

M2REG-AOD-2425-ASM-SET 6-MATH

Suggested solutions

$$1. \quad (a) \text{ Volume of water} = \frac{1}{3}(4)^2\pi(12) \times \left[1 - \frac{(12-h)^3}{12^3}\right] \quad 1M+1M$$

$$= 64\pi - \frac{\pi(12-h)^3}{27} \quad 1A$$

(b) Let $V \text{ cm}^3$ be the volume of water.

$$\frac{dV}{dt} = \frac{\pi}{9}(12-h)^2 \frac{dh}{dt} \quad 1M$$

When $\frac{dV}{dt} = \pi$ and $h = 6$,

$$\pi = \frac{\pi}{9}(12-6)^2 \times \frac{dh}{dt} \quad 1M$$

$$\frac{dh}{dt} = \frac{1}{4}$$

Required rate is $\frac{1}{4} \text{ cm/s}$. 1A

$$2. \quad (a) \text{ Area} = \frac{1}{2}(u)(4 \ln u) = 2u \ln u \quad 1A$$

(b) Let v be the length of PQ and A be the area of $\triangle OPQ$.

$$v = 4 \ln u$$

$$\frac{dv}{dt} = \frac{4}{u} \times \frac{du}{dt}$$

When $u = e$ and $\frac{dv}{dt} = -4$,

$$-4 = \frac{4}{e} \times \frac{du}{dt} \quad 1M$$

$$\frac{du}{dt} = -e \quad 1A$$

$$\frac{dA}{dt} = \left(2u \times \frac{1}{u} + 2 \ln u\right) \frac{du}{dt} \quad 1M$$

$$= -4e$$

Required rate is $-4e/\text{s}$. 1A

3. (a) $A = m(4e^{-m})$
 $= 4me^{-m}$ 1A
- (b) (i) $\frac{dA}{dm} = 4e^{-m} + 4me^{-m}(-1)$ 1M
 $= 4e^{-m}(1 - m)$
 When $\frac{dA}{dm} = 0$, $m = 1$. 1A
- | | | |
|-----------------|-------------|---------|
| m | $0 < m < 1$ | $m > 1$ |
| $\frac{dA}{dm}$ | + | - |
- A attains its maximum when $m = 1$.
 Required value $= 4(1)e^{-1}$
 $= 4e^{-1}$ 1A
- (ii) $\frac{dA}{dt} = \frac{dA}{dm} \cdot \frac{dm}{dt}$ 1M
 $= 4e^{-0.5}(1 - 0.5) \cdot 0.5$
 $= e^{-0.5}$ 1A
 Required rate is $e^{-0.5}$ units per second.

4. (a) Radius of water surface
 $= \sqrt{13^2 - (13 - h)^2}$ 1M
 $= \sqrt{26h - h^2}$
 $A = \pi(\sqrt{26h - h^2})^2$
 $= \pi(26h - h^2)$ 1A
- (b) $V = \frac{1}{3}\pi h^2(39 - h)$
 $\frac{dV}{dt} = \frac{1}{3}\pi \left(78h \frac{dh}{dt} - 3h^2 \frac{dh}{dt} \right)$ 1M
 $= \pi(26h - h^2) \frac{dh}{dt}$
 $= A \frac{dh}{dt}$ 1A
 $\frac{1}{20}A = A \frac{dh}{dt}$
 $\frac{dh}{dt} = \frac{1}{20}$ 1A
 Required rate is $\frac{1}{20}$ cm/s.

5. (a) (i) Let C be a point on OP such that $AC \perp OP$.

$$\begin{aligned} AC &= \sqrt{OA^2 - OC^2} & 1M \\ &= \sqrt{4^2 - (4-x)^2} \\ &= \sqrt{8x - x^2} \text{ cm} \end{aligned}$$

$$DP = AC = \sqrt{8x - x^2} \text{ cm} \quad 1A$$

$$\begin{aligned} \text{(ii) } S &= \frac{(DP)(AD)}{2} \\ &= \frac{x\sqrt{8x - x^2}}{2} \end{aligned} \quad 1A$$

$$\begin{aligned} \text{(b) } \frac{dS}{dx} &= \frac{1}{2} \left[\sqrt{8x - x^2} + \frac{x(8 - 2x)}{2\sqrt{8x - x^2}} \right] & 1M \\ &= \frac{(8x - x^2) + x(4 - x)}{2\sqrt{8x - x^2}} \\ &= \frac{x(6 - x)}{\sqrt{8x - x^2}} \end{aligned}$$

When $\frac{dS}{dx} = 0$, $x = 6$ or 0 (rejected). 1A

x	$0 < x < 6$	$6 < x < 8$
$\frac{dS}{dx}$	+	-

1M

S attains its maximum at $x = 6$.

$$\begin{aligned} \text{Required value} &= \frac{6\sqrt{8(6) - 6^2}}{2} \\ &= 6\sqrt{3} \end{aligned} \quad 1A$$

$$\begin{aligned} \text{(c) } \frac{dS}{dt} &= \frac{dS}{dx} \cdot \frac{dx}{dt} & 1M \\ &= \frac{2(6 - 2)}{\sqrt{8(2) - 2^2}} \left(-\frac{1}{\sqrt{3}} \right) \\ &= -\frac{4}{3} \end{aligned}$$

The area of $\triangle ADP$ decreases at a rate of $\frac{4}{3} \text{ cm}^2/\text{s}$. 1A