1. (a) $\sin^4 x + \cos^4 x = (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x$

$$= 1 - 2\left(\frac{\sin 2x}{2}\right)^2 \quad \text{1M}$$

$$= 1 - \frac{1}{2}\left(\frac{1 - \cos 4x}{2}\right) \quad \text{1M}$$

$$= \frac{3}{4} + \frac{\cos 4x}{4} \quad \text{1A}$$

(b) $\displaystyle\int (\sin^4 x + \cos^4 x)\,\mathrm{d}x = \int \left(\frac{3}{4} + \frac{\cos 4x}{4}\right)\mathrm{d}x$ 1M

$$= \frac{3x}{4} + \frac{\sin 4x}{16} + \text{constant} \quad \text{1A}$$

2. $\displaystyle\int \frac{\left(\sqrt[3]{x} - \frac{2}{\sqrt{x}}\right)^3}{x^2}\,\mathrm{d}x = \int \frac{x - 6x^{\frac{1}{6}} + 6x^{-\frac{2}{3}} - 8x^{-\frac{3}{2}}}{x^2}\,\mathrm{d}x$ 1M

$$= \int \left(\frac{1}{x} - 6x^{-\frac{11}{6}} + 6x^{-\frac{8}{3}} - 8x^{-\frac{7}{2}}\right)\mathrm{d}x$$

$$= \ln|x| + \frac{36}{5}x^{-\frac{5}{6}} - \frac{36}{5}x^{-\frac{5}{3}} + \frac{16}{5}x^{-\frac{5}{2}} + \text{constant} \quad \text{1A}$$

3. $\displaystyle\int \frac{x^2 - 9}{x + 3}\,\mathrm{d}x = \int \frac{(x + 3)(x - 3)}{x + 3}\,\mathrm{d}x$

$$= \int (x - 3)\,\mathrm{d}x \quad \text{1M}$$

$$= \frac{1}{2}x^2 - 3x + \text{constant} \quad \text{1A}$$

4. (a) $\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right) = \cos^2\frac{x}{2} - \sin^2\frac{x}{2}$

$$= \cos\left(2 \times \frac{x}{2}\right)$$

$$= \cos x \quad \text{1}$$

(b) $\displaystyle\int \left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)\mathrm{d}x = \int \cos x\,\mathrm{d}x$ 1M

$$= \sin x + \text{constant} \quad \text{1A}$$

5. $\displaystyle\int \sqrt[3]{4x + 3}\,\mathrm{d}x = \int (4x + 3)^{\frac{1}{3}}\,\mathrm{d}x$

$$= \frac{3}{16}(4x + 3)^{\frac{4}{3}} + \text{constant} \quad \text{1A}$$

6. (a) $\displaystyle\int \frac{\sqrt{x} - 3\sqrt[4]{x}}{\sqrt[3]{x}}\,\mathrm{d}x = \int x^{\frac{1}{6}}\,\mathrm{d}x - 3\int x^{-\frac{1}{12}}\,\mathrm{d}x$ 1M

$$= \frac{6}{7}x^{\frac{7}{6}} - \frac{36}{11}x^{\frac{11}{12}} + \text{constant} \quad \text{1A}$$

(b) $\int \frac{(2x+1)^2}{\sqrt[3]{x^2}}\,dx = \int \frac{4x^2+4x+1}{\sqrt[3]{x^2}}\,dx$

$= 4\int x^{\frac{4}{3}}\,dx + 4\int x^{\frac{1}{3}}\,dx + \int x^{-\frac{2}{3}}\,dx$ 1M

$= \frac{12}{7}x^{\frac{7}{3}} + 3x^{\frac{4}{3}} + 3x^{\frac{1}{3}} + \text{constant}$ 1A

7. (a) $\frac{d}{dx}\left(x^3 \ln x\right) = x^3\left(\frac{1}{x}\right) + (\ln x)(3x^2)$

$= x^2 + 3x^2 \ln x$ 1A

(b) $\int (x^2 + 3x^2 \ln x)\,dx = x^3 \ln x + \text{constant}$ 1M

$\int x^2\,dx + 3\int x^2 \ln x\,dx = x^3 \ln x + \text{constant}$

$3\int x^2 \ln x\,dx = x^3 \ln x - \int x^2\,dx + \text{constant}$

$\int x^2 \ln x\,dx = \frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^2\,dx + \text{constant}$

$= \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + \text{constant}$ 1A

8. (a) $\frac{d}{dx}[(x^4-1)\ln\left(x^2+1\right)] = 4x^3 \ln\left(x^2+1\right) + (x^4-1)\frac{2x}{x^2+1}$ 1M

$= 4x^3 \ln\left(x^2+1\right) + (x^2-1)(x^2+1)\frac{2x}{x^2+1}$

$= 2x^3 - 2x + 4x^3 \ln\left(x^2+1\right)$ 1A

(b) $\int [2x^3 - 2x + 4x^3 \ln\left(x^2+1\right)]\,dx = (x^4-1)\ln\left(x^2+1\right) + \text{constant}$ 1M

$2\int (x^3-x)\,dx + 4\int x^3 \ln\left(x^2+1\right)\,dx = (x^4-1)\ln\left(x^2+1\right) + \text{constant}$

$\int x^3 \ln\left(x^2+1\right)\,dx = \frac{1}{4}(x^4-1)\ln\left(x^2+1\right) - \frac{1}{2}\int (x^3-x)\,dx + \text{constant}$

$= \frac{1}{4}(x^4-1)\ln\left(x^2+1\right) - \frac{1}{8}x^4 + \frac{1}{4}x^2 + \text{constant}$ 1A

9. (a) $\int \cos 4x \cos x\,dx = \frac{1}{2}\int (\cos 5x + \cos 3x)\,dx$ 1M

$= \frac{\sin 5x}{10} + \frac{\sin 3x}{6} + \text{constant}$ 1A

(b) $\frac{\sin 6x - \sin 2x}{\sin x} = \frac{2\cos 4x \sin 2x}{\sin x}$ 1M

$= \frac{2\cos 4x(2\sin x \cos x)}{\sin x}$ 1M

$= 4\cos 4x \cos x$ 1

(c) $\displaystyle\int \frac{\sin 6x}{\sin x}\,dx = \int \left(\frac{\sin 6x - \sin 2x}{\sin x} + \frac{\sin 2x}{\sin x}\right) dx$

$\displaystyle = 4\int \cos 4x \cos x\,dx + \int \frac{\sin 2x}{\sin x}\,dx$ 1M

$\displaystyle = 4\int \cos 4x \cos x\,dx + 2\int \cos x\,dx$

$\displaystyle = 4\left(\frac{\sin 5x}{10} + \frac{\sin 3x}{6}\right) + 2\sin x + \text{constant}$

$\displaystyle = 2\sin x + \frac{2\sin 5x}{5} + \frac{2\sin 3x}{3} + \text{constant}$ 1A

10. (a) $\displaystyle\int e^{\ln x^{-1}}\,dx = \int x^{-1}\,dx$ 1M

$= \ln|x| + \text{constant}$ 1A

(b) $\displaystyle\int 3^{2x}7^{x}\,dx = \int 63^{x}\,dx$ 1M

$\displaystyle = \frac{1}{\ln 63}63^{x} + \text{constant}$ 1A

11. $\displaystyle\int \left(x - \frac{2}{x}\right)^3 dx = \int \left(x^3 - 6x + \frac{12}{x} - \frac{8}{x^3}\right) dx$ 1M

$\displaystyle = \frac{x^4}{4} - 3x^2 + 12\ln|x| + 4x^{-2} + \text{constant}$ 1A

12. $\displaystyle\int \frac{x^2(2x^2+3x-3)}{\sqrt{x}}\,dx = \int (2x^{\frac{7}{2}} + 3x^{\frac{5}{2}} - 3x^{\frac{3}{2}})\,dx$ 1M

$\displaystyle = \frac{4}{9}x^{\frac{9}{2}} + \frac{6}{7}x^{\frac{7}{2}} - \frac{6}{5}x^{\frac{5}{2}} + \text{constant}$ 1A

13. $\displaystyle y = \int \left(x - \frac{1}{2x^2}\right)^2 dx$

$\displaystyle = \int \left(x^2 - \frac{1}{x} + \frac{1}{4x^4}\right) dx$ 1M

$\displaystyle = \frac{x^3}{3} - \ln|x| - \frac{1}{12}x^{-3} + \text{constant}$ 1A

14. $\displaystyle\int \frac{dx}{\sqrt[3]{(16x-56)^4}} = \int (16x-56)^{-\frac{4}{3}}\,dx$

$\displaystyle = -\frac{3}{16}(16x-56)^{-\frac{1}{3}} + \text{constant}$ 1A

15. $\displaystyle\int \frac{x^4-9}{x(x^2-3)}\,\mathrm{d}x = \int \frac{(x^2-3)(x^2+3)}{x(x^2-3)}\,\mathrm{d}x$

$\displaystyle = \int \frac{x^2+3}{x}\,\mathrm{d}x$

$\displaystyle = \int \left(x+\frac{3}{x}\right)\mathrm{d}x$ 1M

$\displaystyle = \frac{x^2}{2} + 3\ln|x| + \text{constant}$ 1A

16. $\displaystyle\int (\sqrt{x}+\sqrt[3]{x})^4\,\mathrm{d}x = \int (x^{\frac{1}{2}}+x^{\frac{1}{3}})^4\,\mathrm{d}x$

$\displaystyle = \int \left(x^2+4x^{\frac{11}{6}}+6x^{\frac{5}{3}}+4x^{\frac{3}{2}}+x^{\frac{4}{3}}\right)\mathrm{d}x$ 1M

$\displaystyle = \frac{1}{3}x^3+\frac{24}{17}x^{\frac{17}{6}}+\frac{9}{4}x^{\frac{8}{3}}+\frac{8}{5}x^{\frac{5}{2}}+\frac{3}{7}x^{\frac{7}{3}}+\text{constant}$ 1A

17. $\displaystyle y = \int (2+\sqrt{x})^2\,\mathrm{d}x$

$\displaystyle = \int (4+4\sqrt{x}+x)\,\mathrm{d}x$ 1M

$\displaystyle = 4\int \mathrm{d}x + 4\int x^{\frac{1}{2}}\,\mathrm{d}x + \int x\,\mathrm{d}x$

$\displaystyle = 4x + \frac{8}{3}x^{\frac{3}{2}} + \frac{x^2}{2} + \text{constant}$ 1A

18. $\displaystyle\int 2^{3-5x}\,\mathrm{d}x = \int e^{(3-5x)\ln 2}\,\mathrm{d}x$ 1M

$\displaystyle = \frac{1}{-5\ln 2}e^{(3-5x)\ln 2} + \text{constant}$

$\displaystyle = -\frac{1}{5\ln 2}2^{3-5x} + \text{constant}$ 1A

19. $\displaystyle\int (2\sqrt{x}-1)(\sqrt[3]{x}+1)\,\mathrm{d}x = \int (2x^{\frac{1}{2}}-1)(x^{\frac{1}{3}}+1)\,\mathrm{d}x$

$\displaystyle = \int (2x^{\frac{5}{6}}+2x^{\frac{1}{2}}-x^{\frac{1}{3}}-1)\,\mathrm{d}x$ 1M

$\displaystyle = \frac{12}{11}x^{\frac{11}{6}}+\frac{4}{3}x^{\frac{3}{2}}-\frac{3}{4}x^{\frac{4}{3}}-x+\text{constant}$ 1A

20. $\displaystyle\int \frac{\sqrt{x}(3x^3-2x+1)}{x^3}\,\mathrm{d}x = \int \frac{3x^{\frac{7}{2}}-2x^{\frac{3}{2}}+x^{\frac{1}{2}}}{x^3}\,\mathrm{d}x$

$\displaystyle = \int (3x^{\frac{1}{2}}-2x^{-\frac{3}{2}}+x^{-\frac{5}{2}})\,\mathrm{d}x$ 1M

$\displaystyle = 2x^{\frac{3}{2}}+4x^{-\frac{1}{2}}-\frac{2}{3}x^{-\frac{3}{2}}+\text{constant}$ 1A

21. (a) $\int (e^{4x} + e^{-x})^5 \, dx$

$= \int \left[\sum_{r=0}^{5} \left(C_r^5 (e^{4x})^{5-r} (e^{-x})^r \right) \right] dx$ 1M

$= \int (e^{20x} + 5e^{15x} + 10e^{10x} + 10e^{5x} + 5 + e^{-5x}) \, dx$ 1A

$= \frac{e^{20x}}{20} + \frac{e^{15x}}{3} + e^{10x} + 2e^{5x} + 5x - \frac{1}{5e^{5x}} + \text{constant}$ 1M+1A

(b) Let $u = \ln x$. Then $du = \frac{1}{x} \, dx$. 1M

$$\int \frac{\ln x}{x} \, dx = \int u \, du$$
$$= \frac{u^2}{2} + \text{constant}$$
$$= \frac{(\ln x)^2}{2} + \text{constant}$$ 1A

22. Let $u = x^3 - 2x + 3$. Then $du = (3x^2 - 2) \, dx$. 1M

$$\int (6x^2 - 4)(x^3 - 2x + 3)^{\frac{1}{3}} \, dx = 2 \int u^{\frac{1}{3}} \, du$$
$$= \frac{3}{2} u^{\frac{4}{3}} + \text{constant}$$
$$= \frac{3}{2} (x^3 - 2x + 3)^{\frac{4}{3}} + \text{constant}$$ 1A

23. Let $u = x + x^2 + \frac{1}{x}$. Then $du = \left(1 + 2x - \frac{1}{x^2}\right) dx$. 1M

$$\int \left(1 + 2x - \frac{1}{x^2}\right) \left(x + x^2 + \frac{1}{x}\right)^{-1} dx = \int u^{-1} \, du$$
$$= \ln |u| + \text{constant}$$
$$= \ln \left| x + x^2 + \frac{1}{x} \right| + \text{constant}$$ 1A

24. Let $u = \ln x$. Then $du = \frac{1}{x} \, dx$. 1M

$$\int \frac{\sqrt{\ln \sqrt{x}}}{x} \, dx = \frac{1}{\sqrt{2}} \int u^{\frac{1}{2}} \, du$$
$$= \frac{\sqrt{2}}{3} u^{\frac{3}{2}} + \text{constant}$$
$$= \frac{\sqrt{2}}{3} (\ln x)^{\frac{3}{2}} + \text{constant}$$ 1A

25. (a) $\int \cos^6 \frac{x}{2} \sin 2x \, dx = \int \left(\frac{1 + \cos x}{2} \right)^3 (2 \sin x \cos x) \, dx$ 1M

Let $u = \cos x$. Then $\mathrm{d}u = -\sin x\,\mathrm{d}x$. 1M

$$\int \cos^6 \frac{x}{2} \sin 2x\,\mathrm{d}x = -\frac{1}{4}\int (1+u)^3 u\,\mathrm{d}u$$

$$= -\frac{1}{4}(u + 3u^2 + 3u^3 + u^4)\,\mathrm{d}u \quad \text{1A}$$

$$= -\frac{1}{4}\left(\frac{\cos^2 x}{2} + \cos^3 x + \frac{3\cos^4 x}{4} + \frac{\cos^5 x}{5}\right) + \text{constant}$$

$$= -\frac{\cos^2 x}{8} - \frac{\cos^3 x}{4} - \frac{3\cos^4 x}{16} - \frac{\cos^5 x}{20} + \text{constant} \quad \text{1A}$$

(b) Let $u = 2^x$. Then $\mathrm{d}u = 2^x \ln 2\,\mathrm{d}x$. 1M

$$\int 2^x \cos^6 2^{x-1} \sin 2^{x+1}\,\mathrm{d}x = \frac{1}{\ln 2}\int \cos^6 \frac{u}{2} \sin 2u\,\mathrm{d}u \quad \text{1A}$$

$$= \frac{1}{\ln 2}\left(\frac{\cos^2 u}{8} - \frac{\cos^3 u}{4} - \frac{3\cos^4 u}{16} - \frac{\cos^5 u}{20}\right) + \text{constant}$$

$$= -\frac{\cos^2 2^x}{8\ln 2} - \frac{\cos^3 2^x}{4\ln 2} - \frac{3\cos^4 2^x}{16\ln 2} - \frac{\cos^5 2^x}{20\ln 2} + \text{constant} \quad \text{1A}$$

26. Let $u = 2\sqrt{x} + 1$. Then $\mathrm{d}u = \frac{1}{\sqrt{x}}\,\mathrm{d}x$. 1M

$$\int \frac{(2\sqrt{x}+1)^5}{\sqrt{x}}\,\mathrm{d}x = \int u^5\,\mathrm{d}u$$

$$= \frac{u^6}{6} + \text{constant}$$

$$= \frac{(2\sqrt{x}+1)^6}{6} + \text{constant} \quad \text{1A}$$

27. Let $u = \sqrt{x} + 1$. Then $\mathrm{d}u = \frac{1}{2\sqrt{x}}\,\mathrm{d}x$. 1M

$$\int \frac{\sin(\sqrt{x}+1)}{\sqrt{x}}\,\mathrm{d}x = 2\int \sin u\,\mathrm{d}u$$

$$= -2\cos u + \text{constant}$$

$$= -2\cos(\sqrt{x}+1) + \text{constant} \quad \text{1A}$$

28. Let $u = e^x + 1$. Then $\mathrm{d}u = e^x\,\mathrm{d}x$. 1M

$$\int \frac{e^x}{e^{2x} + 2e^x + 1}\,\mathrm{d}x = \int \frac{e^x}{(e^x+1)^2}\,\mathrm{d}x$$

$$= \int \frac{\mathrm{d}u}{u^2}$$

$$= -\frac{1}{u} + \text{constant}$$

$$= -\frac{1}{e^x + 1} + \text{constant} \quad \text{1A}$$

29. Let $u = 3 + \sin(\ln x)$. Then $\mathrm{d}u = \dfrac{\cos(\ln x)}{x}\,\mathrm{d}x$. 1M

$$\int \frac{\cos(\ln x)}{x[3+\sin(\ln x)]}\,\mathrm{d}x = \int \frac{\mathrm{d}u}{u} \qquad \text{1A}$$
$$= \ln|3+\sin(\ln x)| + \text{constant}$$
$$= \ln(3+\sin(\ln x)) + \text{constant} \qquad \text{1A}$$

30. Let $u = \log_2 x$. Then $\mathrm{d}u = \dfrac{1}{x\ln 2}\,\mathrm{d}x$. 1M

$$\int \frac{3(\log_2 x)^2 + 2\log_2 x - 4}{x}\,\mathrm{d}x = \int (\ln 2)(3u^2+2u-4)\,\mathrm{d}u \qquad \text{1A}$$
$$= (\ln 2)(u^3+u^2-4u) + \text{constant}$$
$$= (\ln 2)[(\log_2 x)^3 + (\log_2 x)^2 - 4\log_2 x] + \text{constant} \qquad \text{1A}$$

31. Let $u = 1 + \dfrac{3}{x^2}$. Then $\mathrm{d}u = -\dfrac{6}{x^3}\,\mathrm{d}x$.

$$\int \frac{\sqrt{x^2+3}}{x^4}\,\mathrm{d}x = -\frac{1}{6}\int \left(\frac{\sqrt{x^2+3}}{x}\right)\left(-\frac{6}{x^3}\right)\mathrm{d}x$$
$$= -\frac{1}{6}\int \sqrt{1+\frac{3}{x^2}}\left(-\frac{6}{x^3}\right)\mathrm{d}x \qquad \text{1M}$$
$$= -\frac{1}{6}\int \sqrt{u}\,\mathrm{d}u \qquad \text{1A}$$
$$= -\frac{1}{9}u^{\frac{3}{2}} + \text{constant}$$
$$= -\frac{1}{9}\left(1+\frac{3}{x^2}\right)^{\frac{3}{2}} + \text{constant} \qquad \text{1A}$$

32. Let $u = x^2 - 3x + 4$. Then $\mathrm{d}u = (2x-3)\,\mathrm{d}x$. 1M

$$\int (2x-3)(x^2-3x+4)^8\,\mathrm{d}x = \int u^8\,\mathrm{d}u$$
$$= \frac{u^9}{9} + \text{constant}$$
$$= \frac{1}{9}(x^2-3x+4)^9 + \text{constant} \qquad \text{1A}$$

33. (a) $(1+ax)^7 = \displaystyle\sum_{r=0}^{7} C_r^7 a^r x^r$ 1M

$$\frac{\beta_1}{\beta_3} = \frac{C_1^7 a}{C_3^7 a^3} = \frac{1}{45} \qquad \text{1M}$$

$$\frac{7}{35a^2} = \frac{1}{45}$$

$$a^2 = 9$$

$$a = 3 \quad \text{or} \quad -3 \text{ (rejected)} \qquad \text{1A}$$

(b) $\displaystyle\int \sum_{r=0}^{7} \beta_r x^r \, \mathrm{d}x = \int (1+3x)^7 \, \mathrm{d}x$ 1M

$\displaystyle = \frac{(1+3x)^8}{24} + \text{constant}$ 1A

34. Let $= \sqrt{x} - 1$. Then $\mathrm{d}u = \dfrac{1}{2\sqrt{x}} \, \mathrm{d}x$. 1M

$$\int \sqrt{\sqrt{x} - 1} \, \mathrm{d}x = \int 2\sqrt{x}\sqrt{\sqrt{x} - 1} \cdot \frac{1}{2\sqrt{x}} \, \mathrm{d}x$$

$$= \int 2(u+1)\sqrt{u} \, \mathrm{d}u$$

$$= \int (2u^{\frac{3}{2}} + 2u^{\frac{1}{2}}) \, \mathrm{d}u \qquad \text{1M}$$

$$= \frac{4}{5}u^{\frac{5}{2}} + \frac{4}{3}u^{\frac{3}{2}} + \text{constant}$$

$$= \frac{4}{5}(\sqrt{x} - 1)^{\frac{5}{2}} + \frac{4}{3}(\sqrt{x} - 1)^{\frac{3}{2}} + \text{constant} \qquad \text{1A}$$

35. (a) $\displaystyle P + \frac{Q}{x+4} + \frac{R}{2x-5} = \frac{P(x+4)(2x-5) + Q(2x-5) + R(x+4)}{(x+4)(2x-5)}$

$$= \frac{2Px^2 + (3P + 2Q + R)x + (-20P - 5Q + 4R)}{(x+4)(2x-5)}$$

We have $2P = 4$, $3P + 2Q + R = 11$ and $-20P - 5Q + 4R = -59$. 1M

Solving, we have $P = 2$, $Q = 3$ and $R = -1$. 1A+1A

(b) $\displaystyle\int \frac{4x^2 + 11x - 59}{2x^2 + 3x - 20} \, \mathrm{d}x = \int \frac{4x^2 + 11x - 59}{(x+4)(2x-5)} \, \mathrm{d}x$

$\displaystyle = \int \left(2 + \frac{3}{x+4} - \frac{1}{2x-5}\right) \mathrm{d}x$ 1M

$\displaystyle = 2x + 3\ln|x+4| - \frac{1}{2}\ln|2x-5| + \text{constant}$ 1A

36. Let $u = e^x$. Then $\mathrm{d}u = e^x\,\mathrm{d}x$. 1M

$$\int e^x 5^{e^x}\,\mathrm{d}x = \int 5^u\,\mathrm{d}u$$

$$= \int e^{u\ln 5}\,\mathrm{d}u \qquad \text{1M}$$

$$= \frac{1}{\ln 5}e^{u\ln 5} + \text{constant}$$

$$= \frac{1}{\ln 5}5^{e^x} + \text{constant} \qquad \text{1A}$$

37. Let $u = \sin x$. Then $\mathrm{d}u = \cos x\,\mathrm{d}x$. 1M

$$\int \frac{\cos^5 x}{\sin^6 x}\,\mathrm{d}x = \int \frac{(1-u^2)^2}{u^6}\,\mathrm{d}u$$

$$= \int (u^{-6} - 2u^{-4} + u^{-2})\,\mathrm{d}u \qquad \text{1A}$$

$$= -\frac{1}{5u^5} + \frac{2}{3u^3} - \frac{1}{u} + \text{constant}$$

$$= -\frac{1}{5\sin^5 x} + \frac{2}{3\sin^3 x} - \frac{1}{\sin x} + \text{constant} \qquad \text{1A}$$

38. Let $u = 2e^x + 1$. Then $\mathrm{d}u = 2e^x\,\mathrm{d}x$. 1M

$$\int e^{2x}\sqrt{2e^x+1}\,\mathrm{d}x = \int \frac{1}{2}\left(\frac{u-1}{2}\right)\sqrt{u}\,\mathrm{d}u$$

$$= \int \left(\frac{1}{4}u^{\frac{3}{2}} - \frac{1}{4}u^{\frac{1}{2}}\right)\mathrm{d}u \qquad \text{1M}$$

$$= \frac{1}{10}u^{\frac{5}{2}} - \frac{1}{6}u^{\frac{3}{2}} + \text{constant}$$

$$= \frac{1}{10}(2e^x+1)^{\frac{5}{2}} - \frac{1}{6}(2e^x+1)^{\frac{3}{2}} + \text{constant} \qquad \text{1A}$$

39. Let $u = e^x + e^{-x}$. Then $\mathrm{d}u = (e^x - e^{-x})\,\mathrm{d}x$. 1M

$$\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,\mathrm{d}x = \int \frac{\mathrm{d}u}{u}$$

$$= \ln|u| + \text{constant}$$

$$= \ln|e^x + e^{-x}| + \text{constant} \qquad \text{1A}$$

40. Let $u = 1 - e^{-5x}$. Then $du = 5e^{-5x}\,dx$. 1M

$$\int \frac{dx}{e^{5x}-1} = \int \frac{e^{-5x}}{1-e^{-5x}}\,dx$$

$$= \frac{1}{5}\int \frac{du}{u} \qquad \text{1M}$$

$$= \frac{1}{5}\ln|u| + \text{constant}$$

$$= \frac{1}{5}\ln|1-e^{-5x}| + \text{constant} \qquad \text{1A}$$

Alternative solution

Let $u = e^{5x} - 1$. Then $du = 5e^{5x}\,dx$. 1M

$$\int \frac{dx}{e^{5x}-1} = \int \frac{1-e^{5x}+e^{5x}}{e^{5x}-1}\,dx$$

$$= \int \frac{1-e^{5x}}{e^{5x}-1}\,dx + \int \frac{e^{5x}}{e^{5x}-1}\,dx$$

$$= -\int dx + \frac{1}{5}\int \frac{du}{u} \qquad \text{1M}$$

$$= -x + \frac{1}{5}\ln|u| + \text{constant}$$

$$= -x + \frac{1}{5}\ln|e^{5x}-1| + \text{constant} \qquad \text{1A}$$

41. (a) $A + B = \int \frac{\sin\theta}{\sin\theta+\cos\theta}\,d\theta + \int \frac{\cos\theta}{\sin\theta+\cos\theta}\,d\theta$

$$= \int \frac{\sin\theta+\cos\theta}{\sin\theta+\cos\theta}\,d\theta$$

$$= \int d\theta$$

$$= \theta + \text{constant} \qquad \text{1A}$$

Let $u = \sin\theta + \cos\theta$. Then $du = (\cos\theta - \sin\theta)\,d\theta$. 1M

$B - A = \int \frac{\cos\theta}{\sin\theta+\cos\theta}\,d\theta - \int \frac{\sin\theta}{\sin\theta+\cos\theta}\,d\theta$

$$= \int \frac{\cos\theta-\sin\theta}{\sin\theta+\cos\theta}\,d\theta$$

$$= \int \frac{du}{u}$$

$$= \ln|u| + \text{constant}$$

$$= \ln|\sin\theta+\cos\theta| + \text{constant} \qquad \text{1A}$$

(b) Let $x = \sin\theta$. Then $dx = \cos\theta\, d\theta$.

$$\int \frac{1}{x+\sqrt{1-x^2}}\,dx = \int \frac{\cos\theta}{\sin\theta+\sqrt{1-\sin^2\theta}}\,d\theta \qquad \text{1M}$$
$$= \int \frac{\cos\theta}{\sin\theta+\cos\theta}\,d\theta$$
$$= \frac{(A+B)+(B-A)}{2}$$
$$= \frac{\theta}{2}+\frac{1}{2}\ln|\sin\theta+\cos\theta| + \text{constant} \qquad \text{1M}$$
$$= \frac{1}{2}\sin^{-1}x+\frac{1}{2}\ln|x+\sqrt{1-x^2}| + \text{constant} \qquad \text{1A}$$

42. (a) Let $u = 25 + x^2$. Then $du = 2x\,dx$. 1M

$$\int \frac{x}{25+x^2}\,dx = \frac{1}{2}\int \frac{1}{u}\,du \qquad \text{1A}$$
$$= \frac{1}{2}\ln|u| + \text{constant}$$
$$= \frac{1}{2}\ln\left(25+x^2\right) + \text{constant} \qquad \text{1A}$$

(b) Let $x = 5\tan\theta$. Then $dx = 5\sec^2\theta\,d\theta$. 1M

$$\int \frac{x^2}{25+x^2}\,dx = \int \frac{25\tan^2\theta}{25+25\tan^2\theta}\cdot 5\sec^2\theta\,d\theta$$
$$= 5\int \tan^2\theta\,d\theta \qquad \text{1A}$$
$$= 5\int (\sec^2\theta - 1)\,d\theta \qquad \text{1M}$$
$$= 5\tan\theta - 5\theta + \text{constant}$$
$$= x - 5\tan^{-1}\frac{x}{5} + \text{constant} \qquad \text{1A}$$

43. (a) $x^2 + 6x + 10 = (x+3)^2 + 1^2$ 1A

(b) Let $x + 3 = \tan\theta$. Then $dx = \sec^2\theta\,d\theta$. 1M

$$\int \frac{1}{x^2+6x+10}\,dx = \int \frac{\sec^2\theta}{\tan^2\theta+1}\,d\theta$$
$$= \int d\theta \qquad \text{1A}$$
$$= \theta + \text{constant}$$
$$= \tan^{-1}(x+3) + \text{constant} \qquad \text{1A}$$

44. (a) Let $x = 3\tan\theta$. Then $\mathrm{d}x = 3\sec^2\theta\,\mathrm{d}\theta$. 1M

$$\int \frac{1}{x^2+9}\,\mathrm{d}x = \int \frac{3\sec^2\theta}{9\tan^2\theta+9}\,\mathrm{d}\theta$$

$$= \frac{1}{3}\int \mathrm{d}\theta \qquad \text{1A}$$

$$= \frac{1}{3}\theta + \text{constant}$$

$$= \frac{1}{3}\tan^{-1}\frac{x}{3} + \text{constant} \qquad \text{1A}$$

(b) Let $u = x^2 + 9$. Then $\mathrm{d}u = 2x\,\mathrm{d}x$. 1M

$$\int \frac{2x+1}{x^2+9}\,\mathrm{d}x = \int \frac{2x}{x^2+9}\,\mathrm{d}x + \int \frac{1}{x^2+9}\,\mathrm{d}x \qquad \text{1M}$$

$$= \int \frac{\mathrm{d}u}{u} + \frac{1}{3}\tan^{-1}\frac{x}{3}$$

$$= \ln|u| + \frac{1}{3}\tan^{-1}\frac{x}{3} + \text{constant}$$

$$= \ln\left(x^2+9\right) + \frac{1}{3}\tan^{-1}\frac{x}{3} + \text{constant} \qquad \text{1A}$$

45. (a) $6x - x^2 - 8 = 1^2 - (x-3)^2$ 1A

(b) Let $x - 3 = \sin\theta$. Then $\mathrm{d}x = \cos\theta\,\mathrm{d}\theta$. 1M

$$\int \sqrt{6x - x^2 - 8}\,\mathrm{d}x = \int \sqrt{1-\sin^2\theta}\cos\theta\,\mathrm{d}\theta$$

$$= \int \cos^2\theta\,\mathrm{d}\theta \qquad \text{1A}$$

$$= \frac{1}{2}\int (1+\cos 2\theta)\,\mathrm{d}\theta \qquad \text{1M}$$

$$= \frac{\theta}{2} + \frac{\sin 2\theta}{4} + \text{constant}$$

$$= \frac{\theta}{2} + \frac{1}{2}\sin\theta\cos\theta + \text{constant}$$

$$= \frac{1}{2}\sin^{-1}(x-3) + \frac{1}{2}(x-3)\sqrt{6-x^2-8} + \text{constant} \qquad \text{1A}$$

46. (a) $x^2 + 2x\sin\frac{5\pi}{12} + 1 = x^2 + 2x\sin\frac{5\pi}{12} + \sin^2\frac{5\pi}{12} + \cos^2\frac{5\pi}{12}$

$$= \left(x + \sin\frac{5\pi}{12}\right)^2 + \cos^2\frac{5\pi}{12} \qquad \text{1A}$$

(b) Let $x + \sin\frac{5\pi}{12} = \cos\frac{5\pi}{12}\tan\theta$. Then $dx = \cos\frac{5\pi}{12}\sec^2\theta\, d\theta$. 1M

$$\begin{aligned}\int \frac{\cos\frac{5\pi}{12}}{x^2 + 2x\sin\frac{5\pi}{12} + 1}\,dx &= \int \frac{\cos\frac{5\pi}{12}}{\cos^2\frac{5\pi}{12}\tan^2\theta + \cos^2\frac{5\pi}{12}} \cdot \cos\frac{5\pi}{12}\sec^2\theta\, d\theta \\ &= \int d\theta && \text{1A} \\ &= \theta + \text{constant} \\ &= \tan^{-1}\left(\frac{x + \sin\frac{5\pi}{12}}{\cos\frac{5\pi}{12}}\right) + \text{constant} && \text{1A} \\ &= \tan^{-1}\left(\frac{x\csc\frac{5\pi}{12} + 1}{\cot\frac{5\pi}{12}}\right) + \text{constant} \\ &= \tan^{-1}\frac{x(\sqrt{6} - \sqrt{2}) + 1}{2 - \sqrt{3}} + \text{constant} && \text{1A}\end{aligned}$$

47. (a) $f(x) = 4^2 - (x-3)^2$ 1A

(b) Let $x - 3 = 4\sin\theta$. Then $dx = 4\cos\theta\, d\theta$. 1M

$$\begin{aligned}\int \frac{1}{\sqrt{7 + 6x - x^2}}\,dx &= \int \frac{4\cos\theta}{\sqrt{16 - 16\sin^2\theta}}\,d\theta \\ &= \int d\theta && \text{1A} \\ &= \theta + \text{constant} \\ &= \sin^{-1}\frac{x-3}{4} + \text{constant} && \text{1A}\end{aligned}$$

48. (a)
$$\begin{aligned}x^2 + 2x\cos\frac{7\pi}{12} + 1 &= x^2 + 2x\cos\frac{7\pi}{12} + \sin^2\frac{7\pi}{12} + \cos^2\frac{7\pi}{12} \\ &= \left(x + \cos\frac{7\pi}{12}\right)^2 + \sin^2\frac{7\pi}{12} && \text{1A}\end{aligned}$$

(b) Let $x + \cos\frac{7\pi}{12} = \sin\frac{7\pi}{12}\tan\theta$. Then $dx = \sin\frac{7\pi}{12}\sec^2\theta\, d\theta$. 1M

$$\begin{aligned}\int \frac{1}{x^2 + 2x\cos\frac{7\pi}{12} + 1}\,dx &= \int \frac{\sin\frac{7\pi}{12}\sec^2\theta}{\sin^2\frac{7\pi}{12}\tan^2\theta + \sin^2\frac{7\pi}{12}}\,d\theta \\ &= \csc\frac{7\pi}{12}\int d\theta && \text{1A} \\ &= \csc\frac{7\pi}{12}\theta + \text{constant} \\ &= \csc\frac{7\pi}{12}\tan^{-1}\left(x\csc\frac{7\pi}{12} + \cot\frac{7\pi}{12}\right) + \text{constant} && \text{1A} \\ &= (\sqrt{6} - \sqrt{2})\tan^{-1}\left[(\sqrt{6} - \sqrt{2})x + \sqrt{3} - 2\right] + \text{constant} && \text{1A}\end{aligned}$$

49. Let $\sqrt{2}x = \sin\theta$. Then $\sqrt{2}\,\mathrm{d}x = \cos\theta\,\mathrm{d}\theta$. 1M

$$\int \frac{x^2}{\sqrt{1-2x^2}}\,\mathrm{d}x = \int \frac{\frac{1}{2}\sin^2\theta}{\sqrt{1-\sin^2\theta}} \cdot \frac{\cos\theta}{\sqrt{2}}\,\mathrm{d}\theta$$

$$= \frac{\sqrt{2}}{4}\int \sin^2\theta\,\mathrm{d}\theta \qquad \text{1A}$$

$$= \frac{\sqrt{2}}{8}\int (1-\cos 2\theta)\,\mathrm{d}\theta \qquad \text{1M}$$

$$= \frac{\sqrt{2}}{8}\theta - \frac{\sqrt{2}}{16}\sin 2\theta + \text{constant}$$

$$= \frac{\sqrt{2}}{8}\theta - \frac{\sqrt{2}}{8}\sin\theta\cos\theta + \text{constant}$$

$$= \frac{\sqrt{2}}{8}\sin^{-1}(\sqrt{2}x) - \frac{1}{4}x\sqrt{1-2x^2} + \text{constant} \qquad \text{1A}$$

50. $x^2 + 8x + 17 = (x+4)^2 + 1$

Let $x + 4 = \tan\theta$. Then $\mathrm{d}x = \sec^2\theta\,\mathrm{d}\theta$. 1M

$$\int \frac{x}{x^2+8x+17}\,\mathrm{d}x = \int \frac{(\tan\theta - 4)\sec^2\theta}{\tan^2\theta + 1}\,\mathrm{d}\theta$$

$$= \int (\tan\theta - 4)\,\mathrm{d}\theta \qquad \text{1A}$$

Let $u = \cos\theta$. Then $\mathrm{d}u = -\sin\theta\,\mathrm{d}\theta$. 1M

$$\int \frac{x}{x^2+8x+17}\,\mathrm{d}x = -\int \frac{\mathrm{d}u}{u} - \int 4\,\mathrm{d}\theta$$

$$= -\ln|\cos\theta| - 4\theta + \text{constant}$$

$$= -\ln\left|\frac{1}{\sqrt{x^2+8x+17}}\right| - 4\tan^{-1}(x+4) + \text{constant}$$

$$= \ln\sqrt{x^2+8x+17} - 4\tan^{-1}(x+4) + \text{constant} \qquad \text{1A}$$

51. (a) $4x - x^2 = 4 - (x-2)^2$ 1A

(b) Let $x - 2 = 2\sin\theta$. Then $\mathrm{d}x = 2\cos\theta\,\mathrm{d}\theta$. 1M

$$\int \frac{\mathrm{d}x}{\sqrt{4x-x^2}} = \int \frac{2\cos\theta}{\sqrt{4-4\sin^2\theta}}\,\mathrm{d}\theta$$

$$= \int \frac{\cos\theta}{\cos\theta}\,\mathrm{d}\theta$$

$$= \int \mathrm{d}\theta \qquad \text{1A}$$

$$= \theta + \text{constant}$$

$$= \sin^{-1}\frac{x-2}{2} + \text{constant} \qquad \text{1A}$$

52. Let $\ln x = \sin\theta$. Then $\frac{1}{x}\,\mathrm{d}x = \cos\theta\,\mathrm{d}\theta$. 1M

$$\int \frac{\sqrt{1-(\ln x)^2}}{x}\,\mathrm{d}x = \int \sqrt{1-\sin^2\theta}\cos\theta\,\mathrm{d}\theta$$

$$= \int \cos^2\theta\,\mathrm{d}\theta \qquad \text{1A}$$

$$= \frac{1}{2}\int (1+\cos 2\theta)\,\mathrm{d}\theta \qquad \text{1M}$$

$$= \frac{\theta}{2} + \frac{\sin 2\theta}{4} + \text{constant}$$

$$= \frac{\theta}{2} + \frac{\sin\theta\cos\theta}{2} + \text{constant}$$

$$= \frac{1}{2}\sin^{-1}(\ln x) + \frac{1}{2}(\ln x)\sqrt{1-(\ln x)^2} + \text{constant} \qquad \text{1A}$$

53. (a) $\sin 3\theta = \sin(2\theta + \theta)$

$$= \sin 2\theta\cos\theta + \cos 2\theta\sin\theta \qquad \text{1M}$$

$$= (2\sin\theta\cos\theta)\cos\theta + (2\cos^2\theta - 1)\sin\theta$$

$$= \sin\theta(4\cos^2\theta - 1) \qquad \text{1}$$

(b) Let $u = \cos\theta$. Then $\mathrm{d}u = -\sin\theta\,\mathrm{d}\theta$. 1M

$$\int \frac{\sin 3\theta}{1+\cos^2\theta}\,\mathrm{d}\theta = \int \frac{\sin\theta(4\cos^2\theta - 1)}{1+\cos^2\theta}\,\mathrm{d}\theta$$

$$= \int \frac{1-4u^2}{1+u^2}\,\mathrm{d}u$$

$$= \int \left(\frac{5}{1+u^2} - 4\right)\mathrm{d}u \qquad \text{1M}$$

Let $u = \tan\phi$. Then $\mathrm{d}u = \sec^2\phi\,\mathrm{d}\phi$. 1M

$$\int \frac{\sin 3\theta}{1+\cos^2\theta}\,\mathrm{d}\theta = \int \frac{5\sec^2\phi}{1+\tan^2\phi}\,\mathrm{d}\phi - 4u$$

$$= 5\int \mathrm{d}\phi - 4u$$

$$= 5\phi - 4u + \text{constant}$$

$$= 5\tan^{-1}(\cos\theta) - 4\cos\theta + \text{constant} \qquad \text{1A}$$

54. Let $x = e\sin\theta$. Then $\mathrm{d}x = e\cos\theta\,\mathrm{d}\theta$. 1M

$$\begin{aligned}
\int x^2\sqrt{e^2-x^2}\,\mathrm{d}x &= \int (e\sin\theta)^2\sqrt{e^2-e^2\sin^2\theta}\,e\cos\theta\,\mathrm{d}\theta \\
&= e^4\int \sin^2\theta\cos^2\theta\,\mathrm{d}\theta && \text{1A} \\
&= \frac{e^4}{4}\int (2\sin\theta\cos\theta)^2\,\mathrm{d}\theta \\
&= \frac{e^4}{4}\int \sin^2 2\theta\,\mathrm{d}\theta && \text{1M} \\
&= \frac{e^4}{8}\int (1-\cos 4\theta)\,\mathrm{d}\theta && \text{1M} \\
&= \frac{e^4\theta}{8} - \frac{e^4\sin 4\theta}{32} + \text{constant} \\
&= \frac{e^4\theta}{8} - \frac{e^4}{16}\sin 2\theta\cos 2\theta + \text{constant} \\
&= \frac{e^4\theta}{8} - \frac{e^4}{8}\sin\theta\cos\theta(1-2\sin^2\theta) + \text{constant} \\
&= \frac{e^4\theta}{8} - \frac{e^4}{8}\left(\frac{x}{e}\right)\left(\frac{\sqrt{e^2-x^2}}{e}\right)\left[1-2\left(\frac{x}{e}\right)^2\right] + \text{constant} \\
&= \frac{e^4}{8}\sin^{-1}\frac{x}{e} - \frac{1}{8}x(e^2-2x^2)\sqrt{e^2-x^2} + \text{constant} && \text{1A}
\end{aligned}$$

55. Let $x = 5\sin\theta$. Then $\mathrm{d}x = 5\cos\theta\,\mathrm{d}\theta$. 1M

$$\begin{aligned}
\int \frac{1}{(25-x^2)^{\frac{5}{2}}}\,\mathrm{d}x &= \int \frac{5\cos\theta}{(25-25\sin^2\theta)^{\frac{5}{2}}}\,\mathrm{d}\theta \\
&= \frac{1}{625}\int \sec^4\theta\,\mathrm{d}\theta && \text{1A}
\end{aligned}$$

Let $u = \tan\theta$. Then $\mathrm{d}u = \sec^2\theta\,\mathrm{d}\theta$. 1M

$$\begin{aligned}
\int \frac{1}{(25-x^2)^{\frac{5}{2}}}\,\mathrm{d}x &= \frac{1}{625}\int (u^2+1)\,\mathrm{d}u \\
&= \frac{1}{1875}u^3 + \frac{1}{625}u + \text{constant} \\
&= \frac{1}{1875}\tan^3\theta + \frac{1}{625}\tan\theta + \text{constant} \\
&= \frac{x^3}{1875(25-x^2)^{\frac{3}{2}}} + \frac{x}{625\sqrt{25-x^2}} + \text{constant} && \text{1A}
\end{aligned}$$

56. (a)
$$\begin{aligned}
\frac{\sec^2\theta}{\sec^2\theta + 2\tan^2\theta} &= \frac{\frac{1}{\cos^2\theta}}{\frac{1}{\cos^2\theta} + 2\cdot\frac{\sin^2\theta}{\cos^2\theta}} \times \frac{\cos^2\theta}{\cos^2\theta} \\
&= \frac{1}{1+2\sin^2\theta} && \text{1}
\end{aligned}$$

(b) Let $t = \tan x$. Then $\mathrm{d}t = \sec^2 x\,\mathrm{d}x$. 1M

$$\begin{aligned}\int \frac{\mathrm{d}x}{1+2\sin^2 x} &= \int \frac{\sec^2 x}{\sec^2 x + 2\tan^2 x}\,\mathrm{d}x \\ &= \int \frac{\mathrm{d}t}{(t^2+1)+2t^2} \qquad \text{1A} \\ &= \int \frac{\mathrm{d}t}{3t^2+1}\end{aligned}$$

Let $\sqrt{3}t = \tan\theta$. Then $\sqrt{3}\,\mathrm{d}t = \sec^2\theta\,\mathrm{d}\theta$. 1M

$$\begin{aligned}\int \frac{\mathrm{d}x}{1+2\sin^2 x} &= \frac{1}{\sqrt{3}}\int \frac{\sec^2\theta}{\tan^2\theta+1}\,\mathrm{d}\theta \\ &= \frac{\sqrt{3}}{3}\int \mathrm{d}\theta \\ &= \frac{\sqrt{3}}{3}\theta + \text{constant} \\ &= \frac{\sqrt{3}}{3}\tan^{-1}(\sqrt{3}\tan x) + \text{constant} \qquad \text{1A}\end{aligned}$$

57. Let $x = 2\tan\theta$. Then $\mathrm{d}x = 2\sec^2\theta\,\mathrm{d}\theta$. 1M

$$\begin{aligned}\int \frac{x^4}{x^2+4}\,\mathrm{d}x &= \frac{1}{2}\int \frac{(2\tan\theta)^4\cdot 2\sec^2\theta}{4\tan^2\theta+4}\,\mathrm{d}\theta \\ &= 8\int \tan^4\theta\,\mathrm{d}\theta \qquad \text{1M} \\ &= 8\int (\sec^2\theta-1)^2\,\mathrm{d}\theta \\ &= 8\int \sec^4\theta\,\mathrm{d}\theta - 16\int \sec^2\theta\,\mathrm{d}\theta + 8\int \mathrm{d}\theta\end{aligned}$$

Let $u = \tan\theta$. Then $\mathrm{d}u = \sec^2\theta\,\mathrm{d}\theta$. 1M

$$\begin{aligned}\int \frac{x^4}{x^2+4}\,\mathrm{d}x &= 8\int (u^2+1)\,\mathrm{d}u - 16\tan\theta + 8\theta \\ &= \frac{8u^3}{3} + 8u - 16\tan\theta + 8\theta + \text{constant} \\ &= \frac{8\tan^3\theta}{3} - 8\tan\theta + 8\theta + \text{constant} \\ &= \frac{x^3}{3} - 4x + 8\tan^{-1}\frac{x}{2} + \text{constant} \qquad \text{1A}\end{aligned}$$

58. (a) $$\begin{aligned}x^2 - 4x\cos\frac{\pi}{12} + 4 &= x^2 - 4x\cos\frac{\pi}{12} + 4\cos^2\frac{\pi}{12} + 4\sin^2\frac{\pi}{12} \\ &= \left(x - 2\cos\frac{\pi}{12}\right)^2 + 4\sin^2\frac{\pi}{12} \qquad \text{1A}\end{aligned}$$

(b) Let $x - 2\cos\frac{\pi}{12} = 2\sin\frac{\pi}{12}\tan\theta$. Then $\mathrm{d}x = 2\sin\frac{\pi}{12}\sec^2\theta\,\mathrm{d}\theta$. 1M

$$\begin{aligned}\int \frac{\sin\frac{\pi}{12}}{x^2 - 4\cos\frac{\pi}{12} + 4}\,\mathrm{d}x &= \int \frac{\sin\frac{\pi}{12}}{\left(x - 2\cos\frac{\pi}{12}\right)^2 + 4\sin^2\frac{\pi}{12}}\,\mathrm{d}x \\ &= \int \frac{\sin\frac{\pi}{12}}{4\sin^2\frac{\pi}{12}\tan^2\theta + 4\sin^2\frac{\pi}{12}} \cdot 2\sin\frac{\pi}{12}\sec^2\theta\,\mathrm{d}\theta \\ &= \frac{1}{2}\int \mathrm{d}\theta \qquad \text{1A} \\ &= \frac{1}{2}\theta + \text{constant} \\ &= \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\csc\frac{\pi}{12} - \cot\frac{\pi}{12}\right) + \text{constant} \\ &= \frac{1}{2}\tan^{-1}\left(\frac{\sqrt{6}+\sqrt{2}}{2}x - 2 - \sqrt{3}\right) + \text{constant} \qquad \text{1A}\end{aligned}$$

59. Let $e^x = \tan\theta$. Then $e^x\,\mathrm{d}x = \sec^2\theta\,\mathrm{d}\theta$. 1M

$$\begin{aligned}\int \frac{e^x\,\mathrm{d}x}{(1+e^{2x})^2} &= \int \frac{\sec^2\theta}{(1+\tan^2\theta)^2}\,\mathrm{d}\theta \\ &= \int \cos^2\theta\,\mathrm{d}\theta \qquad \text{1A} \\ &= \frac{1}{2}\int (1+\cos 2\theta)\,\mathrm{d}\theta \qquad \text{1M} \\ &= \frac{\theta}{2} + \frac{\sin 2\theta}{4} + \text{constant} \\ &= \frac{\theta}{2} + \frac{1}{2}\sin\theta\cos\theta + \text{constant} \\ &= \frac{1}{2}\tan^{-1}e^x + \frac{1}{2}\left(\frac{e^x}{\sqrt{1+e^{2x}}}\right)\left(\frac{1}{\sqrt{1+e^{2x}}}\right) + \text{constant} \\ &= \frac{1}{2}\tan^{-1}e^x + \frac{e^x}{2(1+e^{2x})} + \text{constant} \qquad \text{1A}\end{aligned}$$

60. Let $x = 7\sin\theta$. Then $\mathrm{d}x = 7\cos\theta\,\mathrm{d}\theta$. 1M

$$\begin{aligned}\int \sqrt{49 - x^2}\,\mathrm{d}x &= \int \sqrt{49 - 49\sin^2\theta}\,7\cos\theta\,\mathrm{d}\theta \\ &= 49\int \cos^2\theta\,\mathrm{d}\theta \qquad \text{1A} \\ &= \frac{49}{2}\int (1+\cos 2\theta)\,\mathrm{d}\theta \qquad \text{1M} \\ &= \frac{49\theta}{2} + \frac{49}{4}\sin 2\theta + \text{constant} \\ &= \frac{49\theta}{2} + \frac{49}{2}\sin\theta\cos\theta + \text{constant} \\ &= \frac{49}{2}\sin^{-1}\frac{x}{7} + \frac{x}{2}\sqrt{49 - x^2} + \text{constant} \qquad \text{1A}\end{aligned}$$

61. (a) $\displaystyle\int x\cos 2x\,\mathrm{d}x = \frac{x\sin 2x}{2} - \frac{1}{2}\int \sin 2x\,\mathrm{d}x$ 1M

$\displaystyle = \frac{x\sin 2x}{2} + \frac{\cos 2x}{4} + \text{constant}$ 1A

(b) $\displaystyle\int x\sin^2 x\,\mathrm{d}x = \frac{1}{2}\int x(1-\cos 2x)\,\mathrm{d}x$ 1M

$\displaystyle = \frac{1}{2}\int x\,\mathrm{d}x - \frac{1}{2}\int x\cos 2x\,\mathrm{d}x$

$\displaystyle = \frac{x^2}{4} - \frac{x\sin 2x}{4} - \frac{\cos 2x}{8} + \text{constant}$ 1A

62. (a) $\displaystyle\int e^x\sin x\,\mathrm{d}x = e^x\sin x - \int e^x\cos x\,\mathrm{d}x$ 1M

$\displaystyle = e^x\sin x - e^x\cos x - \int e^x\sin x\,\mathrm{d}x$ 1M

$\displaystyle 2\int e^x\sin x\,\mathrm{d}x = e^x\sin x - e^x\cos x + \text{constant}$

$\displaystyle \int e^x\sin x\,\mathrm{d}x = \frac{e^x\sin x}{2} - \frac{e^x\cos x}{2} + \text{constant}$ 1A

(b) Let $u = \frac{\pi}{2} - x$. Then $\mathrm{d}u = -\,\mathrm{d}x$. 1M

$\displaystyle\int \frac{\cos x}{e^x}\,\mathrm{d}x = -\int \frac{\cos\left(\frac{\pi}{2}-u\right)}{e^{\frac{\pi}{2}-u}}\,\mathrm{d}u$

$\displaystyle = -\frac{1}{e^{\frac{\pi}{2}}}\int e^u\sin u\,\mathrm{d}u$

$\displaystyle = -\frac{1}{e^{\frac{\pi}{2}}}\left(\frac{e^u\sin u}{2} - \frac{e^u\cos u}{2}\right) + \text{constant}$ 1M

$\displaystyle = -\frac{1}{2e^{\frac{\pi}{2}}}\left[e^{\frac{\pi}{2}-x}\sin\left(\frac{\pi}{2}-x\right) - e^{\frac{\pi}{2}-x}\cos\left(\frac{\pi}{2}-x\right)\right] + \text{constant}$ 1A

$\displaystyle = -\frac{1}{2}(e^{-x}\cos x - e^{-x}\sin x) + \text{constant}$

$\displaystyle = \frac{\sin x - \cos x}{2e^x} + \text{constant}$ 1A

63. Let $u = \sqrt{x}$. Then $\mathrm{d}u = \frac{1}{2\sqrt{x}}\,\mathrm{d}x$. 1M

$\displaystyle\int e^{\sqrt{x}}\,\mathrm{d}x = 2\int \sqrt{x}\cdot\frac{1}{2\sqrt{x}}e^{\sqrt{x}}\,\mathrm{d}x$

$\displaystyle = 2\int ue^u\,\mathrm{d}u$

$\displaystyle = 2ue^u - 2\int e^u\,\mathrm{d}u$ 1M

$= 2ue^u - 2e^u + \text{constant}$

$= 2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + \text{constant}$ 1A

64. $\int (x^2 + e^x)e^x \, dx = x^2 e^x - 2\int xe^x \, dx + \int e^{2x} \, dx$ 1M

$= x^2 e^x - 2xe^x + 2\int e^x \, dx + \int e^{2x} \, dx$ 1M

$= x^2 e^x - 2xe^x + 2e^x + \frac{1}{2}e^{2x} + \text{constant}$ 1A

65. $\int (2x^2 + 3)e^x \, dx = (2x^2 + 3)e^x - 4\int xe^x \, dx$ 1M

$= (2x^2 + 3)e^x - 4xe^x + 4\int e^x \, dx$ 1M

$= (2x^2 - 4x + 7)e^x + \text{constant}$ 1A

66. $\int \ln(1 - x) \, dx = x\ln(1 - x) + \int \frac{x}{1 - x} \, dx$ 1M

$= x\ln(1 - x) + \int \frac{1 - (1 - x)}{1 - x} \, dx$

$= x\ln(1 - x) + \int \frac{dx}{1 - x} - \int dx$ 1M

$= x\ln(1 - x) - \ln|1 - x| - x + \text{constant}$ 1A

67. (a) $\int e^x \sec^2 x \, dx = e^x \tan x - \int \frac{d}{dx}(e^x) \tan x \, dx$

$= e^x \tan x - \int e^x \tan x \, dx$ 1

(b) $\int e^x \sec^2 x \, dx = e^x \tan x - \int e^x \tan x \, dx$

$\int e^x(\tan^2 x + 1) \, dx = e^x \tan x - \int e^x \tan x \, dx$ 1M

$\int e^x(\tan^2 x + 1) \, dx + \int e^x \tan x \, dx = e^x \tan x + \text{constant}$

$\int e^x(\tan^2 x + \tan x + 1) \, dx = e^x \tan x + \text{constant}$ 1A

68. $\int e^{mx} \sin nx \, dx = \frac{1}{m}e^{mx} \sin nx - \frac{n}{m}\int e^{mx} \cos nx \, dx$ 1M

$= \frac{1}{m}e^{mx} \sin nx - \frac{n}{m^2}e^{mx} \cos nx - \frac{n^2}{m^2}\int e^{mx} \sin nx \, dx$ 1M

$\left(1 + \frac{n^2}{m^2}\right)\int e^{mx} \sin nx \, dx = \frac{1}{m}e^{mx} \sin nx - \frac{n}{m^2}e^{mx} \cos nx + \text{constant}$

$\int e^{mx} \sin nx \, dx = \frac{e^{mx}}{m^2 + n^2}(m \sin nx - n \cos nx) + \text{constant}$ 1A

69. Let $u = \sqrt{2x}$. Then $\mathrm{d}u = \dfrac{\sqrt{2}}{2}x^{-\frac{1}{2}}\,\mathrm{d}x = \dfrac{1}{u}\,\mathrm{d}x$. 1M

$$\begin{aligned}\int \sin\sqrt{2x}\,\mathrm{d}x &= \int u\sin u\,\mathrm{d}u \\ &= -u\cos u + \int \cos u\,\mathrm{d}u && \text{1M} \\ &= -u\cos u + \sin u + \text{constant} \\ &= -\sqrt{2x}\cos\sqrt{2x} + \sin\sqrt{2x} + \text{constant} && \text{1A}\end{aligned}$$

70.
$$\begin{aligned}\int e^{-3x}\cos 4x\,\mathrm{d}x &= -\frac{1}{3}e^{-3x}\cos 4x - \frac{4}{3}\int e^{-3x}\sin 4x\,\mathrm{d}x && \text{1M} \\ &= -\frac{1}{3}e^{-3x}\cos 4x + \frac{4}{9}e^{-3x}\sin 4x - \frac{16}{9}\int e^{-3x}\cos 4x\,\mathrm{d}x && \text{1M} \\ \frac{25}{9}\int e^{-3x}\cos 4x\,\mathrm{d}x &= -\frac{1}{3}e^{-3x}\cos 4x + \frac{4}{9}e^{-3x}\sin 4x + \text{constant} \\ \int e^{-3x}\cos 4x\,\mathrm{d}x &= -\frac{3}{25}e^{-3x}\cos 4x + \frac{4}{25}e^{-3x}\sin 4x + \text{constant} && \text{1A}\end{aligned}$$

71.
$$\begin{aligned}\int x\cdot 7^x\,\mathrm{d}x &= \int xe^{x\ln 7}\,\mathrm{d}x && \text{1M} \\ &= \frac{1}{\ln 7}xe^{x\ln 7} - \frac{1}{\ln 7}\int e^{x\ln 7}\,\mathrm{d}x && \text{1M} \\ &= \frac{1}{\ln 7}xe^{x\ln 7} - \frac{1}{(\ln 7)^2}e^{x\ln 7} + \text{constant} \\ &= \frac{1}{\ln 7}x\cdot 7^x - \frac{1}{(\ln 7)^2}7^x + \text{constant} && \text{1A}\end{aligned}$$

72.
$$\begin{aligned}\int (\ln x)^2\,\mathrm{d}x &= x(\ln x)^2 - 2\int \ln x\,\mathrm{d}x && \text{1M} \\ &= x(\ln x)^2 - 2x\ln x + 2\int \mathrm{d}x && \text{1M} \\ &= x(\ln x)^2 - 2x\ln x + 2x + \text{constant} && \text{1A}\end{aligned}$$

73. (a) Let $u = \dfrac{\sqrt{x}}{2}$. Then $\mathrm{d}u = \dfrac{1}{4\sqrt{x}}\,\mathrm{d}x = \dfrac{1}{8u}\,\mathrm{d}x$. 1M

$$\begin{aligned}\int e^{\frac{\sqrt{x}}{2}}\,\mathrm{d}x &= 8\int ue^u\,\mathrm{d}u \\ &= 8ue^u - 8\int e^u\,\mathrm{d}u && \text{1M} \\ &= 8ue^u - 8e^u + \text{constant} \\ &= 4\sqrt{x}e^{\frac{\sqrt{x}}{2}} - 8e^{\frac{\sqrt{x}}{2}} + \text{constant} && \text{1A}\end{aligned}$$

(b) Let $w = \dfrac{\sqrt{x}}{2}$. Then $\mathrm{d}w = \dfrac{1}{8w}\,\mathrm{d}x$.

$$\begin{aligned}\int \sqrt{x}e^{\frac{\sqrt{x}}{2}}\,\mathrm{d}x &= 16\int w^2e^w\,\mathrm{d}w \\ &= 16w^2e^w - 32\int we^w\,\mathrm{d}w && \text{1M} \\ &= 16w^2e^w - 32we^w + 32\int e^w\,\mathrm{d}w && \text{1M} \\ &= 16w^2e^w - 32we^w + 32e^w + \text{constant} \\ &= 4xe^{\frac{\sqrt{x}}{2}} - 16\sqrt{x}e^{\frac{\sqrt{x}}{2}} + 32e^{\frac{\sqrt{x}}{2}} + \text{constant} && \text{1A}\end{aligned}$$

74. $$\begin{aligned}\int x\cos 2x\cos 4x\,\mathrm{d}x &= \frac{1}{2}\int x(\cos 6x + \cos 2x)\,\mathrm{d}x && \text{1M} \\ &= \frac{1}{2}\int x\cos 6x\,\mathrm{d}x + \frac{1}{2}\int x\cos 2x\,\mathrm{d}x \\ &= \frac{1}{12}x\sin 6x - \frac{1}{12}\int \sin 6x\,\mathrm{d}x + \frac{1}{4}x\sin 2x - \frac{1}{4}\int \sin 2x\,\mathrm{d}x && \text{1M} \\ &= \frac{1}{12}x\sin 6x + \frac{1}{72}\cos 6x + \frac{1}{4}x\sin 2x + \frac{1}{8}\cos 2x + \text{constant} && \text{1A}\end{aligned}$$

75. $$\begin{aligned}\int e^x\sin(x+1)\,\mathrm{d}x &= e^x\sin(x+1) - \int e^x\cos(x+1)\,\mathrm{d}x && \text{1M} \\ &= e^x[\sin(x+1) - \cos(x+1)] - \int e^x\sin(x+1)\,\mathrm{d}x && \text{1M} \\ 2\int e^x\sin(x+1)\,\mathrm{d}x &= e^x[\sin(x+1) - \cos(x+1)] + \text{constant} \\ \int e^x\sin(x+1)\,\mathrm{d}x &= \frac{1}{2}e^x[\sin(x+1) - \cos(x+1)] + \text{constant} && \text{1A}\end{aligned}$$

76. $$\begin{aligned}\int x^5(\ln x)^2\,\mathrm{d}x &= \frac{1}{6}x^6(\ln x)^2 - \frac{1}{3}\int x^5\ln x\,\mathrm{d}x && \text{1M} \\ &= \frac{1}{6}x^6(\ln x)^2 - \frac{1}{18}x^6\ln x + \frac{1}{18}\int x^5\,\mathrm{d}x && \text{1M} \\ &= \frac{1}{6}x^6(\ln x)^2 - \frac{1}{18}x^6\ln x + \frac{1}{108}x^6 + \text{constant} && \text{1A}\end{aligned}$$

77. $$\begin{aligned}\int e^{2x}\sin 5x\,\mathrm{d}x &= \frac{1}{2}e^{2x}\sin 5x - \frac{5}{2}\int e^{2x}\cos 5x\,\mathrm{d}x && \text{1M} \\ &= \frac{1}{2}e^{2x}\sin 5x - \frac{5}{4}e^{2x}\cos 5x - \frac{25}{4}\int e^{2x}\sin 5x\,\mathrm{d}x && \text{1M} \\ \frac{29}{4}\int e^{2x}\sin 5x\,\mathrm{d}x &= \frac{1}{2}e^{2x}\sin 5x - \frac{5}{4}e^{2x}\cos 5x + \text{constant} \\ \int e^{2x}\sin 5x\,\mathrm{d}x &= \frac{e^{2x}}{29}(2\sin 5x - 5\cos 5x) + \text{constant} && \text{1A}\end{aligned}$$

78. $\int \cos x \ln(\sin x)\,\mathrm{d}x = \sin x \ln(\sin x) - \int \cos x$ 1M

$= \sin x \ln(\sin x) - \sin x + \text{constant}$ 1A

79. $\int \ln\left(\sqrt{x^2+1}-x\right)\mathrm{d}x$

$= x\ln\left(\sqrt{x^2+1}-x\right) - \int \frac{x}{\sqrt{x^2+1}-x}\cdot\left(\frac{2x}{2\sqrt{x^2+1}}-1\right)\mathrm{d}x$ 1M+1M

$= x\ln\left(\sqrt{x^2+1}-x\right) - \int \frac{x}{\sqrt{x^2+1}-x}\cdot\frac{x-\sqrt{x^2+1}}{\sqrt{x^2+1}}\,\mathrm{d}x$

$= x\ln\left(\sqrt{x^2+1}-x\right) + \int \frac{x}{\sqrt{x^2+1}}\,\mathrm{d}x$

Let $u = x^2+1$. Then $\mathrm{d}u = 2x\,\mathrm{d}x$. 1M

$\int \ln\left(\sqrt{x^2+1}-x\right)\mathrm{d}x$

$= x\ln\left(\sqrt{x^2+1}-x\right) + \frac{1}{2}\int \frac{\mathrm{d}u}{\sqrt{u}}$

$= x\ln\left(\sqrt{x^2+1}-x\right) + \sqrt{u} + \text{constant}$

$= x\ln\left(\sqrt{x^2+1}-x\right) + \sqrt{x^2+1} + \text{constant}$ 1A

80. $\int x^2\cos(\ln x)\,\mathrm{d}x = x^3\sin(\ln x) - 3\int x^2\sin(\ln x)\,\mathrm{d}x$ 1M

$= x^3\sin(\ln x) + 3\int x^3\cdot\left[-\frac{\sin(\ln x)}{x}\right]\mathrm{d}x$

$= x^3\sin(\ln x) + 3x^3\cos(\ln x) - 9\int x^2\cos(\ln x)\,\mathrm{d}x$ 1M

$10\int x^2\cos(\ln x)\,\mathrm{d}x = x^3\sin(\ln x) + 3x^3\cos(\ln x) + \text{constant}$

$\int x^2\cos(\ln x)\,\mathrm{d}x = \frac{1}{10}x^3\sin(\ln x) + \frac{3}{10}x^3\cos(\ln x) + \text{constant}$ 1A

81. (a) $\frac{\mathrm{d}y}{\mathrm{d}x} = kx^2 - 18x + 23$

$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2kx - 18$

$(3,\ 33)$ is the point of inflexion of the curve.

$2k(3) - 18 = 0$ 1M

$k = 3$ 1A

(b) $y = \int (3x^2 - 18x + 23)\,\mathrm{d}x$ 1M

$= x^3 - 9x^2 + 23x + C$

The curve passes through $(3,\ 33)$.

$$3^3 - 9(3)^2 + 23(3) + C = 33 \qquad \text{1M}$$
$$C + 18$$

The equation of the curve is $y = x^3 - 9x^2 + 23x + 18$. 1A

82. (a) Let $x - 1 = 6\sin\theta$. Then $\mathrm{d}x = 6\cos\theta\,\mathrm{d}\theta$. 1M

$$\begin{aligned}\int \frac{1}{\sqrt{35 + 2x - x^2}}\,\mathrm{d}x &= \int \frac{1}{\sqrt{36 - (x-1)^2}}\,\mathrm{d}x \\ &= \int \frac{6\cos\theta}{\sqrt{36 - 36\sin^2\theta}}\,\mathrm{d}\theta \\ &= \int \mathrm{d}\theta && \text{1M} \\ &= \theta + \text{constant} \\ &= \sin^{-1}\frac{x-1}{6} + \text{constant} && \text{1A}\end{aligned}$$

(b) Let $u = 35 + 2x - x^2$. Then $\mathrm{d}u = (2 - 2x)\,\mathrm{d}x$. 1M

$$\begin{aligned}y &= \int \frac{-2x + 3}{\sqrt{35 + 2x - x^2}}\,\mathrm{d}x \\ &= \int \frac{-2x + 2}{\sqrt{35 + 2x - x^2}}\,\mathrm{d}x + \int \frac{1}{\sqrt{35 + 2x - x^2}}\,\mathrm{d}x \\ &= \int u^{-\frac{1}{2}}\,\mathrm{d}u + \sin^{-1}\frac{x-1}{6} + C_1 && \text{1M} \\ &= 2u^{\frac{1}{2}} + \sin^{-1}\frac{x-1}{6} + C \\ &= 2\sqrt{35 + 2x - x^2} + \sin^{-1}\frac{x-1}{6} + C\end{aligned}$$

G passes through the point $\left(4,\ 6\sqrt{3}\right)$.

$$6\sqrt{3} = 2\sqrt{35 + 8 - 16} + \sin^{-1}\frac{4-1}{6} + C \qquad \text{1M}$$
$$C = -\frac{\pi}{6}$$

Put $x = 1$.

$$\begin{aligned}y &= 2\sqrt{35 + 2 - 1} + \sin^{-1}\frac{1-1}{6} - \frac{\pi}{6} \\ &= 12 - \frac{\pi}{6}\end{aligned}$$

Thus, G passes through $\left(1,\ 12 - \frac{\pi}{6}\right)$. 1A

83. (a) $f(x) = \int (2x - 4)\,\mathrm{d}x$

$$= x^2 - 4x + C \qquad \text{1M}$$

Γ passes through $(5,\ 15)$.

$$15 = 5^2 - 4(5) + C \qquad \text{1M}$$

$$C = 10$$

Required equation is $y = x^2 - 4x + 10$. 1A

(b) (i) Let $(a,\ b)$ be the coordinates of P.

$$\frac{b+1}{a+1} = 2a - 4 \qquad \text{1M}$$

$$\frac{(a^2 - 4a + 10) + 1}{a+1} = 2a - 4$$

$$a^2 - 4a + 11 = 2a^2 - 2a - 4$$

$$0 = a^2 + 2a - 15$$

$$a = 3 \quad \text{or} \quad -5$$

Note that the slope of Γ is positive, we have $a = 3$ only.

The coordinates of P are $(3,\ 7)$. 1A

(ii) Slope of $L = f'(3) = 2$

Slope of required straight line $= -\dfrac{1}{2}$ 1M

Required equation is

$$y - 7 = -\frac{1}{2}(x - 3) \qquad \text{1M}$$

$$x + 2y - 17 = 0 \qquad \text{1A}$$

84. (a) Let $u = 1 + x^3$. Then $\mathrm{d}u = 3x^2\,\mathrm{d}x$. 1M

$$\int \frac{x^5}{\sqrt{1+x^3}}\,\mathrm{d}x = \frac{1}{3}\int \frac{u-1}{\sqrt{u}}\,\mathrm{d}u \qquad \text{1M}$$

$$= \frac{1}{3}\int (u^{\frac{1}{2}} - u^{-\frac{1}{2}})\,\mathrm{d}u$$

$$= \frac{2}{9}u^{\frac{3}{2}} - \frac{2}{3}u^{\frac{1}{2}} + \text{constant}$$

$$= \frac{2}{9}(1+x^3)^{\frac{3}{2}} - \frac{2}{3}(1+x^3)^{\frac{1}{2}} + \text{constant} \qquad \text{1A}$$

(b) $$y = \int \frac{9x^5}{2\sqrt{1+x^3}}\,\mathrm{d}x \qquad \text{1M}$$

$$= \frac{9}{2}\left[\frac{2}{9}(1+x^3)^{\frac{3}{2}} - \frac{2}{3}(1+x^3)^{\frac{1}{2}}\right] + C \qquad \text{1M}$$

$$= (1+x^3)^{\frac{3}{2}} - 3\sqrt{1+x^3} + C$$

$$3 = 1 - 3 + C \qquad \text{1M}$$

$$C = 5$$

Required equation is $y = (1+x^3)^{\frac{3}{2}} - 3\sqrt{1+x^3} + 5$. 1A

85. (a) (i) Let r cm be the base radius of the inverted circular cone.

$$\frac{r}{12} = \frac{24-h}{24} \qquad \text{1M}$$

$$r = \frac{24-h}{2}$$

Consider the capacity of the container.

$$V = \frac{1}{3}\pi\left(\frac{24-h}{2}\right)^2 h$$

$$= \frac{\pi}{12}(h^3 - 48h^2 + 576h) \qquad \text{1A}$$

(ii) $$\frac{dV}{dh} = \frac{\pi}{12}(3h^2 - 96h + 576) \qquad \text{1M}$$

$$= \frac{\pi}{4}(h-8)(h-24)$$

When $\frac{dV}{dh} = 0$, we have $h = 8$ or 24 (rejected). 1M

h	$0 < h < 8$	$8 < h < 24$
$\frac{dV}{dh}$	+	−

1M

When V is a maximum, $h = 8$. 1A

(b) (i) $$\frac{dW}{dt} = 60e^{-\frac{t}{30}} - 55e^{-\frac{t}{25}} \qquad \text{1A}$$

$$W = \int\left(60e^{-\frac{t}{30}} - 55e^{-\frac{t}{25}}\right)ddt$$

$$= -1800e^{-\frac{t}{30}} + 1375e^{-\frac{t}{25}} + C$$

When $t = 0$, $W = 0$.

$$0 = -1800e^0 + 1375e^0 + C$$

$$C = 425$$

Thus, $W = -1800e^{-\frac{t}{30}} + 1375e^{-\frac{t}{25}} + 425$. 1A

(ii) Capacity of the container

$$= \frac{\pi}{12}[8^3 - 48(8)^2 + 576(8)]$$

$$= \frac{512\pi}{3}\ \text{cm}^3 \qquad \text{1A}$$

When $\frac{dW}{dt} = 0$.

$$60e^{-\frac{t}{30}} - 55e^{-\frac{t}{25}} = 0 \qquad \text{1M}$$

$$60e^{-\frac{t}{30}} = 55e^{-\frac{t}{25}}$$

$$e^{\frac{t}{150}} = \frac{11}{12}$$

$$\frac{t}{150} = \ln\frac{11}{12}$$

$$t = 150\ln\frac{11}{12}\ \text{(rejected)}$$

When $t = 0$, $W = 0$.

$$\lim_{t\to\infty} W = \lim_{t\to\infty}\left(-1800e^{-\frac{t}{30}} + 1375e^{-\frac{t}{25}} + 425\right) \quad \text{1M}$$

$$= -0 + 0 + 425$$

$$= 425$$

$$< \frac{512\pi}{3}$$

The water will not overflow.

The claim is agreed. 1A

86. (a) Note that $x^2 + 6x + 34 = (x+3)^2 + 25$.

Let $x + 3 = 5\tan\theta$. Then $\mathrm{d}x = 5\sec^2\theta\,\mathrm{d}\theta$. 1M

$$\int \frac{1}{x^2+6x+34}\,\mathrm{d}x = \int \frac{5\sec^2\theta}{25\tan^2\theta + 25}\,\mathrm{d}\theta \quad \text{1M}$$

$$= \frac{1}{5}\int \mathrm{d}\theta$$

$$= \frac{\theta}{5} + \text{constant}$$

$$= \frac{1}{5}\tan^{-1}\frac{x+3}{5} + \text{constant} \quad \text{1}$$

(b) Let $u = x^2 + 6x + 34$. Then $\mathrm{d}u = (2x+6)\,\mathrm{d}x$. 1M

$$y = \int \frac{2x+5}{x^2+6x+34}\,\mathrm{d}x$$

$$= \int \frac{2x+6}{x^2+6x+34}\,\mathrm{d}x - \int \frac{1}{x^2+6x+34}\,\mathrm{d}x$$

$$= \int \frac{\mathrm{d}u}{u} - \frac{1}{5}\tan^{-1}\frac{x+3}{5}$$

$$= \ln|u| - \frac{1}{5}\tan^{-1}\frac{x+3}{5} + C$$

$$= \ln\left(x^2+6x+34\right) - \frac{1}{5}\tan^{-1}\frac{x+3}{5} + C \quad \text{1M}$$

G passes through the point $(-8,\ \ln 2)$.

$$\ln 2 = \ln(64 - 48 + 34) - \frac{1}{5}\tan^{-1}\frac{-8+3}{5} + C \quad \text{1M}$$

$$\ln 2 = \ln 50 - \frac{1}{5}\left(-\frac{\pi}{4}\right) + C$$

$$C = -2\ln 5 - \frac{\pi}{20}$$

Put $x = -3$.

$$y = \ln(9 - 18 + 34) - 0 - 2\ln 5 - \frac{\pi}{20}$$

$$= -\frac{\pi}{20}$$

G passes through the point $\left(-3,\ -\frac{\pi}{20}\right)$. 1A

87. (a) Slope of $L = \cos\frac{\pi}{3}\sin^2\frac{\pi}{2} = \frac{3}{8}$ 1M

Equation of L is

$$y - \frac{\sqrt{3}}{4} = \frac{3}{8}\left(x - \frac{\pi}{3}\right)$$

$3x - 8y + 2\sqrt{3} - \pi = 0$ 1A

(b) Let $u = \sin x$. Then $du = \cos x \, dx$. 1M

$$\begin{aligned} y &= \int \cos x \sin^2 x \, dx \\ &= \int u^2 \, du \\ &= \frac{u^3}{3} + C \\ &= \frac{1}{3}\sin^3 x + C \end{aligned}$$

Γ passes through P.

$$\frac{\sqrt{3}}{4} = \frac{1}{3}\sin^3 \frac{\pi}{3} + C$$

$$C = \frac{\sqrt{3}}{8}$$

The equation of Γ is $y = \frac{1}{3}\sin^3 x + \frac{\sqrt{3}}{8}$. 1A

(c) $\frac{d^2y}{dx^2} = 2\sin x \cos^2 x - \sin^3 x$ 1M

$= 2\sin x(2\cos^2 x - \sin^2 x)$

When $\frac{d^2y}{dx^2} = 0$, $\sin x = 0$ or $\tan x = \pm\sqrt{2}$.

Thus, we have $x = \tan^{-1}\sqrt{2}$ or $x = \pi - \tan^{-1}\sqrt{2}$.

x	$0 < x < \tan^{-1}\sqrt{2}$	$\tan^{-1}\sqrt{2} < x < \pi - \tan^{-1}\sqrt{2}$	$\pi - \tan^{-1}\sqrt{2} < x < \pi$
$\frac{d^2y}{dx^2}$	+	−	+

1M

Required number is 2. 1A

88. (a) $h'(x) = \frac{2x^2 - 8x + 18}{x}$

$= \frac{2}{x}[(x^2 - 4x + 4 - 4) + 9]$ 1M

$= \frac{2}{x}[(x-2)^2 + 5]$

> 0 (for all $x > 0$)

Thus, $h(x)$ is an increasing function. 1A

(b) (i) $y = \int \frac{2x^2 - 8x + 18}{x}\,dx$

$= \int \left(2x - 8 + \frac{18}{x}\right) dx$ 1M

$= x^2 - 8x + 18\ln x + C$

When $x = 1$, $y = 5$.

$5 = 1^2 - 8(1) + 18\ln 1 + C$ 1M

$C = 12$

The equation of H is $y = x^2 - 8x + 18\ln x + 12$. 1A

(ii) $h''(x) = 2 - \frac{18}{x^2}$ 1A

$= \frac{2(x+3)(x-3)}{x^2}$

When $h''(x) = 0$, $x = 3$ or -3 (rejected).

x	$0 < x < 3$	$x > 3$
$h''(x)$	$-$	$+$

1A

Thus, $(3,\ 18\ln 3)$ is the point of inflexion of H. 1A

89. (a) Let $x = 4\sin\theta$. Then $dx = 4\cos\theta\,d\theta$. 1M

$y = \int \frac{2 - kx}{\sqrt{16 - x^2}}\,dx$

$= \int \frac{2 - 4k\sin\theta}{\sqrt{16 - 16\sin^2\theta}}(4\cos\theta)\,d\theta$ 1M

$= \int (2 - 4k\sin\theta)\,d\theta$

$= 2\theta - 4k\cos\theta + C$

$= 2\sin^{-1}\frac{x}{4} + k\sqrt{16 - x^2} + C$

Γ passes through the origin.

$0 = 2\sin^{-1}\frac{0}{4} + k\sqrt{16 - 0^2} + C$

$C = -4k$

Thus, $y = 2\sin^{-1}\frac{x}{4} + k\sqrt{16 - x^2} - 4k$. 1A

(b) (i) When $\frac{dy}{dx} = 0$, $x = \frac{2}{k}$. 1M

Since $-4 < \frac{2}{k} < 4$, we have $k < -\frac{1}{2}$ or $k > \frac{1}{2}$. 1A

(ii) $\frac{d^2y}{dx^2} = \frac{-k\sqrt{16 - x^2} - (2 - kx)\left(\frac{1}{2\sqrt{16 - x^2}}\right)(-2x)}{16 - x^2}$ 1M

$= \frac{2x - 16k}{(16 - x^2)^{\frac{3}{2}}}$

By (b)(i), we have $2x - 16k < 2x - 8$ or $2x - 16k > 2x + 8$ for $-4 < x < 4$.

The sign of $\frac{d^2y}{dx^2}$ on the interval $(-4,\ 4)$ may be positive or negative.

The claim is disagreed. 1A

90. (a) $y = \int x \ln x^4 \, dx$

$= 4 \int x \ln x \, dx$

$= 2x^2 \ln x - 2 \int x \, dx$ 1M

$= 2x^2 \ln x - x^2 + C$ 1A

Γ passes through the point $P\left(e^2,\ 3e^4 + 7\right)$.

$3e^4 + 7 = 2(e^2)^2 \ln e^2 - (e^2)^2 + C$ 1M

$C = 7$

Required equation is $y = 2x^2 \ln x - x^2 + 7$. 1A

(b) $f'(x) = 4x \ln x$

$f''(x) = 4(\ln x + 1)$ 1A

When $f''(x) = 0$, $x = \dfrac{1}{e}$.

x	$0 < x < \dfrac{1}{e}$	$x > \dfrac{1}{e}$
$f''(x)$	$-$	$+$

1M

The point of inflexion of Γ is $\left(\dfrac{1}{e},\ -3e^{-2} + 7\right)$. 1A

91. (a) Slope of tangent $= f'(5) = 5e$ 1A

Required equation is

$y - 13 = 5e(x - 5)$

$y = 5ex - 25e + 13$ 1A

(b) $f(x) = \int x e^{\frac{x}{5}} \, dx$

$= 5xe^{\frac{x}{5}} - 5 \int e^{\frac{x}{5}} \, dx$ 1M

$= 5xe^{\frac{x}{5}} - 25e^{\frac{x}{5}} + C$

Γ passes through $(5,\ 13)$.

$13 = 5(5)e^1 - 25e^1 + C$ 1M

$C = 13$

Required equation is $y = 5xe^{\frac{x}{5}} - 25e^{\frac{x}{5}} + 13$. 1A

(c) $f''(x) = \dfrac{1}{5}xe^{\frac{x}{5}} + e^{\frac{x}{5}}$ 1M

$= \dfrac{1}{5}e^{\frac{x}{5}}(x + 5)$

When $f''(x) = 0$, $x = -5$.

x	$x < -5$	$x > -5$	
$f''(x)$	$-$	$+$	1M

The point of inflexion of Γ is $\left(-5,\ -\dfrac{50}{e} + 13\right)$. 1A

92. (a) $\displaystyle\int \frac{\ln x}{x^2}\,dx = -\frac{\ln x}{x} + \int \frac{1}{x^2}\,dx$ 1M+1A

$\displaystyle = -\frac{\ln x}{x} - \frac{1}{x} + C_1$ 1A

(b) $\displaystyle y = \int \frac{e - \ln x}{x^2}\,dx$ 1M

$\displaystyle = \int \frac{e}{x^2}\,dx - \int \frac{\ln x}{x^2}\,dx$

$\displaystyle = -\frac{e}{x} + \frac{\ln x}{x} + \frac{1}{x} + C_2$ 1M

C passes through $(1,\ 1)$.

$1 = -e + 0 + 1 + C_2$ 1M

$C_2 = e$

Required equation is $y = -\dfrac{e}{x} + \dfrac{\ln x}{x} + \dfrac{1}{x} + e$. 1A

93. (a) Let $u = \ln x$. Then $du = \dfrac{1}{x}\,dx$. 1M

$\displaystyle\int \frac{(\ln x)^2}{x}\,dx = \int u^2\,du$ 1M

$\displaystyle = \frac{u^3}{3} + C$

$\displaystyle = \frac{(\ln x)^3}{3} + C$ 1A

(b) $\displaystyle y = \int \frac{(\ln x^3)^2}{x}\,dx$

$\displaystyle = \int \frac{(3\ln x)^2}{x}\,dx$ 1M

$\displaystyle = 9\int \frac{(\ln x)^2}{x}\,dx$

$= 3(\ln x)^3 + C$

Γ passes through the point $(1,\ 0)$.

$0 = 3(\ln 1)^3 + C$ 1M

$C = 0$

The equation of Γ is $y = 3(\ln x)^3$. 1A

94. (a) $y = \int (-e^{-x} + 1)\,\mathrm{d}x$

$= e^{-x} + x + C$ 1A

The curve passes through the point $(-1,\ e-2)$.

$$e - 2 = e - 1 + C$$ 1M

$$C = -1$$

Required equation is $y = e^{-x} + x - 1$. 1A

(b) When $x = 0$, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -e^0 + 1 = 0$. 1M

Required equation is

$$y - 0 = 0(x - 0)$$

$$y = 0$$ 1A

95. (a) $y = \int \dfrac{x+1}{x+3}\,\mathrm{d}x$ 1M

$= \int \dfrac{x+3-2}{x+3}\,\mathrm{d}x$

$= \int \left(1 - \dfrac{2}{x+3}\right)\mathrm{d}x$

$= x - 2\ln|x+3| + C$ 1A

Γ passes through $(e-3,\ 2e-5)$.

$$2e - 5 = (e - 3) - 2\ln e + C$$

$$C = e$$

Required equation is $x - 2\ln|x+3| + e$. 1A

(b) $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = \dfrac{\mathrm{d}}{\mathrm{d}x}\left(1 - \dfrac{2}{x+3}\right)$

$= \dfrac{2}{(x+3)^2}$ 1A

> 0

Γ does not have a point of inflexion. 1A

96. (a) $x = \int (-7t + a)\,\mathrm{d}t$

$= -\dfrac{7}{2}t^2 + at + C$ 1M

When $t = 0$, $x = 6280$.

$$6280 = 0 + 0 + C$$ 1M

$$C = 6280$$

When $t = 4$, $x = \dfrac{6280}{2} = 3140$.

$$3140 = -\frac{7}{2}(4)^2 + a(4) + 6280$$
$$a = -771$$

Thus, we have $x = -\dfrac{7}{2}t^2 - 771t + 6280$.
At 2:00 p.m., $t = 6$.

$$x = -\frac{7}{2}(6)^2 - 771(6) + 6280$$
$$= 1528$$
$$> 1500$$

The claim is disagreed. 1A

(b) Let y °C be the temperature inside the container when $t = 6$.

$$191(y^2 + 7y) = 1528$$ 1M
$$y^2 + 7y - 8 = 0$$
$$y = -8 \quad \text{or} \quad 1 \text{ (rejected)}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 191(2T + 7)\frac{\mathrm{d}T}{\mathrm{d}t}$$ 1M

When $t = 6$ and $T = -8$,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 191(2(-8) + 7)\frac{\mathrm{d}T}{\mathrm{d}t}$$
$$-7(6) - 771 = -1719\frac{\mathrm{d}T}{\mathrm{d}t}$$
$$\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{271}{573}$$

Required rate is $\dfrac{271}{573}$ °C/h. 1A

97. (a) $\displaystyle\int 3x^2 \ln x \, \mathrm{d}x = x^3 \ln x - \int x^2 \, \mathrm{d}x$ 1M+1A

$$= x^3 \ln x - \frac{x^3}{3} + C$$ 1A

(b) $\displaystyle y = \int 3x^2 \ln x \, \mathrm{d}x$ 1M

$$= x^3 \ln x - \frac{x^3}{3} + C$$

The curve passes through the point $\left(e,\ e^3\right)$.

$$e^3 = e^3 \ln e - \frac{e^3}{3} + C$$ 1M
$$C = \frac{e^3}{3}$$

Required equation is $y = x^3 \ln x - \dfrac{x^3}{3} + \dfrac{e^3}{3}$. 1A

98. $y = \int x\cos 2x \, \mathrm{d}x$ 1M

$= \dfrac{x\sin 2x}{2} - \dfrac{1}{2}\int \sin 2x \, \mathrm{d}x$ 1M+1A

$= \dfrac{x\sin 2x}{2} + \dfrac{\cos 2x}{4} + C$ 1A

The curve passes through the point $\left(\dfrac{\pi}{4},\ 0\right)$.

$0 = \dfrac{\frac{\pi}{4}\sin\frac{\pi}{2}}{2} + \dfrac{\cos\frac{\pi}{2}}{4} + C$ 1M

$C = -\dfrac{\pi}{8}$

Required equation is $y = \dfrac{x\sin 2x}{2} + \dfrac{\cos 2x}{4} - \dfrac{\pi}{8}$. 1A

99. (a) Let $u = x^2 + 4$. Then $\mathrm{d}u = 2x\,\mathrm{d}x$. 1M

$\int \dfrac{20x^3}{\sqrt[3]{x^2+4}}\,\mathrm{d}x = 10\int \dfrac{u-4}{u^{\frac{1}{3}}}\,\mathrm{d}u$ 1M

$= 10\int (u^{\frac{2}{3}} - 4u^{-\frac{1}{3}})\,\mathrm{d}u$

$= 6u^{\frac{5}{3}} - 60u^{\frac{2}{3}} + C$

$= 6(x^2+4)^{\frac{5}{3}} - 60(x^2+4)^{\frac{2}{3}} + C$ 1A

(b) $y = \int \dfrac{x^3}{\sqrt[3]{x^2+4}}\,\mathrm{d}x$ 1M

$= \dfrac{1}{20}\int \dfrac{20x^3}{\sqrt[3]{x^2+4}}\,\mathrm{d}x$

$= \dfrac{3}{10}(x^2+4)^{\frac{5}{3}} - 3(x^2+4)^{\frac{2}{3}} + C$

Γ passes through the point $(2,\ 1)$.

$1 = \dfrac{3}{10}(2^2+4)^{\frac{5}{3}} - 3(2^2+4)^{\frac{2}{3}} + C$ 1M

$C = 68$

Required equation is $y = \dfrac{3}{10}(x^2+4)^{\frac{5}{3}} - 3(x^2+4)^{\frac{2}{3}} + \dfrac{17}{5}$. 1A

100. (a) $f'(1) = \dfrac{1}{1^2 - 2(1) + 2} - \dfrac{1}{1} = 0$ 1M

Required equation is $y = 0$. 1A

(b) Let $x - 1 = \tan\theta$. Then $dx = \sec^2\theta\, d\theta$. 1M

$$\begin{aligned} f(x) &= \int \left(\frac{1}{x^2 - 2x + 2} - \frac{1}{x}\right) dx \\ &= \int \frac{1}{(x-1)^2 + 1}\, dx - \int \frac{1}{x}\, dx \\ &= \int \frac{\sec^2\theta}{\tan^2\theta + 1}\, d\theta - \ln|x| \\ &= \int d\theta - \ln x \\ &= \theta - \ln x + C_1 \\ &= \tan^{-1}(x-1) - \ln x + C_1 \end{aligned}$$

C passes through the point $(1,\ 0)$.

$$0 = \tan^{-1}(1-1) - \ln 1 + C_1$$ 1M

$$C_1 = 0$$

Required equation is $y = \tan^{-1}(x-1) - \ln x$. 1A

(c) $$\begin{aligned} f'(x) &= \frac{1}{x^2 - 2x + 2} - \frac{1}{x} \\ &= \frac{-x^2 + 3x - 2}{x(x^2 - 2x + 2)} \\ &= -\frac{(x-1)(x-2)}{x(x^2 - 2x + 2)} \end{aligned}$$

When $f'(x) = 0$, we have $x = 1$ or 2. 1M

x	$0 < x < 1$	$1 < x < 2$	$x > 2$
$f'(x)$	$-$	$+$	$-$

1M

The maximum point of C is $\left(2,\ \frac{\pi}{4} - \ln 2\right)$. 1A

END OF PAPER