

M2REG-DEFINT-2425-ASM-SET 2-MATH**Suggested solutions**

1. Let $u = 3 + 2 \ln x$. Then $du = \frac{2}{x} dx$. 1M

$$\begin{aligned} \int_1^e \frac{3 + \ln x^2}{x} dx &= \int_1^e \frac{3 + 2 \ln x}{x} dx \\ &= \frac{1}{2} \int_3^5 u du && 1A \\ &= \frac{1}{4} \left[u^2 \right]_3^5 \\ &= 4 && 1A \end{aligned}$$

2. Let $u = e^x - e^{-x}$. Then $du = (e^x + e^{-x}) dx$. 1M

$$\begin{aligned} \int_{\ln 2}^{\ln 4} \frac{e^x + e^{-x}}{\sqrt{e^x - e^{-x}}} dx &= \int_{\frac{3}{2}}^{\frac{15}{4}} \frac{du}{\sqrt{u}} && 1A \\ &= \left[2u^{\frac{1}{2}} \right]_{\frac{3}{2}}^{\frac{15}{4}} \\ &= \sqrt{15} - \sqrt{6} && 1A \end{aligned}$$

3. Let $u = x^3 + 1$. Then $du = 3x^2 dx$. 1M

$$\begin{aligned} \int_0^{\sqrt[3]{7}} \frac{x^5}{\sqrt[3]{x^3 + 1}} dx &= \frac{1}{3} \int_0^{\sqrt[3]{7}} \frac{x^3(3x^2)}{\sqrt[3]{x^3 + 1}} dx \\ &= \frac{1}{3} \int_1^8 \frac{u - 1}{\sqrt[3]{u}} du && 1A \\ &= \frac{1}{3} \int_1^8 (u^{\frac{2}{3}} - u^{-\frac{1}{3}}) du \\ &= \frac{1}{3} \left[\frac{3u^{\frac{5}{3}}}{5} - \frac{3u^{\frac{2}{3}}}{2} \right]_1^8 \\ &= \frac{47}{10} && 1A \end{aligned}$$

4. Let $u = \sqrt{x} + 4$. Then $du = \frac{1}{2\sqrt{x}} dx$. 1M

$$\begin{aligned} \int_0^4 \frac{\sqrt{x} - 4}{\sqrt{x} + 4} dx &= \int_0^4 \frac{2\sqrt{x}(\sqrt{x} - 4)}{\sqrt{x} + 4} \cdot \frac{1}{2\sqrt{x}} dx \\ &= \int_4^6 \frac{2(u - 4)(u - 8)}{u} du && 1A \\ &= \int_4^6 \left(2u - 24 + \frac{64}{u} \right) du \\ &= \left[u^2 - 24u + 64 \ln |u| \right]_4^6 \\ &= -28 + 64 \ln \frac{3}{2} && 1A \end{aligned}$$

5. Let $u = 1 - x^2$. Then $du = -2x dx$. 1M

$$\begin{aligned} \int_0^1 x^3 \cdot \sqrt{1 - x^2} dx &= -\frac{1}{2} \int_1^0 (1 - u)\sqrt{u} du && 1A \\ &= \frac{1}{2} \int_0^1 (u^{\frac{1}{2}} - u^{\frac{3}{2}}) du \\ &= \frac{1}{2} \left[\frac{2}{3} u^{\frac{3}{2}} - \frac{2}{5} u^{\frac{5}{2}} \right]_0^1 \\ &= \frac{2}{15} && 1A \end{aligned}$$

6. Let $u = e^{3x} + 3x + 1$. Then $du = (3e^{3x} + 3) dx$. 1M

$$\begin{aligned} \int_0^3 \frac{x}{e^{3x} + 3x + 1} dx &= \frac{1}{3} \int_0^3 \left(1 - \frac{e^{3x} + 1}{e^{3x} + 3x + 1} \right) dx && 1A \\ &= \frac{1}{3} \int_0^3 dx - \frac{1}{9} \int_2^{e^9+10} \frac{du}{u} && 1A \\ &= \frac{1}{3} \left[x \right]_0^3 - \frac{1}{9} \left[\ln |u| \right]_2^{e^9+10} \\ &= 1 - \frac{1}{9} \ln \frac{e^9 + 10}{2} && 1A \end{aligned}$$

7. (a) Let $u = 1 - e^{-x}$. Then $du = e^{-x} dx$. 1M

$$\int_{\ln 2}^{\ln 3} \frac{e^{-x}}{1 - e^{-x}} dx = \int_{\frac{1}{2}}^{\frac{2}{3}} \frac{du}{u} \quad 1A$$

$$= \left[\ln |u| \right]_{\frac{1}{2}}^{\frac{2}{3}} \\ = \ln \frac{4}{3} \quad 1A$$

$$(b) \frac{A}{u} + \frac{B}{u-1} = \frac{A(u-1) + Bu}{u(u-1)} \\ = \frac{(A+B)u - A}{u(u-1)}$$

We have $A + B = 0$ and $-A = 1$. 1M

Solving, we have $A = -1$ and $B = 1$. 1A

$$(c) \int_{\ln 2}^{\ln 3} \frac{1}{e^x(e^x - 1)} dx = \int_{\ln 2}^{\ln 3} \left(-\frac{1}{e^x} + \frac{1}{e^x - 1} \right) dx \quad 1M$$

$$= - \int_{\ln 2}^{\ln 3} e^{-x} dx + \int_{\ln 2}^{\ln 3} \frac{1}{e^x - 1} dx \\ = - \int_{\ln 2}^{\ln 3} e^{-x} dx + \int_{\ln 2}^{\ln 3} \frac{e^{-x}}{1 - e^{-x}} dx \\ = \left[e^{-x} \right]_{\ln 2}^{\ln 3} + \ln \frac{4}{3} \quad 1M$$

$$= -\frac{1}{6} + \ln \frac{4}{3} \quad 1A$$

$$\begin{aligned}
 8. \quad (a) \quad \frac{x}{x^2+4} - \frac{1}{x+4} &= \frac{x(x+4) - (x^2+4)}{(x^2+4)(x+4)} & 1M \\
 &= \frac{4x-4}{(x^2+4)(x+4)} \\
 &= \frac{4(x-1)}{(x^2+4)(x+4)} & 1
 \end{aligned}$$

(b) Let $t = 2 \tan \theta$. Then $dt = 2 \sec^2 \theta d\theta$.

$$\begin{aligned}
 \int_0^{\frac{\pi}{4}} \frac{2 \tan \theta - 1}{\tan \theta + 2} d\theta &= \int_0^{\frac{\pi}{4}} \frac{2 \tan \theta - 1}{2 \sec^2 \theta (\tan \theta + 2)} \cdot 2 \sec^2 \theta d\theta \\
 &= \int_0^2 \frac{t-1}{2 \left(1 + \frac{t^2}{4}\right) \left(\frac{t}{2} + 2\right)} dt & 1A \\
 &= \int_0^2 \frac{4(t-1)}{(t^2+4)(t+4)} dt \\
 &= \int_0^2 \left(\frac{t}{t^2+4} - \frac{1}{t+4} \right) dt & 1M
 \end{aligned}$$

Let $u = t^2 + 4$. Then $du = 2t dt$. 1M

$$\begin{aligned}
 \int_0^{\frac{\pi}{4}} \frac{2 \tan \theta - 1}{\tan \theta + 2} d\theta &= \frac{1}{2} \int_4^8 \frac{du}{u} - \int_0^2 \frac{dt}{t+4} \\
 &= \frac{1}{2} \left[\ln |u| \right]_4^8 - \left[\ln |t+4| \right]_0^2 & 1A \\
 &= \frac{1}{2} (\ln 8 - \ln 4) - (\ln 6 - \ln 2) \\
 &= \frac{5}{2} \ln 2 - \ln 6 & 1A
 \end{aligned}$$

9. Let $x = \sqrt{3} \tan \theta$. Then $dx = \sqrt{3} \sec^2 \theta d\theta$. 1M

$$\begin{aligned}
 \int_{-3}^3 \frac{1}{(x^2+3)^{\frac{3}{2}}} dx &= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{\sqrt{3} \sec^2 \theta}{(3 \tan^2 \theta + 3)^{\frac{3}{2}}} d\theta & 1A \\
 &= \frac{1}{3} \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{\sec^2 \theta}{\sec^3 \theta} d\theta \\
 &= \frac{1}{3} \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \cos \theta d\theta \\
 &= \frac{1}{3} \left[\sin \theta \right]_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \\
 &= \frac{\sqrt{3}}{3} & 1A
 \end{aligned}$$

10. Let $x = \sqrt{8} \sin \theta$. Then $dx = \sqrt{8} \cos \theta d\theta$. 1M

$$\int_{\sqrt{2}}^2 \frac{x^4}{\sqrt{8-x^2}} dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{64 \sin^4 \theta}{\sqrt{8-8 \sin^2 \theta}} \cdot \sqrt{8} \cos \theta d\theta \quad 1A$$

$$= 64 \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sin^4 \theta d\theta$$

$$= 64 \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \left(\frac{1 - \cos 2\theta}{2} \right)^2 d\theta \quad 1M$$

$$= 16 \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} (1 - 2 \cos 2\theta + \cos^2 2\theta) d\theta$$

$$= 16 \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \left(1 - 2 \cos 2\theta + \frac{1 + \cos 4\theta}{2} \right) d\theta \quad 1A$$

$$= 16 \left[\frac{3\theta}{2} - \sin 2\theta + \frac{\sin 4\theta}{8} \right]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$$

$$= 2\pi - 16 + 7\sqrt{3} \quad 1A$$

11. (a) $x^2 - x + 1 = \left(x - \frac{1}{2}\right)^2 + \frac{3}{4}$ 1A

Let $x - \frac{1}{2} = \frac{\sqrt{3}}{2} \tan \theta$. Then $dx = \frac{\sqrt{3}}{2} \sec^2 \theta d\theta$. 1M

$$\int_1^2 \frac{1}{x^2 - x + 1} dx = \int_1^2 \frac{1}{\left(x - \frac{1}{2}\right)^2 + \frac{3}{4}} dx$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\frac{\sqrt{3}}{2} \sec^2 \theta}{\frac{3}{4}(\tan^2 \theta + 1)} d\theta \quad 1A$$

$$= \frac{2\sqrt{3}}{3} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} d\theta$$

$$= \frac{2\sqrt{3}}{3} \left[\theta \right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= \frac{\sqrt{3}\pi}{9} \quad 1A$$

(b) $\int_1^2 \frac{x^2 + x + 1}{x^3 + 1} dx = \int_1^2 \left(\frac{x^2}{x^3 + 1} + \frac{x + 1}{x^3 + 1} \right) dx$

$$= \int_1^2 \frac{x^2}{x^3 + 1} dx + \int_1^2 \frac{x + 1}{x^2 - x + 1} dx \quad 1A$$

Let $u = x^3 + 1$. Then $du = 3x^2 dx$. 1M

$$\int_1^2 \frac{x^2 + x + 1}{x^3 + 1} dx = \frac{1}{3} \int_2^9 \frac{du}{u} + \int_1^2 \frac{1}{x^2 - x + 1} dx$$

$$= \frac{1}{3} \left[\ln |u| \right]_2^9 + \frac{\sqrt{3}\pi}{9} \quad 1M$$

$$= \frac{1}{3} \ln \frac{9}{2} + \frac{\sqrt{3}\pi}{9} \quad 1A$$

$$12. \quad (a) \quad x^2 - x + 1 = \left(x - \frac{1}{2}\right)^2 + \frac{3}{4} \quad 1A$$

$$\text{Let } x - \frac{1}{2} = \frac{\sqrt{3}}{2} \tan \theta. \text{ Then } dx = \frac{\sqrt{3}}{2} \sec^2 \theta \, d\theta. \quad 1M$$

$$\begin{aligned} \int_1^2 \frac{1}{x^2 - x + 1} \, dx &= \int_1^2 \frac{1}{\left(x - \frac{1}{2}\right)^2 + \frac{3}{4}} \, dx \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\frac{\sqrt{3}}{2} \sec^2 \theta}{\frac{3}{4}(\tan^2 \theta + 1)} \, d\theta \quad 1A \\ &= \frac{2\sqrt{3}}{3} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} d\theta \\ &= \frac{2\sqrt{3}}{3} \left[\theta \right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} \\ &= \frac{\sqrt{3}\pi}{9} \quad 1A \end{aligned}$$

$$\begin{aligned} (b) \quad \int_1^2 \frac{x^2 + x + 1}{x^3 + 1} \, dx &= \int_1^2 \left(\frac{x^2}{x^3 + 1} + \frac{x + 1}{x^3 + 1} \right) dx \\ &= \int_1^2 \frac{x^2}{x^3 + 1} \, dx + \int_1^2 \frac{x + 1}{x^2 - x + 1} \, dx \quad 1A \end{aligned}$$

$$\text{Let } u = x^3 + 1. \text{ Then } du = 3x^2 \, dx. \quad 1M$$

$$\begin{aligned} \int_1^2 \frac{x^2 + x + 1}{x^3 + 1} \, dx &= \frac{1}{3} \int_2^9 \frac{du}{u} + \int_1^2 \frac{1}{x^2 - x + 1} \, dx \\ &= \frac{1}{3} \left[\ln |u| \right]_2^9 + \frac{\sqrt{3}\pi}{9} \quad 1M \\ &= \frac{1}{3} \ln \frac{9}{2} + \frac{\sqrt{3}\pi}{9} \quad 1A \end{aligned}$$

$$13. \quad (a) \quad 2x^2 - x + 2 = 2\left(x - \frac{1}{4}\right)^2 + \frac{15}{8} \quad 1A$$

$$\text{Let } x - \frac{1}{4} = \frac{\sqrt{15}}{4} \tan \theta. \text{ Then } dx = \frac{\sqrt{15}}{4} \sec^2 \theta d\theta. \quad 1M$$

$$\begin{aligned} \int \frac{1}{2x^2 - x + 2} dx &= \int \frac{1}{2\left(x - \frac{1}{4}\right)^2 + \frac{15}{8}} dx \\ &= \int \frac{\frac{\sqrt{15}}{4} \sec^2 \theta}{\frac{15}{8} \tan^2 \theta + \frac{15}{8}} d\theta \quad 1A \end{aligned}$$

$$\begin{aligned} &= \frac{2\sqrt{15}}{15} \int d\theta \\ &= \frac{2\sqrt{15}}{15} \theta + \text{constant} \\ &= \frac{2\sqrt{15}}{15} \tan^{-1} \frac{4x - 1}{\sqrt{15}} + \text{constant} \quad 1A \end{aligned}$$

$$(b) \text{ Let } u = \frac{1}{x}. \text{ Then } du = -\frac{1}{x^2} dx.$$

$$\begin{aligned} \int_{\frac{1}{2}}^2 \frac{1}{(x^5 + 1)(2x^2 - x + 2)} dx &= \int_{\frac{1}{2}}^2 \frac{x^2}{(x^5 + 1)(2x^2 - x + 2)} dx \\ &= - \int_2^{\frac{1}{2}} \frac{\frac{1}{u^2}}{\left(\frac{1}{u^5} + 1\right)\left(\frac{2}{u^2} - \frac{1}{u} + 2\right)} du \quad 1A \\ &= \int_{\frac{1}{2}}^2 \frac{u^5}{(u^5 + 1)(2u^2 - u + 2)} du \\ &= \int_{\frac{1}{2}}^2 \frac{x^5}{(x^5 + 1)(2x^2 - x + 2)} dx \end{aligned}$$

$$\begin{aligned} &\int_{\frac{1}{2}}^2 \frac{1}{(x^5 + 1)(2x^2 - x + 2)} dx \\ &= \frac{1}{2} \left[\int_{\frac{1}{2}}^2 \frac{1}{(x^5 + 1)(x^2 - x + 2)} dx + \int_{\frac{1}{2}}^2 \frac{x^5}{(x^5 + 1)(x^2 - x + 2)} dx \right] \\ &= \frac{1}{2} \int_{\frac{1}{2}}^2 \frac{1 + x^5}{(x^5 + 1)(x^2 - x + 2)} dx \\ &= \frac{1}{2} \int_{\frac{1}{2}}^2 \frac{1}{2x^2 - x + 2} dx \quad 1A \end{aligned}$$

$$= \frac{\sqrt{15}}{15} \left[\tan^{-1} \frac{4x - 1}{\sqrt{15}} \right]_{\frac{1}{2}}^2 \quad 1M$$

$$= \frac{\sqrt{15}}{15} \left(\tan^{-1} \frac{7\sqrt{15}}{15} - \tan^{-1} \frac{\sqrt{15}}{15} \right) \quad 1A$$

14. (a) Let $x = \sin \theta$. Then $dx = \cos \theta d\theta$

When $x = 0$, $\theta = 0$; when $x = 1$, $\theta = \frac{\pi}{2}$.

$$\int_0^1 \sqrt{1-x^2} dx = \int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2 \theta} \cos \theta d\theta \quad 1M$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} (1 + \cos 2\theta) d\theta \quad 1M$$

$$= \frac{1}{2} \left[\theta + \frac{\sin 2\theta}{2} \right]_0^{\frac{\pi}{2}} \quad 1A$$

$$= \frac{1}{2} \left(\frac{\pi}{2} + 0 - 0 - 0 \right) \quad 1A$$

$$= \frac{\pi}{4}$$

(b) $\frac{d}{dx} [(1-x^2)^{\frac{3}{2}}x] = (1-x^2)^{\frac{3}{2}} + \frac{3}{2}(1-x^2)^{\frac{1}{2}}(-2x)(x) \quad 1M$

$$= \sqrt{1-x^2} [(1-x^2) - 3x^2]$$

$$= \sqrt{1-x^2} - 4x^2\sqrt{1-x^2} \quad 1$$

(c) By (b), $4x^2\sqrt{1-x^2} = \sqrt{1-x^2} - \frac{d}{dx} [(1-x^2)^{\frac{3}{2}}x]$

$$\int_0^1 x^2\sqrt{1-x^2} dx = \frac{1}{4} \int_0^1 \sqrt{1-x^2} dx - \frac{1}{4} \left[(1-x^2)^{\frac{3}{2}}x \right]_0^1 \quad 1M$$

$$= \frac{1}{4} \times \frac{\pi}{4} - \frac{1}{4}(0-0)$$

$$= \frac{\pi}{16} \quad 1A$$

15. (a) (i) $\tan \alpha = \frac{\cos \frac{2\pi}{11} - 1}{\sin \frac{2\pi}{11}}$
 $= \frac{1 - 2 \sin^2 \frac{\pi}{11} - 1}{2 \sin \frac{\pi}{11} \cos \frac{\pi}{11}}$ 1M
 $= -\tan \frac{\pi}{11}$
 $= \tan \left(-\frac{\pi}{11} \right)$
 $\alpha = -\frac{\pi}{11}$ 1

(ii) $\tan \beta = \frac{\cos \frac{2\pi}{11} + 1}{\sin \frac{2\pi}{11}}$
 $= \frac{2 \cos^2 \frac{\pi}{11} - 1 + 1}{2 \sin \frac{\pi}{11} \cos \frac{\pi}{11}}$ 1M
 $= \cot \frac{\pi}{11}$
 $= \tan \left(\frac{\pi}{2} - \frac{\pi}{11} \right)$
 $\beta = \frac{9\pi}{22}$ 1A

(b) (i) $x^2 + 2x \cos \frac{2\pi}{11} + 1 = x^2 + 2x \cos \frac{2\pi}{11} + \cos^2 \frac{2\pi}{11} + \sin^2 \frac{2\pi}{11}$
 $= \left(x + \cos \frac{2\pi}{11} \right)^2 + \sin^2 \frac{2\pi}{11}$ 1A

(ii) Let $x + \cos \frac{2\pi}{11} = \sin \frac{2\pi}{11} \tan \theta$. Then $dx = \sin \frac{2\pi}{11} \sec^2 \theta d\theta$. 1M
When $x = -1$ and when $x = 1$,
 $\tan \theta = \frac{-1 + \cos \frac{2\pi}{11}}{\sin \frac{2\pi}{11}}$ and $\tan \theta = \frac{1 + \cos \frac{2\pi}{11}}{\sin \frac{2\pi}{11}}$
 $\theta = -\frac{\pi}{11}$ $\theta = \frac{9\pi}{22}$ 1M

$\int_{-1}^1 \frac{\sin \frac{2\pi}{11}}{x^2 + 2x \cos \frac{2\pi}{11} + 1} dx$
 $= \int_{-1}^1 \frac{\sin \frac{2\pi}{11}}{\left(x + \cos \frac{2\pi}{11} \right)^2 + \sin^2 \frac{2\pi}{11}} dx$
 $= \int_{-\frac{\pi}{11}}^{\frac{9\pi}{22}} \frac{\sin^2 \frac{2\pi}{11} \sec^2 \theta}{\sin^2 \frac{2\pi}{11} \tan^2 \theta + \sin^2 \frac{2\pi}{11}} d\theta$ 1A
 $= \int_{-\frac{\pi}{11}}^{\frac{9\pi}{22}} d\theta$
 $= \left[\theta \right]_{-\frac{\pi}{11}}^{\frac{9\pi}{22}}$
 $= \frac{\pi}{2}$ 1A