1. (a) Let the radius of the water surface be a cm. By considering similar triangles,

$$\frac{a-3}{4-3} = \frac{h}{10} \qquad \text{1M}$$

$$a = \frac{h+30}{10}$$

Therefore,

$$V = \frac{\pi}{3}h\left[3^2 + 3\left(\frac{h+30}{10}\right) + \left(\frac{h+30}{10}\right)^2\right] \qquad \text{1M}$$

$$= \frac{\pi}{300}h[900 + 30(h+30) + (h^2 + 60h + 900)]$$

$$= \frac{\pi}{300}(h^3 + 90h^2 + 2700h) \qquad \text{1}$$

(b) $$\frac{dV}{dt} = \frac{\pi}{300}(3h^2 + 180h + 2700)\frac{dh}{dt} \qquad \text{1M+1A}$$

When $h = 5$,

$$7\pi = \frac{\pi}{300}[3(5)^2 + 180(5) + 2700]\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{4}{7} \qquad \text{1A}$$

The rate of increase of depth of water is $\frac{4}{7}$ cm/s.

2. Let V cm^3 and r cm be the volume and radius of the spherical balloon respectively.

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt} \qquad \text{1M}$$

$$-100 = 4\pi \cdot 10^2\frac{dr}{dt} \qquad \text{1M}$$

$$\frac{dr}{dt} = -\frac{1}{4\pi} \qquad \text{1A}$$

Hence the rate of change of the radius is $-\frac{1}{4\pi}$ cm/s.

3. (a) Curved surface area of the cone $= \pi(15)(\sqrt{20^2 + 15^2}) = 375\pi \text{ cm}^2$ 1A

$$\frac{A}{375\pi} = \left(\frac{h}{20}\right)^2$$ 1M

$$A = \frac{15}{16}\pi h^2$$ 1

(b) Volume of the cone $= \frac{1}{3}(15)^2(20) = 1500\pi \text{ cm}^3$

When the volume of water is $96\pi \text{ cm}^3$,

$$\frac{96\pi}{1500\pi} = \left(\frac{h}{20}\right)^3$$ 1M

$$h = 8$$ 1A

By (a), $A = \frac{15}{16}\pi h^2$. We have $\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{15\pi}{8} h \frac{\mathrm{d}h}{\mathrm{d}t}$. 1M

Put $\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{3}{\pi}$, $h = 8$,

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{15\pi}{8} \times 8 \times \frac{3}{\pi} = 45$$ 1A

Required rate is 45 cm²/s.

4. (a) Note that the coordinates of Q are $(0,\ 2e^u)$.

Area of $\triangle OPQ = \frac{u(2e^u)}{2} = ue^u$ 1A

(b) Let v be the y-coordinate of P.

$$v = 2e^u$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} = 2e^u \frac{\mathrm{d}u}{\mathrm{d}t}$$ 1M

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{1}{2e^u} \cdot \frac{\mathrm{d}v}{\mathrm{d}t}$$

Let A square units be the area of $\triangle OPQ$, then $A = ue^u$.

$$\frac{\mathrm{d}A}{\mathrm{d}t} = (e^u + ue^u)\frac{\mathrm{d}u}{\mathrm{d}t}$$ 1M+1M

$$= \left(\frac{1+u}{2}\right)\frac{\mathrm{d}v}{\mathrm{d}t}$$

When $u = 4$, $\frac{\mathrm{d}A}{\mathrm{d}t} = \left(\frac{1+4}{2}\right)(6) = 15$ 1A

Thus, the required rate of change is 15 square units per second.

5. (a) $y = \frac{1}{2}\ln x$

$\frac{dy}{dx} = \frac{1}{2x}$ 1M

Slope of tangent to C at $P = \frac{1}{2r}$. Slope of normal to C at $P = -2r$.

Let the coordinates of Q be $(q,\ 0)$.

$$\frac{0 - \frac{1}{2}\ln r}{q - r} = -2r$$ 1M

$$\ln r = -4qr + 4r^2$$

$$q = \frac{4r^2 + \ln r}{4r}$$ 1

The x-coordinate of Q is $\frac{4r^2 + \ln r}{4r}$.

(b) Let the area of $\triangle PQR$ be A.

$$A = \frac{1}{2}\left(\frac{4r^2 + \ln r}{4r} - r\right)\left(\frac{1}{2}\ln r\right)$$ 1M

$$= \frac{(\ln r)^2}{16r}$$ 1A

$$\frac{dA}{dr} = \frac{2}{16r}(\ln r)\left(\frac{1}{r}\right) - \frac{(\ln r)^2}{16r^2}$$ 1M

$$= \frac{\ln r(2 - \ln r)}{16r^2}$$

When $\frac{dA}{dt} = 0$,

$$\ln r = 2 \quad \text{or} \quad 0$$

$$r = e^2 \quad \text{or} \quad 1 \text{ (rejected)}$$

r	$1 < r < e^2$	$r > e^2$
$\frac{dA}{dr}$	+	−

1M

A attains its greatest value when $r = e^2$. Greatest area $= \frac{(\ln e^2)^2}{16e^2} = \frac{1}{4e^2}$ 1A

(c) $OP = \sqrt{r^2 + (\ln\sqrt{r})^2} = \frac{1}{2}\sqrt{4r^2 + (\ln r)^2}$ 1M

$$\frac{dOP}{dt} = \frac{4r^2 + \ln r}{2r\sqrt{4r^2 + (\ln r)^2}} \cdot \frac{dr}{dt}$$ 1M

When $r = e$, $\frac{dOP}{dt} = \frac{1}{2}\left(e^2 + \frac{1}{4}\right)^{-\frac{1}{2}}\left(2e + \frac{1}{2e}\right)\frac{dr}{dt} = \frac{1}{e}\left(e^2 + \frac{1}{4}\right)\frac{dr}{dt}$

As $\frac{dOP}{dt} \le 32e^2$,

$$\frac{1}{e} \cdot \left(e^2 + \frac{1}{4}\right)^{\frac{1}{2}}\frac{dr}{dt} \le 32e^2$$

$$\frac{dr}{dt} \le 32e^3\left(e^2 + \frac{1}{4}\right)^{-\frac{1}{2}}$$

When $r = e$,

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\ln r(2 - \ln r)}{16r^2} \cdot \frac{\mathrm{d}r}{\mathrm{d}t}$$

$$= \frac{1}{16e^2} \cdot \frac{\mathrm{d}r}{\mathrm{d}t} \qquad \text{1M}$$

$$\leq \frac{1}{16e^2} \cdot 32e^3 \left(e^2 + \frac{1}{4}\right)^{-\frac{1}{2}}$$

$$< 2e \cdot (e^2)^{-\frac{1}{2}} = 2$$

The claim is correct. 1A

6. (a) Let A square units be the area of the circle.

$$A = \left(\frac{\sqrt{u^2 + v^2}}{2}\right)^2 \pi \qquad \text{1M}$$

$$= \frac{\pi}{4} S$$

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\pi}{4} \times \frac{\mathrm{d}S}{\mathrm{d}t} \qquad \text{1M}$$

$$5\pi = \frac{\pi}{4} \times \frac{\mathrm{d}S}{\mathrm{d}t}$$

$$\frac{\mathrm{d}S}{\mathrm{d}t} = 20$$

S increases at a constant rate of 20 square units per second. 1A

(b) Coordinates of P and Q are $\left(u,\ 2^{u-1}\right)$ and $(u,\ 2^u)$ respectively. 1A

When $u = 2$,

$$S = u^2 + 2^{2u-2}$$

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \left(2u + 2^{2u-2} \cdot 2\ln 2\right) \frac{\mathrm{d}u}{\mathrm{d}t} \qquad \text{1M}$$

$$20 = (4 + 8\ln 2)\frac{\mathrm{d}u}{\mathrm{d}t}$$

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{5}{1 + 2\ln 2}$$

Let the area of $\triangle OPQ$ be β square units. When $u = 2$,

$$\beta = \frac{u(2^u - 2^{u-1})}{2}$$

$$\frac{\mathrm{d}\beta}{\mathrm{d}t} = \left[(2^{u-1} - 2^{u-2}) + u(2^{u-1} - 2^{u-2})\ln 2\right] \frac{\mathrm{d}u}{\mathrm{d}t}$$

$$= (1 + 2\ln 2)\frac{\mathrm{d}u}{\mathrm{d}t}$$

$$= (1 + 2\ln 2) \times \frac{20}{4 + 8\ln 2} \qquad \text{1M}$$

$$= 5$$

Required rate is 5 square units per second. 1A

7. (a) $PQ = \sqrt{u^2+36} + \sqrt{(20-u)^2+16}$ 1M

$\dfrac{dPQ}{du} = u(u^2+36)^{-\frac{1}{2}} + (u-20)[(20-u)^2+16]^{-\frac{1}{2}}$ 1M

When $\dfrac{dPQ}{du} = 0$,

$$u(u^2+36)^{-\frac{1}{2}} = (20-u)[(20-u)^2+16]^{-\frac{1}{2}}$$ 1M

$$u^2[(20-u)^2+16] = (20-u)^2(u^2+36)$$

$$0 = 20u^2 - 1440u + 14\,400$$

$$u = 12 \quad \text{or} \quad 60 \text{ (rejected)}$$

u	$0 < u < 12$	$12 < u < 20$
$\dfrac{dPQ}{du}$	$-$	$+$

1M

PQ attains its minimum at $u = 12$.

Thus, $a = 12$. 1A

(b) (i) Let β be the area of rectangle $PQSR$.

$$\beta = (PQ)u$$ 1M

$$\frac{d\beta}{du} = u\frac{dPQ}{du} + PQ$$ 1M

$$\left.\frac{d\beta}{du}\right|_{u=12} = (12)(0) + \sqrt{12^2+36} + \sqrt{(20-12)^2+16}$$

$$= 10\sqrt{5} \neq 0$$ 1M

Thus, β does not attain its minimum when $u = 12$.

The claim is disagreed. 1A

(ii) After time t min,

$$OP = \sqrt{u^2 + (u^2+36)}$$

$$\frac{dOP}{dt} = 2u(2u^2+36)^{-\frac{1}{2}}\frac{du}{dt}$$ 1M

$$28 = 2(12)(2(12)^2+36)^{-\frac{1}{2}}\left(\left.\frac{du}{dt}\right|_{u=12}\right)$$ 1M

$$\left.\frac{du}{dt}\right|_{u=12} = 21$$

Let the perimeter of rectangle $PQSR$ be p.

$$p = 2(PQ+u)$$ 1M

$$\frac{dp}{dt} = 2\frac{dPQ}{du}\frac{du}{dt} + 2\frac{du}{dt}$$ 1M

$$\left.\frac{dp}{dt}\right|_{u=12} = 2(0)(21) + 2(21)$$

$$= 42$$

Required rate is 42 units per minute. 1A

8. (a) (i) $T = \frac{PQ}{7} + \frac{BQ}{1.4}$ 1M

$= \frac{x}{7} + \frac{5\sqrt{30^2 + (40 - x)^2}}{7}$

$= \frac{x + 5\sqrt{x^2 - 80x + 2500}}{7}$ 1A

(ii) When T is minimum, $\frac{dT}{dx} = 0$

$$\frac{1}{7} + \frac{5}{7} \cdot \frac{1}{2}(x^2 - 80x + 2500)^{-\frac{1}{2}}(2x - 80) = 0 \quad \text{1M}$$

$$(\sqrt{x^2 - 80x + 2500})^2 = [-5(x - 40)]^2$$

$$2x^2 - 160x + 3125 = 0 \quad \text{1}$$

$$x = 40 - \frac{5\sqrt{6}}{2} \quad \text{or} \quad 40 + \frac{5\sqrt{6}}{2} \text{ (rejected)}$$

x	$0 < x < 40 - \frac{5\sqrt{6}}{2}$	$40 - \frac{5\sqrt{6}}{2} < x < 40$
$\frac{dT}{dx}$	$-$	$+$

When T is minimum, $x = 40 - \frac{5\sqrt{6}}{2}$. 1M

$QB = \sqrt{30^2 + \left[40 - \left(40 - \frac{5\sqrt{6}}{2}\right)\right]^2} = \frac{25\sqrt{6}}{2}$ m 1

(b) (i) $\sin\beta = \frac{30}{\left(\frac{25\sqrt{6}}{2}\right)} = \frac{2\sqrt{6}}{5}$ 1A

$\cos\beta = \frac{40 - \left(40 - \frac{5\sqrt{6}}{2}\right)}{\frac{25\sqrt{6}}{2}} = \frac{1}{5}$ 1A

In $\triangle MAB$,

$$\frac{MB}{\sin\alpha} = \frac{AB}{\sin(\pi - \alpha - \beta)} \quad \text{1M}$$

$$MB = \frac{40\sin\alpha}{\sin(\alpha + \beta)}$$

$$= \frac{40\sin\alpha}{\sin\alpha\cos\beta + \cos\alpha\sin\beta} \quad \text{1M}$$

$$= \frac{40\tan\alpha}{\frac{1}{5}\tan\alpha + \frac{2\sqrt{6}}{5}}$$

$$= \frac{200\tan\alpha}{\tan\alpha + 2\sqrt{6}} \quad \text{1}$$

(ii) $\frac{dMB}{dt} = 200 \cdot \frac{\sec^2\alpha(\tan\alpha + 2\sqrt{6}) - \tan\alpha\sec^2\alpha}{(\tan\alpha + 2\sqrt{6})^2} \cdot \frac{d\alpha}{dt}$ 1M

$$-1.4 = \frac{400\sqrt{6}\sec^2 0.2}{(\tan 0.2 + 2\sqrt{6})^2}\frac{d\alpha}{dt}$$

$\frac{d\alpha}{dt} \approx -0.0357$ rad/s 1A

9. (a) $\ell = 2\sqrt{1^2 + x^2 + (\sqrt{2})^2} + 2 - x = 2\sqrt{x^2+3} + 2 - x$ 1A

$\dfrac{\mathrm{d}\ell}{\mathrm{d}t} = (x^2+3)^{-\frac{1}{2}}(2x) - 1 = \dfrac{2x}{\sqrt{x^2+3}} - 1$ 1

(b) (i) When $\dfrac{\mathrm{d}\ell}{\mathrm{d}x} = 0$,

$\dfrac{2x}{\sqrt{x^2+3}} = 1$ 1M

$4x^2 = x^2 + 3$

$x = 1$ or -1 (rejected)

x	$0 < x < 1$	$1 < x < \frac{3}{2}$
$\frac{\mathrm{d}\ell}{\mathrm{d}x}$	$-$	$+$

1M

ℓ attains the least value at $x = 1$. 1A

(ii) ℓ attains its greatest when $x = 0$ or $x = \dfrac{3}{2}$.

When $x = 0$, $\ell = 2\sqrt{3} + 2$; when $x = \dfrac{3}{2}$, $\ell = \sqrt{21} + \dfrac{1}{2} < 2\sqrt{3} + 2$ 1M

So, ℓ attains the greatest value at $x = 0$. 1A

(c) (i) $E = \dfrac{k}{(2-x)^2} + \dfrac{2k}{x^2+3}$ 1A

(ii) $\dfrac{\mathrm{d}E}{\mathrm{d}x} = \dfrac{2k}{2-x)^3} - \dfrac{4kx}{(x^2+3)^2}$

From (b), ℓ attains the least when $x = 1$.

When $x = 1$, $\dfrac{\mathrm{d}E}{\mathrm{d}x} = 2k - \dfrac{4k}{16} = \dfrac{7k}{4} > 0$ 1M

So, E does not attain its greatest value at $x = 1$. The student is incorrect. 1A

10. (a) $S = \frac{r^2}{2}\sin 2\theta + \frac{r^2}{2}\sin(\beta - \theta) + \frac{r^2}{2}\sin(2\pi - 2\beta) + \frac{r^2}{2}\sin(\beta - \theta)$ 1M+1A

$= \frac{r^2}{2}[\sin 2\theta - \sin 2\beta + 2\sin(\beta - \theta)]$ 1

(b) $S = \frac{r^2}{2}[\sin 2\theta - \sin 2\beta + 2\sin(\beta - \theta)]$

$\frac{\mathrm{d}S}{\mathrm{d}\theta} = \frac{r^2}{2}[2\cos 2\theta - 2\cos(\beta - \theta)]$ 1M

When $\frac{\mathrm{d}S}{\mathrm{d}\theta} = 0$,

$\cos 2\theta = \cos(\beta - \theta)$ 1M

$2\theta = \beta - \theta$ or $2\pi - (\beta - \theta)$ (rejected) or $2\pi + (\beta - \theta)$ (rejected)

$\theta = \frac{\beta}{3}$ 1

$\frac{\mathrm{d}^2S}{\mathrm{d}\theta^2} = -r^2[2\sin 2\theta + \sin(\beta - \theta)]$

When $\theta = \frac{\beta}{3}$,

$\frac{\mathrm{d}^2S}{\mathrm{d}\theta^2} = -r^2\left[2\sin\frac{2\beta}{3} + \sin\left(\beta - \frac{\beta}{3}\right)\right]$

$= -3r^2\sin\frac{2\beta}{3} < 0$ 1M

S attains a maximum at $\theta = \frac{\beta}{3}$.

$S_\beta = \frac{r^2}{2}\left[\sin\frac{2\beta}{3} - \sin 2\beta + 2\sin\left(\beta - \frac{\beta}{3}\right)\right]$

$= \frac{r^2}{2}\left(3\sin\frac{2\beta}{3} - \sin 2\beta\right)$

$= \frac{r^2}{2}\left[3\sin\frac{2\beta}{3} - \left(3\sin\frac{2\beta}{3} - 4\sin^3\frac{2\beta}{3}\right)\right]$ 1M

$= 2r^2\sin^3\frac{2\beta}{3}$ 1

(c) Among all possible values of β, S_β attains the greatest value when $\sin\frac{2\beta}{3} = 1$. 1A

$\frac{2\beta}{3} = \frac{\pi}{2}$

$\beta = \frac{3\pi}{4}$

So, $\theta = \frac{\beta}{3} = \frac{\pi}{4}$ 1M

$\angle COD = 2\theta = \frac{\pi}{2}$.

$\angle BOC = \angle AOD = \beta - \theta = \frac{\pi}{2}$

So, COA and BOD are straight lines and $CA \perp BD$.

Thus, $ABCD$ is a square. The student is correct. 1

11. (a) (i) $A = \pi(1)^2 - \pi(\cos\theta)^2 - \left(\pi(1)^2 \times \frac{2\theta}{2\pi} - \frac{1}{2}(1)^2 \sin 2\theta\right)$ 1M+1A

$= \pi \sin^2\theta - \theta + \frac{1}{2}\sin 2\theta$ 1

(ii) $\frac{\mathrm{d}A}{\mathrm{d}\theta} = \pi(2\sin\theta\cos\theta) - 1 + \frac{1}{2}(\cos 2\theta)(2)$ 1A

$= \pi\sin 2\theta - 1 + (1 - 2\sin^2\theta)$

$= \pi\sin 2\theta + 2\sin\theta(\cos\theta\tan\theta)$

$= (\pi - \tan\theta)\sin 2\theta$ 1

(b) $\frac{\mathrm{d}A}{\mathrm{d}\theta} = (\pi - \tan\theta)\sin 2\theta = 0$ 1M

$\tan\theta = \pi$ or $\sin 2\theta = 0$ (rejected) 1M

$\theta = \tan^{-1}\pi$

θ	$0 < \theta < \tan^{-1}\pi$	$\tan^{-1}\pi < \theta < \frac{\pi}{2}$
$\frac{\mathrm{d}A}{\mathrm{d}\theta}$	+	−

A attains its greatest value when $\tan\theta = \pi$. 1A

(c) Perimeter $s = 2\pi(1) \times \frac{2\pi - 2\theta}{2\pi} + 2\sin\theta + 2\pi(\cos\theta)$

$= 2\pi - 2\theta + 2\sin\theta + 2\pi\cos\theta$ 1A

When A is maximum, $\tan\theta = \pi$.

We have $\cos\theta = \frac{1}{\sqrt{1+\pi^2}}$ and $\sin\theta = \frac{\pi}{\sqrt{1+\pi^2}}$.

$\frac{\mathrm{d}s}{\mathrm{d}\theta} = -2 + 2\cos\theta - 2\pi\sin\theta$ 1M

$\left.\frac{\mathrm{d}s}{\mathrm{d}\theta}\right|_{\tan\theta=\pi} = -2 + \frac{2}{\sqrt{1+\pi^2}} - \frac{2\pi^2}{\sqrt{1+\pi^2}} \approx -7.38 \neq 0$ 1M

Hence the perimeter will not attain its greatest value when A attains its greater value.

The student is incorrect. 1A

12. (a) (i) $\angle BOC = 4\theta$

$\angle AOB = \angle AOC = \pi - 2\theta$

$S = \frac{1}{2}(1)(1)\sin 4\theta + \frac{1}{2}(1)(1)\sin(\pi - 2\theta) + \frac{1}{2}(1)(1)\sin(\pi - 2\theta)$ 1M

$= \frac{\sin 4\theta}{2} + \sin 2\theta$ 1

(ii) $\frac{\mathrm{d}S}{\mathrm{d}\theta} = 2\cos 4\theta + 2\cos 2\theta$ 1A

When $\frac{\mathrm{d}S}{\mathrm{d}\theta} = 0$,

$\cos 3\theta \cos\theta =$ 1M

$\cos 3\theta = 0$ or $\cos\theta = 0$ (rejected)

$\theta = \frac{\pi}{6}$ 1A

$\frac{\mathrm{d}^2S}{\mathrm{d}\theta^2} = -8\sin 4\theta - 4\sin 2\theta$

$\left.\frac{\mathrm{d}^2S}{\mathrm{d}\theta^2}\right|_{\theta=\frac{\pi}{6}} = -6\sqrt{3} < 0$ 1M

S attains its maximum when $\theta = \frac{\pi}{6}$.

Maximum $S = \frac{1}{2}\sin\frac{4\pi}{6} + \sin\frac{2\pi}{6}$

$= \frac{3\sqrt{3}}{4}$ 1A

(b) $\angle BOD = \angle DOA = \angle AOE = \angle EOC = \frac{\angle AOC}{2} = \frac{\pi - 2\theta}{2}$

Area of $\triangle AOD = \frac{1}{2}(1)(1)\sin\frac{\pi - 2\theta}{2}$ 1A

$= \frac{1}{2}\cos\theta$

$P = \frac{1}{2}(1)(1)\sin 4\theta + 4 \times \frac{1}{2}\cos\theta$ 1M

$= \frac{\sin 4\theta}{2} + 2\cos\theta$ 1A

$\frac{\mathrm{d}P}{\mathrm{d}\theta} = 2\cos 4\theta - 2\sin\theta$

When $\theta = \frac{\pi}{6}$, $\frac{\mathrm{d}P}{\mathrm{d}\theta} = -2 \neq 0$ 1M

Hence, P will not attain the maximum when S attains the maximum. 1

13. (a) $A = \dfrac{(PQ)(a)}{2} + \dfrac{(QR)(a)}{2} + \dfrac{(RP)(a)}{2}$ 1M

$= \dfrac{a}{2}(2s)$

$a = \dfrac{A}{s}$ 1

(b) (i) $EF = 2\sin\theta$

$r = \dfrac{\frac{1}{2}(1)(1)\sin 2\theta}{\frac{1}{2}(1+1+2\sin\theta)}$ 1M

$= \dfrac{\sin\theta\cos\theta}{1+\sin\theta}$

$= \dfrac{\cos\theta(1+\sin\theta)-\cos\theta}{1+\sin\theta}$

$= \cos\theta - \dfrac{\cos\theta}{1+\sin\theta}$ 1

(ii) $\dfrac{\mathrm{d}r}{\mathrm{d}\theta} = -\sin\theta - \dfrac{-\sin\theta(1+\sin\theta)-\cos\theta(\cos\theta)}{(1+\sin\theta)^2}$ 1M

$= -\sin\theta - \dfrac{-\sin\theta - 1}{(1+\sin\theta)^2}$

$= \dfrac{1}{1+\sin\theta} - \sin\theta$ 1A

When $\dfrac{\mathrm{d}r}{\mathrm{d}\theta} = 0$,

$\dfrac{1}{1+\sin\theta} - \sin\theta = 0$

$\sin^2\theta + \sin\theta - 1 = 0$ 1A

$\sin\theta = \dfrac{-1+\sqrt{5}}{2}$ or $\dfrac{-1-\sqrt{5}}{2}$ (rejected)

$\theta \approx 0.666\,\text{rad}$

$\dfrac{\mathrm{d}^2r}{\mathrm{d}\theta^2} = \dfrac{-\cos\theta}{(1+\sin\theta)^2} - \cos\theta < 0$ for $0 < \theta < \dfrac{\pi}{2}$ 1M

Required value is 0.666 rad. 1A

(iii) Perimeter is the least when $\theta = \dfrac{\pi}{12}$. 1A

By (b)(ii), r has only one maximum value when $\dfrac{\pi}{12} \le \theta \le \dfrac{5\pi}{12}$.

Hence r is the least when $\theta = \dfrac{\pi}{12}$ or $\dfrac{5\pi}{12}$. 1M

When $\theta = \dfrac{\pi}{12}$, $r \approx 0.199$; when $\theta = \dfrac{5\pi}{12}$, $r \approx 0.127$.

Thus, r and the area of the circle is the least when $\theta = \dfrac{5\pi}{12}$.

The claim is disagreed. 1A

14. (a) (i) $PB = PC = \sqrt{x^2 + 4}$ and $PA = 3 - x$

$r = PA + PB + PC = 2\sqrt{x^2 + 4} + (3 - x)$ 1A

$\frac{dr}{dx} = 2 \times \frac{1}{2}(x^2 + 4)^{-\frac{1}{2}}(2x) - 1$

$= \frac{2x}{\sqrt{x^2 + 4}} - 1$ 1

(ii) (1) When $\frac{dr}{dx} = 0$,

$\frac{2x}{\sqrt{x^2 + 4}} = 1$ 1M

$4x^2 = x^2 + 4$

$x = \frac{2}{\sqrt{3}}$ or $-\frac{2}{\sqrt{3}}$ (rejected)

x	$0 < x < \frac{2}{\sqrt{3}}$	$\frac{2}{\sqrt{3}} < x < 3$
$\frac{dr}{dx}$	$-$	$+$

r is increasing when $\frac{2}{\sqrt{3}} \le x < 3$. (Accept $x \ge \frac{2}{\sqrt{3}}$) 1A

(2) r is decreasing on $0 \le x \le \frac{2}{\sqrt{3}}$. 1A

Therefore, r is the least at $x = \frac{2}{\sqrt{3}}$. 1M

Least value of $r = 2\sqrt{\left(\frac{2}{\sqrt{3}}\right)^2 + 4} + 3 - \frac{2}{\sqrt{3}} = 2\sqrt{3} + 3$ 1A

(iii) r is greatest at $x = 0$ or $x = 3$.

When $x = 0$, $r = \sqrt{4} + 3 = 7$; when $x = 3$, $r = \sqrt{3^2 + 4} + 3 - 3 = 2\sqrt{13}$ 1M

The greatest value of r is $2\sqrt{13}$. 1A

(b) $r = 2\sqrt{x^2 + 4} + 1 - x$. So, $\frac{dr}{dx} = \frac{2x}{\sqrt{x^2 + 4}} - 1$. 1A

From (a), r is decreasing on $0 \le x \le 1$. Therefore, r is the least at $x = 1$. 1M

Least value of $r = 2\sqrt{1 + 4} + 1 - 1 = 2\sqrt{5}$ 1A

15. (a) (i) $f'(x) = -(x^2 + 9) + (14 - x)(2x)$

$= -3x^2 + 28x - 9$ 1A

$f''(x) = -6x + 28$

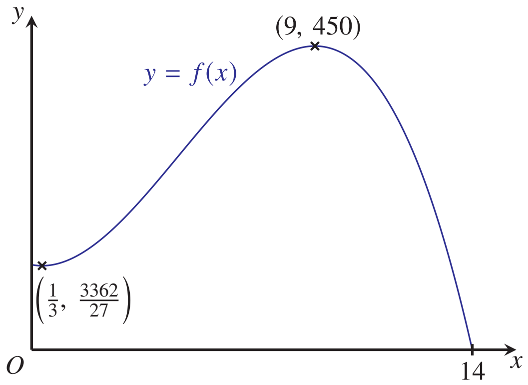
When $f'(x) = 0$, $x = \dfrac{1}{3}$ or 9. 1M

$f''\left(\dfrac{1}{3}\right) = 26 > 0$ and $f''(9) = -26 < 0$ 1M

Thus, the minimum point is $\left(\dfrac{1}{3}, \dfrac{3362}{27}\right)$ and the maximum point is $(9, 450)$. 1A

(ii) (Correct shape) 1M

(All correct) 1A



(b) (i) Since $\triangle APQ \sim \triangle BQR$,

$$\frac{x}{3} = \frac{BR}{14 - x}$$

$$BR = \frac{x(14 - x)}{3}$$ 1A

$$g(x) = \sqrt{x^2 + 3^2} \times \sqrt{(14 - x)^2 + \left[\frac{x(14 - x)}{3}\right]^2}$$ 1M

$$= \sqrt{x^2 + 9}(14 - x)\sqrt{1 + \frac{x^2}{9}}$$

$$= \frac{(14 - x)(x^2 + 9)}{3}$$ 1

(ii) Let T be a point on BC such that $ST \perp BC$.

Since S lies inside the cardboard, $BR + RT \leq BC$,

$$\frac{x(14 - x)}{3} + 3 \leq 11$$ 1M

$$x^2 - 14x + 24 \geq 0$$

$$x \leq 2 \quad \text{or} \quad x \geq 12$$

Since $0 \leq x \leq 14$, we have $0 \leq x \leq 2$ or $12 \leq x \leq 14$. 1

(iii) From (a)(ii), the curve $y = f(x)$ has no maximum points in the above range.

Therefore, the greatest value of $f(x)$ is attained at one of the boundaries. 1M

Since $f(0) = 126$, $f(2) = 156$, $f(12) = 306$ and $f(14) = 0$,

the greatest value of $g(x)$ is $\dfrac{f(12)}{3} = 102$. 1A

16. (a) $\left(\frac{x}{2}\right)^2 + y^2 = 5^2$

$\frac{x^2}{4} + y^2 = 25$ 1A

(b) By (a), differentiate both side with respect to time t,

$$\frac{x}{2} \cdot \frac{\mathrm{d}x}{\mathrm{d}t} + 2y\frac{\mathrm{d}y}{\mathrm{d}t} = 0$$ 1M+1A

When H is 3 m from the ground, $y = 3$ and $x = 8$.

$$\frac{8}{2} \cdot \frac{\mathrm{d}x}{\mathrm{d}t} + 2(3)(-2) = 0$$ 1M

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 3$$

Required rate is 3 m/s. 1A

17. (a) $V = \frac{1}{3}\pi r^2 h$

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{2\pi}{3}rh\frac{\mathrm{d}r}{\mathrm{d}t} + \frac{\pi}{3}r^2\frac{\mathrm{d}h}{\mathrm{d}t}$$ 1M

Therefore,

$$0 = \frac{2\pi}{3}rh\frac{\mathrm{d}r}{\mathrm{d}t} + \frac{\pi}{3}r^2(-2)$$ 1A

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{r}{h}$$ 1

(b) (i) $S = \pi r\sqrt{r^2 + h^2}$. So, $S^2 = \pi^2(r^4 + r^2h^2)$ 1A

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(S^2\right) = \pi^2\left(4r^3\frac{\mathrm{d}r}{\mathrm{d}t} + 2rh^2\frac{\mathrm{d}r}{\mathrm{d}t} + 2r^2h\frac{\mathrm{d}h}{\mathrm{d}t}\right)$$ 1A

$$= \pi^2\left(4r^3 \cdot \frac{r}{h} + 2rh^2 \cdot \frac{r}{h} + 2r^2h(-2)\right)$$ 1M

$$= \frac{2\pi^2r^2}{h}(2r^2 - h^2)$$ 1

(ii) When $t = 0$, $h = 1.2r$,

$$\left.\frac{\mathrm{d}S^2}{\mathrm{d}t}\right|_{t=0} = \frac{2\pi^2r^2}{1.2r}(2r^2 - 1.2^2r^2) = \frac{14\pi^2}{15}r^3 > 0$$ 1M

Since $\frac{\mathrm{d}h}{\mathrm{d}t} < 0$ and $\frac{\mathrm{d}r}{\mathrm{d}t} > 0$, $(2r^2 - h^2)$ is increasing.

Therefore, $\frac{\mathrm{d}}{\mathrm{d}t}\left(S^2\right)$ increases for all $t \geq 0$. 1M

Thus, $\frac{\mathrm{d}S^2}{\mathrm{d}t} > 0$ for all $t \geq 0$.

So, $\frac{\mathrm{d}}{\mathrm{d}t}(S) = \frac{1}{2S} \times \frac{\mathrm{d}S^2}{\mathrm{d}t} > 0$ for all $t \geq 0$.

The claims is agreed. 1A

– 試卷完 –