 1 & 2 & 1 \\ 1 & 1 & 6b-1 \\ 2 & 3 & 2b+1 \end{vmatrix}}{2b(a+1)}$	
$= \dfrac{4b-1}{2b(a+1)}$	
$z = \dfrac{\begin{vmatrix} 1 & a+2 & 2 \\ 1 & -a & 1 \\ 2 & a+3 & 3 \end{vmatrix}}{2b(a+1)}$	
$= \dfrac{a+1}{2b(a+1)}$	
$= \dfrac{1}{2b}$	1A
(ii) (1) The augmented matrix of (E) is	
$\left(\begin{array}{ccc\|c} 1 & 1 & 1 & 2 \\ 1 & 1 & 6b-1 & 1 \\ 2 & 2 & 2b+1 & 3 \end{array}\right) \sim \left(\begin{array}{ccc\|c} 1 & 1 & 1 & 2 \\ 0 & 0 & 6b-2 & -1 \\ 0 & 0 & 2b-1 & -1 \end{array}\right) \sim \left(\begin{array}{ccc\|c} 1 & 1 & 1 & 2 \\ 0 & 0 & 6b-2 & -1 \\ 0 & 0 & -\frac{1}{3} & -\frac{2}{3} \end{array}\right)$	1M
We have $z = 2$.	
$(6b-2)(2) = -1$	
$b = \dfrac{1}{4}$	1A
(2) The augmented matrix of (E) is	
$\left(\begin{array}{ccc\|c} 1 & 1 & 1 & 2 \\ 0 & 0 & -\frac{1}{2} & -1 \\ 0 & 0 & -\frac{1}{3} & -\frac{2}{3} \end{array}\right) \sim \left(\begin{array}{ccc\|c} 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$	
The solution set is $\{(-t,\ t,\ 2) : t \in \mathbb{R}\}$.	1A
(b) When $a = -1$ and $b = \dfrac{1}{4}$, the system (F) is equivalent to (E).	

Solution	Marks
The solution set of (F) is $\{(-t,\ t,\ 2) : t \in \mathbb{R}\}$.	
$4(-t) + t^2 + 2 < -2$	1M
$t^2 - 4t + 4 < 0$	
$(t - 2)^2 < 0$	1M
Note that $(t - 2)^2 \geq 0$ for all real values of t.	
There is no real solutions of (F) satisfying $4x + y^2 + z < -2$.	1A

Solution	Marks
10. (a) Let $x = \sin\theta$. Then $\mathrm{d}x = \cos\theta\,\mathrm{d}\theta$.	1M
$\displaystyle\int_0^1 x^2\sqrt{1-x^2}\,\mathrm{d}x = \int_0^{\frac{\pi}{2}} \sin^2\theta\sqrt{1-\sin^2 x}\cos\theta\,\mathrm{d}\theta$	
$\displaystyle= \int_0^{\frac{\pi}{2}} \sin^2\theta\cos^2\theta\,\mathrm{d}\theta$	
$\displaystyle= \frac{1}{4}\int_0^{\frac{\pi}{2}} \sin^2 2\theta\,\mathrm{d}\theta$	1M
$\displaystyle= \frac{1}{8}\int_0^{\frac{\pi}{2}} (1-\cos 4\theta)\,\mathrm{d}\theta$	1M
$\displaystyle= \frac{1}{8}\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\frac{\pi}{2}}$	
$\displaystyle= \frac{\pi}{16}$	1A
(b) (i) Let $u = k - x$. Then $\mathrm{d}u = -\mathrm{d}x$.	1M
$\displaystyle\int_0^k f(x)g(x)\,\mathrm{d}x = -\int_k^0 f(k-u)g(k-u)\,\mathrm{d}u$	
$\displaystyle= \int_0^k f(u)g(k-u)\,\mathrm{d}u$	
$\displaystyle= \int_0^k f(x)g(k-x)\,\mathrm{d}x$	1
(ii) $\displaystyle\int_0^k f(x)g(x)\,\mathrm{d}x = \frac{1}{2}\left[\int_0^k f(x)g(x)\,\mathrm{d}x + \int_0^k f(x)g(k-x)\,\mathrm{d}x\right]$	1M
$\displaystyle= \frac{1}{2}\int_0^k f(x)[g(x)+g(k-x)]\,\mathrm{d}x$	
$\displaystyle= \frac{a}{2}\int_0^k f(x)\,\mathrm{d}x$	1
(c) Put $f(x) = \sin^2 x\cos^2 x$ and $g(x) = \dfrac{1}{1+e^{\sin 8x}}$.	1M
$\displaystyle f\left(\frac{\pi}{2}-x\right) = \sin^2\left(\frac{\pi}{2}-x\right)\cos^2\left(\frac{\pi}{2}-x\right) = \cos^2 x\sin^2 x = f(x)$	
$\displaystyle g(x) + g\left(\frac{\pi}{2}-x\right) = \frac{1}{1+e^{\sin 8x}} + \frac{1}{1+e^{\sin(4\pi-8x)}}$	
$\displaystyle= \frac{1}{1+e^{\sin x}} + \frac{1}{1+e^{-\sin 8x}}\cdot\frac{e^{\sin 8x}}{e^{\sin 8x}}$	
$= 1$	
$\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin^2 x\cos^2 x}{1+e^{\sin 8x}}\,\mathrm{d}x = \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin^2 x\cos^2 x\,\mathrm{d}x$	1M
$\displaystyle= \frac{1}{2}\left(\frac{\pi}{16}\right)$	1M
$\displaystyle= \frac{\pi}{32}$	1A

Solution	Marks
11. (a) $f(x)=\dfrac{(x-6)^3}{(x+6)^2}$	
$=x-30+\dfrac{432}{x+6}-\dfrac{1728}{(x+6)^2}$	
$f'(x)=1-\dfrac{432}{(x+6)^2}+\dfrac{3456}{(x+6)^3}$	1M
$=\dfrac{(x+30)(x-6)^2}{(x+6)^3}$	1A
$f''(x)=\dfrac{864}{(x+6)^3}-\dfrac{10\,368}{(x+6)^4}$	
$=\dfrac{864(x-6)}{(x+6)^4}$	1A
(b) When $f'(x)=0$, $x=-30$ or 6.	
<table><tr><td>x</td><td>$x<-30$</td><td>$-30<x<-6$</td><td>$-6<x<6$</td><td>$x>6$</td></tr><tr><td>$f'(x)$</td><td>$+$</td><td>$-$</td><td>$+$</td><td>$+$</td></tr></table>	1M
Maximum point is $(-30,\ -81)$.	
H has 1 turning point.	1A
(c) Vertical asymptote is $x=-6$.	1A
$f(x)=x-30+\dfrac{432}{x+6}-\dfrac{1728}{(x+6)^2}$	1M
Oblique asymptote is $y=x-30$.	1A
(d) (i) We have $f(3)=-\dfrac{1}{3}$ and $f'(3)=\dfrac{11}{27}$.	1M
Required equation is	
$\dfrac{y+\frac{1}{3}}{x-3}=\dfrac{11}{27}$	
$y=\dfrac{11x}{27}-\dfrac{14}{9}$	1A
(ii) Required area	
$=\displaystyle\int_3^{\frac{42}{11}}\left[\left(\frac{11x}{27}-\frac{14}{9}\right)-f(x)\right]\mathrm{d}x+\int_{\frac{42}{11}}^{6}[-f(x)]\,\mathrm{d}x$	1M
$=\displaystyle\int_3^{\frac{42}{11}}\left(\frac{11x}{27}-\frac{14}{9}\right)\mathrm{d}x-\int_3^6\left(x-30+\frac{432}{x+6}-\frac{1728}{(x+6)^2}\right)\mathrm{d}x$	
$=\left[\dfrac{11x^2}{54}-\dfrac{14x}{9}\right]_3^{\frac{42}{11}}-\left[\dfrac{x^2}{2}-30x+432\ln\lvert x+6\rvert+\dfrac{1728}{x+6}\right]_3^6$	1M
$=\dfrac{1368}{11}+432\ln\dfrac{3}{4}$	1A

Solution	Marks
12. (a) (i) $\overrightarrow{CM} = \overrightarrow{OM} - \overrightarrow{OC}$	
$= \frac{1}{2}\mathbf{a} - (r\mathbf{a} + s\mathbf{b})$	
$= \left(\frac{1}{2} - r\right)\mathbf{a} - s\mathbf{b}$	1A
$\overrightarrow{OA} \cdot \overrightarrow{CM} = 0$	
$\mathbf{a} \cdot \left[\left(\frac{1}{2} - r\right)\mathbf{a} - s\mathbf{b}\right] = 0$	1M
$\left(\frac{1}{2} - r\right)\mathbf{a} \cdot \mathbf{a} - s\mathbf{a} \cdot \mathbf{b} = 0$	
$\left(\frac{1}{2} - r\right)(2^2) - s\left(\frac{1}{4}\right) = 0$	
$s = 8 - 16r$	1
(ii) $\overrightarrow{CN} = \overrightarrow{ON} - \overrightarrow{OC}$	
$= \frac{1}{2}\mathbf{b} - (r\mathbf{a} + s\mathbf{b})$	
$= -r\mathbf{a} + \left(\frac{1}{2} - s\right)\mathbf{b}$	1A
$\overrightarrow{OB} \cdot \overrightarrow{CN} = 0$	
$\mathbf{b} \cdot \left[-r\mathbf{a} + \left(\frac{1}{2} - s\right)\mathbf{b}\right] = 0$	
$-r\mathbf{a} \cdot \mathbf{b} + \left(\frac{1}{2} - s\right)\mathbf{b} \cdot \mathbf{b} = 0$	
$r + 4s = 2$	
Solving, we have $r = \frac{10}{21}$ and $s = \frac{8}{21}$.	1A+1A
(iii) $\overrightarrow{CP} = \overrightarrow{OP} - \overrightarrow{OC}$	
$= \frac{1}{2}(\mathbf{a} + \mathbf{b}) - \left(\frac{10}{21}\mathbf{a} + \frac{8}{21}\mathbf{b}\right)$	
$= \frac{1}{42}\mathbf{a} + \frac{5}{42}\mathbf{b}$	
Since $OH // CP$, there is a real number λ such that $\overrightarrow{OH} = \lambda\overrightarrow{CP} = \frac{\lambda}{42}\mathbf{a} + \frac{5\lambda}{42}\mathbf{b}$.	1A
$\overrightarrow{BH} = \overrightarrow{OH} - \overrightarrow{OB}$	
$= \frac{\lambda}{42}\mathbf{a} + \left(\frac{5\lambda}{42} - 1\right)\mathbf{b}$	1A
$\overrightarrow{BH} \cdot \overrightarrow{OA} = 0$	
$\left[\frac{\lambda}{42}\mathbf{a} + \left(\frac{5\lambda}{42} - 1\right)\mathbf{b}\right] \cdot \mathbf{a} = 0$	1M
$\frac{\lambda}{42}(2^2) + \left(\frac{5\lambda - 42}{42}\right)\frac{1}{4} = 0$	
$\lambda = 2$	
Thus, $\overrightarrow{OH} = \frac{1}{21}\mathbf{a} + \frac{5}{21}\mathbf{b}$.	1A

Solution	Marks
(b) $(AB)^2 = \overrightarrow{AB} \cdot \overrightarrow{AB}$	
$= (\mathbf{b} - \mathbf{a}) \cdot (\mathbf{b} - \mathbf{a})$	
$= \lvert\mathbf{b}\rvert^2 - 2\mathbf{a} \cdot \mathbf{b} + \lvert\mathbf{a}\rvert^2$	
$= 5 - 2m$	
$AB = \sqrt{5 - 2m}$	1A
Suppose $AB = OA = 2$.	
$\sqrt{5 - 2m} = 2$	
$5 - 2m = 4$	
$m = \dfrac{1}{2}$	
Suppose $AB = OB = 1$, then $AB + OB = 1 + 1 = 2 = OA$ and $\triangle OAB$ does not exist.	1M
There is only 1 value of m such that $\triangle OAB$ is an isosceles triangle.	
The claim is disagreed.	1

END OF PAPER