1. (a) $\frac{\mathrm{d}}{\mathrm{d}x}\left[(x^6+1)\ln\left(x^2+1\right)\right]$

$= (x^6+1)\frac{2x}{x^2+1} + 6x^5\ln\left(x^2+1\right)$ 1M+1A

$= (x^2+1)(x^4-x^2+1)\frac{2x}{x^2+1} + 6x^5\ln\left(x^2+1\right)$ 1M

$= 2x(x^4-x^2+1) + 6x^5\ln\left(x^2+1\right)$ 1A

(b) By (a), $\int [2x(x^4-x^2+1)+6x^5\ln\left(x^2+1\right)]\,\mathrm{d}x = (x^6+1)\ln\left(x^2+1\right) + \text{constant}$ 1M

$$6\int x^5\ln\left(x^2+1\right)\mathrm{d}x = (x^6+1)\ln\left(x^2+1\right) - 2\int (x^5-x^3+x)\,\mathrm{d}x$$

$$= (x^6+1)\ln\left(x^2+1\right) - 2\left(\frac{x^6}{6}-\frac{x^4}{4}+\frac{x^2}{2}\right) + \text{constant}$$ 1A

$$\int x^5\ln\left(x^2+1\right)\mathrm{d}x = \frac{x^6+1}{6}\ln\left(x^2+1\right) - \frac{x^6}{18} + \frac{x^4}{12} - \frac{x^2}{6} + \text{constant}$$ 1A

2. (a) $\int \frac{x+1}{x}\,\mathrm{d}x = \int \left(1+\frac{1}{x}\right)\mathrm{d}x$ 1M

$= x + \ln|x| + \text{constant}$ 1A

(b) Let $u = x^2-1$. Then $\mathrm{d}u = 2x\,\mathrm{d}x$.

$$\int \frac{x^3}{x^2-1}\,\mathrm{d}x = \frac{1}{2}\int \frac{u+1}{u}\,\mathrm{d}u$$ 1A

$$= \frac{u}{2} + \frac{\ln|u|}{2} + \text{constant}$$ 1M

$$= \frac{x^2-1}{2} + \frac{\ln|x^2-1|}{2} + \text{constant}$$ 1A

3. (a) Let $u = 9 - x$. Then $\mathrm{d}u = -\,\mathrm{d}x$. 1M

$$\int \frac{\mathrm{d}x}{\sqrt{9-x}} - = \int \frac{\mathrm{d}u}{\sqrt{u}}$$ 1A

$$= -2u^{\frac{1}{2}} + \text{constant}$$

$$= -2\sqrt{9-x} + \text{constant}$$ 1A

(b) Let $x = 3\sin\theta$. Then $\mathrm{d}x = 3\cos\theta\,\mathrm{d}\theta$. 1M

$$\int \frac{\mathrm{d}x}{\sqrt{9-x^2}} = \int \frac{3\cos\theta\,\mathrm{d}\theta}{\sqrt{9-9\sin^2\theta}}$$

$$= \int \mathrm{d}\theta$$ 1A

$$= \theta + \text{constant}$$

$$= \sin^{-1}\frac{x}{3} + \text{constant}$$ 1A

4. $$\int \frac{x^4+1}{x^2}\,\mathrm{d}x = \int \left(x^2 + \frac{1}{x^2}\right)\mathrm{d}x$$ 1M

$$= \frac{x^3}{3} - \frac{1}{x} + \text{constant}$$ 1A

5. $$\int \frac{2}{\cot\frac{x}{2} + \tan\frac{x}{2}}\,\mathrm{d}x = \int \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\,\mathrm{d}x$$

$$= \int 2\tan\frac{x}{2}\cos^2\frac{x}{2}\,\mathrm{d}x$$ 1M

$$= \int 2\sin\frac{x}{2}\cos\frac{x}{2}\,\mathrm{d}x$$

$$= \int \sin x\,\mathrm{d}x$$ 1M

$$= -\cos x + \text{constant}$$ 1A

6. $\dfrac{dy}{d\theta} = \tan^2\theta\sec^2\theta - \sec^2\theta$ 1M

$= (\tan^2\theta - 1)(\tan^2\theta + 1)$

$= \tan^4\theta - 1$ 1A

$\displaystyle\int \tan^4\theta \, d\theta = \int (\tan^4\theta - 1 + 1) \, d\theta$ 1M

$= \dfrac{\tan^3\theta}{3} - \tan\theta + \theta + \text{constant}$ 1A

7. $\displaystyle\int (1 + \cos\theta)^2 \, d\theta = \int (1 + 2\cos\theta + \cos^2\theta) \, d\theta$

$\displaystyle= \theta + 2\sin\theta + \frac{1}{2}\int (1 + \cos 2\theta) \, d\theta$ 1M+1A

$= \dfrac{3\theta}{2} + 2\sin\theta + \dfrac{\sin 2\theta}{4} + \text{constant}$ 1A

8. (a) $\displaystyle\int \cos^2 2x \, dx = \frac{1}{2}\int (1 + \cos 4x) \, dx$ 1M

$= \dfrac{x}{2} + \dfrac{\sin 4x}{8} + \text{constant}$ 1A

(b) $\displaystyle\int \sin^2 2x \, dx = \int (1 - \cos^2 2x) \, dx$ 1M

$= x - \left(\dfrac{x}{2} + \dfrac{\sin 4x}{8}\right) + \text{constant}$ 1M

$= \dfrac{x}{2} - \dfrac{\sin 4x}{8} + \text{constant}$ 1A

9. $\int (\sin x - \cos x)^2 \, dx = \int (\sin^2 x + \cos^2 x - 2\sin x \cos x) \, dx$ 1A

$= \int (1 - \sin 2x) \, dx$ 1M+1M

$= x + \dfrac{\cos 2x}{2} + \text{constant}$ 1A

10. $\int \cos^2 \theta \, d\theta = \dfrac{1}{2} \int (1 + \cos 2\theta) \, d\theta$ 1M+1A

$= \dfrac{\theta}{2} + \dfrac{\sin 2\theta}{4} + \text{constant}$

11. (a) $\int (4x+1)^2 \, dx = \dfrac{(4x+1)^3}{3(4)} + \text{constant}$

$= \dfrac{(4x+1)^3}{12} + \text{constant}$ 1M+1A

(b) $\int \sin 3\theta \cos \theta \, d\theta = \dfrac{1}{2} \int (\sin 4\theta + \sin 2\theta) \, d\theta$ 1M

$= -\dfrac{\cos 4\theta}{8} - \dfrac{\cos 2\theta}{4} + \text{constant}$ 1M+1A

12. $\int \sqrt{2x+1} \, dx = \dfrac{(2x+1)^{\frac{3}{2}}}{3} + \text{constant}$ 1M+1A

13. Let $u = x - 1$. Then $du = dx$. 1M

$\int (x+2)\sqrt{x-1} \, dx = \int (u+3)\sqrt{u} \, du$

$= \int (u^{\frac{3}{2}} + 3u^{\frac{1}{2}}) \, du$ 1M

$= \dfrac{2u^{\frac{5}{2}}}{5} + 2u^{\frac{3}{2}} + \text{constant}$ 1A

$= \dfrac{2}{5}(x-1)^{\frac{5}{2}} + 2(x-1)^{\frac{3}{2}} + \text{constant}$ 1A

14. Let $u = x - 1$. $\mathrm{d}u = \mathrm{d}x$. 1M

$$\int x\sqrt{x-1}\,\mathrm{d}x = \int (u+1)\sqrt{u}\,\mathrm{d}u \quad \text{1A}$$
$$= \int (u^{\frac{3}{2}} + u^{\frac{1}{2}})\,\mathrm{d}u$$
$$= \frac{2u^{\frac{5}{2}}}{5} + \frac{2u^{\frac{3}{2}}}{3} + \text{constant} \quad \text{1A}$$
$$= \frac{2(x-1)^{\frac{5}{2}}}{5} + \frac{2(x-1)^{\frac{3}{2}}}{3} + \text{constant} \quad \text{1A}$$

15. Let $u = x + 2$. Then $\mathrm{d}u = \mathrm{d}x$ 1M

$$\int x(x+2)^{99}\,\mathrm{d}x = \int (u-2)u^{99}\,\mathrm{d}u \quad \text{1A}$$
$$= \int (u^{100} - 2u^{99})\,\mathrm{d}u$$
$$= \frac{u^{101}}{101} - \frac{u^{100}}{50} + \text{constant} \quad \text{1A}$$
$$= \frac{(x+2)^{101}}{101} - \frac{(x+2)^{100}}{50} + \text{constant} \quad \text{1A}$$

16. Let $u = 3x^2 + 1$. Then $\mathrm{d}u = 6x\,\mathrm{d}x$. 1M

$$\int \frac{x}{\sqrt{3x^2+1}}\,\mathrm{d}x = \frac{1}{6}\int \frac{6x}{\sqrt{3x^2+1}}\,\mathrm{d}x$$
$$= \frac{1}{6}\int \frac{\mathrm{d}u}{\sqrt{u}} \quad \text{1A}$$
$$= \frac{1}{3}u^{\frac{1}{2}} + \text{constant} \quad \text{1}$$
$$= \frac{1}{3}(3x^2+1)^{\frac{1}{2}} + \text{constant} \quad \text{1A}$$

17. (a) $\displaystyle\int \cos(3x+1)\,\mathrm{d}x = \frac{\sin(3x+1)}{3} + \text{constant}$ 1M+1A

(b) Let $u = 2 - x$. Then $\mathrm{d}u = -\,\mathrm{d}x$.

$$\int (2-x)^{2004}\,\mathrm{d}x = -\int u^{2004}\,\mathrm{d}u \qquad \text{1M}$$

$$= -\frac{u^{2005}}{2005} + \text{constant}$$

$$= -\frac{(2-x)^{2005}}{2005} + \text{constant} \qquad \text{1A}$$

18. $\displaystyle\int (8x+5)^{250}\,\mathrm{d}x = \frac{(8x+5)^{251}}{251(8)} + \text{constant}$ 1M

$$= \frac{(8x+5)^{251}}{2008} + \text{constant} \qquad \text{1A}$$

19. Let $u = \sqrt{x+9}$. Then $\mathrm{d}u = \frac{1}{2}(x+9)^{-\frac{1}{2}}\,\mathrm{d}x$. 1M

$$\int \frac{x}{\sqrt{x+9}}\,\mathrm{d}x = 2\int \frac{x}{2(x+9)^{\frac{1}{2}}}\,\mathrm{d}x$$

$$= 2\int (u^2 - 9)\,\mathrm{d}u \qquad \text{1A}$$

$$= \frac{2u^3}{3} - 18u + \text{constant} \qquad \text{1A}$$

$$= \frac{2}{3}(x+9)^{\frac{3}{2}} - 18(x+9)^{\frac{1}{2}} + \text{constant} \qquad \text{1A}$$

20. Let $u = \sin^2 x$. Then $\mathrm{d}u = 2\sin x\cos x\,\mathrm{d}x$. 1M

$$\int \frac{\sin x\cos x}{\sqrt{9\sin^2 x + 4\cos^2 x}}\,\mathrm{d}x = \frac{1}{2}\int \frac{2\sin x\cos x}{\sqrt{5\sin^2 x + 4}}\,\mathrm{d}x$$

$$= \frac{1}{2}\int \frac{\mathrm{d}u}{\sqrt{5u+4}} \qquad \text{1A}$$

$$= \frac{1}{5}\sqrt{5u+4} + \text{constant} \qquad \text{1A}$$

$$= \frac{1}{5}\sqrt{5\sin^2 x + 4} + \text{constant} \qquad \text{1A}$$

21. Let $u = x^2$. Then $\mathrm{d}u = 2x\,\mathrm{d}x$. 1M

$$\int x\sin^2(x^2)\,\mathrm{d}x = \frac{1}{2}\int \sin^2 u\,\mathrm{d}u$$

$$= \frac{1}{4}\int (1 - \cos 2u)\,\mathrm{d}u \qquad \text{1M}$$

$$= \frac{u}{4} - \frac{\sin 2u}{8} + \text{constant} \qquad \text{1A}$$

$$= \frac{x^2}{4} - \frac{\sin 2x^2}{8} + \text{constant} \qquad \text{1A}$$

22. (a) $$\int \tan^3 x\,\mathrm{d}x = \int \tan x(\sec^2 x - 1)\,\mathrm{d}x$$

$$= \int \tan x\,\mathrm{d}(\tan x) + \int \frac{\mathrm{d}(\cos x)}{\cos x} \qquad \text{1M}$$

$$= \frac{\tan^2 x}{2} + \ln|\cos x| + \text{constant} \qquad \text{1A}$$

(b) (i) $$\frac{A}{x} + \frac{B}{x-2} + \frac{C}{(x-2)^2} = \frac{A(x-2)^2 + Bx(x-2) + Cx}{x(x-2)^2}$$

$$= \frac{(A+B)x^2 + (-4A - 2B + C)x + 4A}{x(x-2)^2}$$

So, $A + B = 1$, $-4A - 2B + C = -1$ and $4A = 2$. 1M

Solving, we have $A = \frac{1}{2}$, $B = \frac{1}{2}$ and $C = 2$. 1A+1A

(ii) $$\int \frac{x^2 - x + 2}{x(x-2)^2}\,\mathrm{d}x = \int \left[\frac{1}{2x} + \frac{1}{2(x-2)} + \frac{2}{(x-2)^2}\right]\mathrm{d}x \qquad \text{1M}$$

$$= \frac{1}{2}\ln|x| + \frac{1}{2}\ln|x-2| - \frac{2}{x-2} + \text{constant} \qquad \text{1A}$$

23. (a) $3I + 4J = \int \frac{3\sin x + 4\cos x}{3\sin x + 4\cos x}\,\mathrm{d}x$

$= \int \mathrm{d}x$

$= x + \text{constant}$ 1A

(b) $4I - 3J = \int \frac{4\sin x - 3\cos x}{3\sin x + 4\cos x}\,\mathrm{d}x.$

Let $u = 3\sin x + 4\cos x$. Then $\mathrm{d}u = (3\cos x - 4\sin x)\,\mathrm{d}x$.

$4I - 3J = -\int \frac{\mathrm{d}u}{u}$ 1M

$= -\ln|u| + \text{constant}$

$= -\ln|3\sin x + 4\cos x| + \text{constant}$ 1A

(c) By (a) and (b),

$3(3I + 4J) + 4(4I - 3J) = 3x - 4\ln|3\sin x + 4\cos x| + \text{constant}$ 1M

$25I = 3x - 4\ln|3\sin x + 4\cos x| + \text{constant}$

$I = \frac{3x}{25} - \frac{4}{25}\ln|3\sin x + 4\cos x| + \text{constant}$ 1A

24. (a) Let $x = \sin^2\theta$. Then $\mathrm{d}x = 2\sin\theta\cos\theta\,\mathrm{d}\theta$

$\int \frac{f(x)}{\sqrt{x(1-x)}}\,\mathrm{d}x = \int \frac{f(\sin^2\theta)\cdot 2\sin\theta\cos\theta}{\sqrt{\sin^2\theta(1-\sin^2\theta)}}\,\mathrm{d}\theta$

$= 2\int f(\sin^2\theta)\,\mathrm{d}\theta$ 1

(b) $\int \frac{\mathrm{d}x}{\sqrt{x(1-x)}} = 2\int \mathrm{d}\theta$

$= 2\theta + \text{constant}$

$= 2\sin^{-1}\sqrt{x} + \text{constant}$ 1A

$\int \sqrt{\frac{x}{1-x}}\,\mathrm{d}x = \int \frac{x}{\sqrt{x(1-x)}}\,\mathrm{d}x$

$= 2\int \sin^2\theta\,\mathrm{d}\theta$ 1A

$= \int (1 - \cos 2\theta)\,\mathrm{d}\theta$

$= \theta - \frac{1}{2}\sin 2\theta + \text{constant}$ 1A

$= \sin^{-1}\sqrt{x} - \sqrt{x(1-x)} + \text{constant}$ 1A

25. $\displaystyle\int e^{2x}(\sin x + \cos x)^2\,\mathrm{d}x = \int e^{2x}(1 + 2\sin x\cos x)\,\mathrm{d}x$ 1M

$\displaystyle = \frac{e^{2x}}{2} + \int e^{2x}\sin 2x\,\mathrm{d}x$ 1M

Consider the integral $\displaystyle\int e^{2x}\sin 2x\,\mathrm{d}x$,

$\displaystyle\int e^{2x}\sin 2x\,\mathrm{d}x = \frac{e^{2x}}{2}\sin 2x - \int \frac{e^{2x}}{2}(2\cos 2x)\,\mathrm{d}x$ 1M

$\displaystyle = \frac{e^{2x}\sin 2x}{2} - \frac{e^{2x}}{2}\cos 2x + \int \frac{e^{2x}}{2}(-2\sin 2x)\,\mathrm{d}x$ 1M

$\displaystyle 2\int e^{2x}\sin 2x\,\mathrm{d}x = \frac{e^{2x}\sin 2x}{2} - \frac{e^{2x}\cos 2x}{2} + \text{constant}$ 1M

$\displaystyle\int e^{2x}\sin 2x\,\mathrm{d}x = \frac{e^{2x}}{4}(\sin 2x - \cos 2x) + \text{constant}$ 1A

Therefore, $\displaystyle\int e^{2x}(\sin x + \cos x)^2\,\mathrm{d}x = \frac{e^{2x}}{2} + \frac{e^{2x}}{4}(\sin 2x - \cos 2x) + \text{constant}$ 1A

26. (a) Let $u = 1 + x^2$. Then $\mathrm{d}u = 2x\,\mathrm{d}x$.

$\displaystyle\int \frac{x^3}{1+x^2}\,\mathrm{d}x = \frac{1}{2}\int \frac{u-1}{u}\,\mathrm{d}u$ 1M

$\displaystyle = \frac{1}{2}\int (1 - u^{-1})\,\mathrm{d}u$

$\displaystyle = \frac{u}{2} - \frac{1}{2}\ln|u| + \text{constant}$

$\displaystyle = \frac{x^2}{2} - \frac{1}{2}\ln\left(1 + x^2\right) + \text{constant}$ 1A

(b) $\displaystyle\int x^2\tan^{-1}x\,\mathrm{d}x = \frac{x^3}{3}\tan^{-1}x - \frac{1}{3}\int \frac{x^3}{1+x^2}\,\mathrm{d}x$ 1M+1A

$\displaystyle = \frac{x^3}{3}\tan^{-1}x - \frac{x^2}{6} + \frac{1}{6}\ln\left(1 + x^2\right) + \text{constant}$ 1A

27. (a) Let $2x = \tan\theta$. Then $2\,\mathrm{d}x = \sec^2\theta\,\mathrm{d}\theta$. 1M

$$\int \frac{1}{1+4x^2}\,\mathrm{d}x = \frac{1}{2}\int \frac{\sec^2\theta}{1+\tan^2\theta}\,\mathrm{d}\theta \qquad \text{1M}$$

$$= \frac{\theta}{2} + \text{constant}$$

$$= \frac{1}{2}\tan^{-1}2x + \text{constant} \qquad \text{1A}$$

(b) $$\int \ln\left(1+4x^2\right)\mathrm{d}x = x\ln\left(1+4x^2\right) - \int x\cdot\frac{8x}{1+4x^2}\,\mathrm{d}x \qquad \text{1M+1M}$$

$$= x\ln\left(1+4x^2\right) - 2\int\left(1-\frac{1}{1+4x^2}\right)\mathrm{d}x \qquad \text{1M}$$

$$= x\ln\left(1+4x^2\right) - 2x + \tan^{-1}2x + \text{constant} \qquad \text{1A}$$