M2REG-INDINT-2425-ASM-SET 3-MATH
Suggested solutions

1. Let $x = 2\sqrt{2}\sin\theta$. Then $\mathrm{d}x = 2\sqrt{2}\cos\theta\,\mathrm{d}\theta$. 1M

$$\int \frac{\mathrm{d}x}{\sqrt{8-x^2}} = \int \frac{2\sqrt{2}\cos\theta}{\sqrt{8-8\sin^2\theta}}\,\mathrm{d}\theta$$

$$= \int \mathrm{d}\theta \qquad \text{1A}$$

$$= \theta + \text{constant}$$

$$= \sin^{-1}\frac{x}{2\sqrt{2}} + \text{constant} \qquad \text{1A}$$

2. Let $x = \sqrt{2}\tan\theta$. Then $\mathrm{d}x = \sqrt{2}\sec^2\theta\,\mathrm{d}\theta$. 1M

$$\int \frac{\mathrm{d}x}{2+x^2} = \int \frac{\sqrt{2}\sec^2\theta}{2\tan^2\theta + 2}\,\mathrm{d}\theta$$

$$= \frac{1}{\sqrt{2}}\int \mathrm{d}\theta \qquad \text{1A}$$

$$= \frac{1}{\sqrt{2}}\theta + \text{constant}$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + \text{constant} \qquad \text{1A}$$

3. Let $x = \sqrt{3}\sin\theta$. Then $\mathrm{d}x = \sqrt{3}\cos\theta\,\mathrm{d}\theta$. 1M

$$\int \sqrt{1-\frac{x^2}{3}}\,\mathrm{d}x = \frac{1}{\sqrt{3}}\int \sqrt{3-3\sin^2\theta}\cdot\sqrt{3}\cos\theta\,\mathrm{d}\theta$$

$$= \sqrt{3}\int \cos^2\theta\,\mathrm{d}\theta \qquad \text{1A}$$

$$= \frac{\sqrt{3}}{2}\int (1+\cos 2\theta)\,\mathrm{d}\theta \qquad \text{1M}$$

$$= \frac{\sqrt{3}\theta}{2} + \frac{\sqrt{3}}{4}\sin 2\theta + \text{constant}$$

$$= \frac{\sqrt{3}\theta}{2} + \frac{\sqrt{3}}{2}\sin\theta\cos\theta + \text{constant}$$

$$= \frac{\sqrt{3}}{2}\sin^{-1}\frac{x}{\sqrt{3}} + \frac{\sqrt{3}}{2}\left(\frac{x}{\sqrt{3}}\right)\left(\frac{\sqrt{3-x^2}}{\sqrt{3}}\right) + \text{constant}$$

$$= \frac{\sqrt{3}}{2}\sin^{-1}\frac{x}{\sqrt{3}} + \frac{x\sqrt{3(3-x^2)}}{6} + \text{constant} \qquad \text{1A}$$

4. Let $11x = 3\sin\theta$. Then $11\,\mathrm{d}x = 3\cos\theta\,\mathrm{d}\theta$. 1M

$$\int \sqrt{9-121x^2}\,\mathrm{d}x = \frac{3}{11}\int \sqrt{9-9\sin^2\theta}\cos\theta\,\mathrm{d}\theta$$

$$= \frac{9}{11}\int \cos^2\theta\,\mathrm{d}\theta \qquad \text{1A}$$

$$= \frac{9}{22}\int (1+\cos 2\theta)\,\mathrm{d}\theta \qquad \text{1M}$$

$$= \frac{9}{22}\theta + \frac{9}{44}\sin 2\theta + \text{constant}$$

$$= \frac{9}{22}\theta + \frac{9}{22}\sin\theta\cos\theta + \text{constant}$$

$$= \frac{9}{22}\sin^{-1}\frac{11x}{3} + \frac{9}{22}\left(\frac{11x}{3}\right)\left(\frac{\sqrt{9-121x^2}}{3}\right) + \text{constant}$$

$$= \frac{9}{22}\sin^{-1}\frac{11x}{3} + \frac{x\sqrt{9-121x^2}}{2} + \text{constant} \qquad \text{1A}$$

5. Let $2\sqrt{2}x = \tan\theta$. Then $2\sqrt{2}\,\mathrm{d}x = \sec^2\theta\,\mathrm{d}\theta$. 1M

$$\int \frac{\mathrm{d}x}{(1+8x^2)^2} = \frac{1}{2\sqrt{2}}\int \frac{\sec^2\theta}{(1+\tan^2\theta)^2}\,\mathrm{d}\theta$$

$$= \frac{1}{2\sqrt{2}}\int \cos^2\theta\,\mathrm{d}\theta \qquad \text{1A}$$

$$= \frac{1}{4\sqrt{2}}\int (1+\cos 2\theta)\,\mathrm{d}\theta \qquad \text{1M}$$

$$= \frac{1}{4\sqrt{2}}\theta + \frac{1}{8\sqrt{2}}\sin 2\theta + \text{constant}$$

$$= \frac{1}{4\sqrt{2}}\theta + \frac{1}{4\sqrt{2}}\sin\theta\cos\theta + \text{constant}$$

$$= \frac{1}{4\sqrt{2}}\tan^{-1}(2\sqrt{2}x) + \frac{1}{4\sqrt{2}}\left(\frac{2\sqrt{2}x}{\sqrt{1+8x^2}}\right)\left(\frac{1}{\sqrt{1+8x^2}}\right) + \text{constant}$$

$$= \frac{1}{4\sqrt{2}}\tan^{-1}(2\sqrt{2}x) + \frac{x}{2(1+8x^2)} + \text{constant} \qquad \text{1A}$$

6. Let $x = 7\sin\theta$. Then $\mathrm{d}x = 7\cos\theta\,\mathrm{d}\theta$. 1M

$$\begin{aligned}\int \frac{x^2}{\sqrt{49-x^2}}\,\mathrm{d}x &= \int \frac{49\sin^2\theta\cdot 7\cos\theta}{\sqrt{49-49\sin^2\theta}}\,\mathrm{d}\theta \\ &= 49\int \sin^2\theta\,\mathrm{d}\theta && \text{1A} \\ &= \frac{49}{2}\int (1-\cos 2\theta)\,\mathrm{d}\theta && \text{1M} \\ &= \frac{49\theta}{2} - \frac{49\sin 2\theta}{4} + \text{constant} \\ &= \frac{49\theta}{2} - \frac{49\sin\theta\cos\theta}{2} + \text{constant} \\ &= \frac{49}{2}\sin^{-1}\frac{x}{7} - \frac{49}{2}\left(\frac{x}{7}\right)\left(\frac{\sqrt{49-x^2}}{7}\right) + \text{constant} \\ &= \frac{49}{2}\sin^{-1}\frac{x}{7} - \frac{x\sqrt{49-x^2}}{2} + \text{constant} && \text{1A}\end{aligned}$$

7. $$\int \frac{\mathrm{d}x}{e^x + e^{-x}} = \int \frac{e^x\,\mathrm{d}x}{e^{2x}+1}$$

Let $e^x = \tan\theta$. Then $e^x\,\mathrm{d}x = \sec^2\theta\,\mathrm{d}\theta$. 1M

$$\begin{aligned}\int \frac{\mathrm{d}x}{e^x + e^{-x}} &= \int \frac{\sec^2\theta\,\mathrm{d}\theta}{\tan^2\theta + 1} && \text{1M} \\ &= \int \mathrm{d}\theta \\ &= \theta + C \\ &= \tan^{-1}(e^x) + C && \text{1A}\end{aligned}$$

8. (a) $$\begin{aligned}x^2 + 2x\sin\frac{5\pi}{12} + 1 &= x^2 + 2x\sin\frac{5\pi}{12} + \sin^2\frac{5\pi}{12} + \cos^2\frac{5\pi}{12} \\ &= \left(x + \sin\frac{5\pi}{12}\right)^2 + \cos^2\frac{5\pi}{12} && \text{1A}\end{aligned}$$

(b) Let $x + \sin\frac{5\pi}{12} = \cos\frac{5\pi}{12}\tan\theta$. Then $\mathrm{d}x = \cos\frac{5\pi}{12}\sec^2\theta\,\mathrm{d}\theta$. 1M

$$\begin{aligned}\int \frac{\cos\frac{5\pi}{12}}{x^2 + 2x\sin\frac{5\pi}{12} + 1}\,\mathrm{d}x &= \int \frac{\cos\frac{5\pi}{12}}{\cos^2\frac{5\pi}{12}\tan^2\theta + \cos^2\frac{5\pi}{12}}\cdot\cos\frac{5\pi}{12}\sec^2\theta\,\mathrm{d}\theta \\ &= \int \mathrm{d}\theta && \text{1A} \\ &= \theta + \text{constant} \\ &= \tan^{-1}\left(\frac{x + \sin\frac{5\pi}{12}}{\cos\frac{5\pi}{12}}\right) + \text{constant} && \text{1A} \\ &= \tan^{-1}\left(\frac{x\csc\frac{5\pi}{12} + 1}{\cot\frac{5\pi}{12}}\right) + \text{constant} \\ &= \tan^{-1}\frac{x(\sqrt{6}-\sqrt{2}) + 1}{2-\sqrt{3}} + \text{constant} && \text{1A}\end{aligned}$$

9. $x^2 + 8x + 17 = (x + 4)^2 + 1$

Let $x + 4 = \tan\theta$. Then $dx = \sec^2\theta\, d\theta$. 1M

$$\int \frac{x}{x^2 + 8x + 17}\, dx = \int \frac{(\tan\theta - 4)\sec^2\theta}{\tan^2\theta + 1}\, d\theta$$

$$= \int (\tan\theta - 4)\, d\theta \quad \text{1A}$$

Let $u = \cos\theta$. Then $du = -\sin\theta\, d\theta$. 1M

$$\int \frac{x}{x^2 + 8x + 17}\, dx = -\int \frac{du}{u} - \int 4\, d\theta$$

$$= -\ln|\cos\theta| - 4\theta + \text{constant}$$

$$= -\ln\left|\frac{1}{\sqrt{x^2 + 8x + 17}}\right| - 4\tan^{-1}(x + 4) + \text{constant}$$

$$= \ln\sqrt{x^2 + 8x + 17} - 4\tan^{-1}(x + 4) + \text{constant} \quad \text{1A}$$

10. Let $x^2 = 4\tan\theta$. Then $2x\, dx = 4\sec^2\theta\, d\theta$. 1M

$$\int \frac{x\, dx}{x^4 + 16} = \frac{1}{2}\int \frac{4\sec^2\theta\, d\theta}{16\tan^2\theta + 16} \quad \text{1M}$$

$$= \frac{1}{8}\int d\theta$$

$$= \frac{\theta}{8} + \text{constant}$$

$$= \frac{1}{8}\tan^{-1}\frac{x^2}{4} + \text{constant} \quad \text{1A}$$

11. (a) $8x - x^2 = -[x^2 - 2(4)(x) + 4^2] + 16$

$= 16 - (x - 4)^2$ 1A

(b) $\displaystyle\int \frac{dx}{\sqrt{8x - x^2}} = \int \frac{dx}{\sqrt{16 - (x - 4)^2}}$.

Let $x - 4 = 4\sin\theta$. Then $dx = 4\cos\theta\, d\theta$. 1M

$$\int \frac{dx}{\sqrt{8x - x^2}} = \int \frac{4\cos\theta\, d\theta}{\sqrt{16 - 16\sin^2\theta}} \quad \text{1M}$$

$$= \int d\theta$$

$$= \theta + \text{constant}$$

$$= \sin^{-1}\frac{x - 4}{4} + \text{constant} \quad \text{1A}$$

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