

M2REG-AOD-2425-ASM-SET 2-MATH

Suggested solutions

1. $f'(x) = 2 \cos x - 1$ 1A

When $f'(x) = 0$,

$$\cos x = \frac{1}{2}$$

$$x = \frac{\pi}{3}$$

1A

x	$0 < x < \frac{\pi}{3}$	$\frac{\pi}{3} < x < \pi$
$f'(x)$	+	-

1M

$$f(0) = 0; f\left(\frac{\pi}{3}\right) = \sqrt{3} - \frac{\pi}{3} \text{ and } f(\pi) = -\pi.$$

The greatest value of $f(x)$ is $\sqrt{3} - \frac{\pi}{3}$

1A

The least value of $f(x)$ is $-\pi$.

1A

2. (a) $\frac{dy}{dx} = 3 \sin^2 \left(x + \frac{\pi}{3}\right) \cos \left(x + \frac{\pi}{3}\right)$ 1M

When $\frac{dy}{dx} = 0$, $x = \frac{\pi}{6}$.

The stationary point is $\left(\frac{\pi}{6}, 1 + \pi\right)$.

1A

(b)

x	$0 < x < \frac{\pi}{6}$	$\frac{\pi}{6} < x < \frac{\pi}{2}$
$\frac{dy}{dx}$	+	-

1M

$\left(\frac{\pi}{6}, 1 + \pi\right)$ is a maximum point.

1A

3. (a) $\frac{dy}{dx} = e^{x+1} + xe^{x+1}$ 1A

$$= e^{x+1}(1 + x)$$

When $\frac{dy}{dx} = 0$, $x = -1$.

Stationary point is $(-1, -1)$.

1A

(b)

x	$x < -1$	$x > -1$
$\frac{dy}{dx}$	-	+

1M

$(-1, -1)$ is a minimum point.

1A

$$4. \quad \frac{dy}{dx} = 2x(1-3x^2)^{\frac{1}{2}} + x^2 \left(\frac{1}{2} \right) (1-3x^2)^{-\frac{1}{2}} (-6x) \quad 1M$$

$$= (1-3x^2)^{-\frac{1}{2}} [2x(1-3x^2) - 3x^3]$$

$$= (1-3x^2)^{-\frac{1}{2}} (-9x^3 + 2x)$$

$$\text{When } \frac{dy}{dx} = 0, x = 0 \text{ or } \pm \frac{\sqrt{2}}{3}.$$

x	$-\frac{\sqrt{3}}{3} < x < -\frac{\sqrt{2}}{3}$	$-\frac{\sqrt{2}}{3} < x < 0$	$0 < x < \frac{\sqrt{2}}{3}$	$\frac{\sqrt{2}}{3} < x < \frac{\sqrt{3}}{3}$
$\frac{dy}{dx}$	+	-	+	-

1M

$$\text{Maximum points are } \left(-\frac{\sqrt{2}}{3}, \frac{2\sqrt{3}}{27} \right) \text{ and } \left(\frac{\sqrt{2}}{3}, \frac{2\sqrt{3}}{27} \right).$$

1A

Minimum point is (0, 0).

1A

$$5. \quad (a) \quad \frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{\frac{(x+h)^2}{(x+h)+1} - \frac{x^2}{x+1}}{h} \quad 1M$$

$$= \lim_{h \rightarrow 0} \frac{(x+h)^2(x+1) - x^2(x+h+1)}{h(x+1)(x+h+1)}$$

$$= \lim_{h \rightarrow 0} \frac{hx^2 + (h^2 + 2h)x + h^2}{h(x+1)(x+h+1)}$$

$$= \lim_{h \rightarrow 0} \frac{x^2 + (h+2)x + h}{(x+1)(x+h+1)} \quad 1M$$

$$= \frac{x(x+2)}{(x+1)^2} \quad 1A$$

$$(b) \quad \frac{x(x+2)}{(x+1)^2} \geq 0 \quad 1M$$

$$x(x+2) \geq 0$$

$$x \leq -2 \quad \text{or} \quad x \geq 0 \quad 1A$$

(Accept $x < -2$ or $x > 0$)

$$6. \quad (a) \quad \frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{-\frac{6}{(x+h)^2} + \frac{6}{x^2}}{h} \quad 1M$$

$$= \lim_{h \rightarrow 0} \frac{6(x+h)^2 - 6x^2}{hx^2(x+h)^2}$$

$$= \lim_{h \rightarrow 0} \frac{12x + 6h}{x^2(x+h)^2} \quad 1M$$

$$= \frac{12x}{x^4}$$

$$= \frac{12}{x^3} \quad 1A$$

$$(b) \quad \text{When the curve is increasing, } \frac{dy}{dx} = \frac{12}{x^3} \geq 0. \quad 1M$$

Required range is $x > 0$. 1A

7. (a) $\frac{dy}{dx} = 3x^2 - 4kx - 4k^2$ 1M
 $= (x - 2k)(3x + 2k)$

$$\frac{d^2y}{dx^2} = 6x - 4k$$

When $\frac{dy}{dx} = 0$, $x = 2k$ or $-\frac{2k}{3}$.

$$\left. \frac{d^2y}{dx^2} \right|_{x=2k} = 6(2k) - 4k = 8k > 0$$
 1M

Thus, y attains its local minimum at $x = 2k$. 1

(b) $(2k)^3 - 2k(2k)^2 - 4k^2(2k) = -1000$

$$-8k^3 = -1000$$

$$k^3 = 125$$

$$k = 5$$
 1A

8. (a) $f'(x) = 12x^2 + 4kx + 15$ 1A

(b) Since $f(x)$ is continuous and there are no turning points, we have $f'(x) \geq 0$ or $f'(x) \leq 0$ for all $x \in \mathbb{R}$.

$$\Delta = (4k)^2 - 4(12)(15) \leq 0$$
 1M

$$16k^2 - 720 \leq 0$$

$$-3\sqrt{5} \leq k \leq 3\sqrt{5}$$
 1A

9. (a) $\frac{dy}{dx} = \frac{2x(x+1) - (x^2+p)}{(x+1)^2}$ 1M

$$= \frac{x^2 + 2x - p}{(x+1)^2}$$
 1A

(b) (i) $\frac{1^2 + 2 - p}{(1+1)^2} = 0$ 1M

$$p = 3$$
 1A

(ii) $x^2 + 2x - 3 = 0$ 1M

$$x = -3 \quad \text{or} \quad 1$$

x	$x < -3$	$-3 < x < -1$	$-1 < x < 1$	$x > 1$
$\frac{dy}{dx}$	+	-	-	+

When $x = -3$, $y = -6$; when $x = 1$, $y = 2$.

The local maximum is -6 and the local minimum is 2 . 1M

The local maximum is -6 and the local minimum is 2 . 1A+1A

10. (a) $y = 1 + \frac{\ln x}{x}$
- $$\frac{dy}{dx} = \frac{\frac{1}{x}(x) - \ln x(1)}{x^2} \quad 1M$$
- $$= \frac{1 - \ln x}{x^2} \quad 1A$$
- $$\frac{d^2y}{dx^2} = \frac{x^2\left(-\frac{1}{x}\right) - (1 - \ln x)(2x)}{x^4}$$
- $$= \frac{-3x + 2x \ln x}{x^4}$$
- $$= \frac{2 \ln x - 3}{x^3} \quad 1A$$
- (b) When $\frac{dy}{dx} = 0$,
- $$1 - \ln x = 0$$
- $$\ln x = 1$$
- $$x = e$$
- When $x = e$, $y = \frac{e + \ln e}{e} = \frac{e + 1}{e}$.
- Stationary point is $\left(e, \frac{e + 1}{e}\right)$. 1A
- $$\left.\frac{d^2y}{dx^2}\right|_{x=e} = \frac{2 \ln e - 3}{e^3} = -\frac{1}{e^3} < 0 \quad 1M$$
- $\left(e, \frac{e + 1}{e}\right)$ is a maximum point. 1A
11. (a) $\frac{dy}{dx} = \frac{e^{2x} - (x + 1)(2e^{2x})}{e^{4x}} \quad 1M$
- $$= \frac{-2x - 1}{e^{2x}} \quad 1A$$
- $$\frac{d^2y}{dx^2} = \frac{-2e^{2x} - (-2x - 1)(2e^{2x})}{e^{4x}}$$
- $$= \frac{4x}{e^{2x}} \quad 1A$$
- (b) $\frac{dy}{dx} = \frac{-2x - 1}{e^{2x}} = 0 \quad 1M$
- $$x = -\frac{1}{2}$$
- Stationary point is $\left(-\frac{1}{2}, \frac{e}{2}\right)$. 1A
- $$\left.\frac{d^2y}{dx^2}\right|_{\left(-\frac{1}{2}, \frac{e}{2}\right)} = \frac{4\left(-\frac{1}{2}\right)}{e^{-1}}$$
- $$= -2e$$
- $$< 0$$
- It is a maximum point. 1A

12. (a) $f'(x) = \frac{1}{x^2} - \frac{\ln x}{x^2}$ 1A
 When $f'(x) = 0$,

$$\ln x = 1$$

$$x = e$$
 1M

x	$0 < x < e$	$x > e$
$f'(x)$	+	-

1M

The maximum value of $f(x)$ is $f(e) = \frac{1}{e}$. 1A

(b) By (a), $f(x) \leq f(e)$.

$$\frac{\ln x}{x} \leq \frac{1}{e}$$
 1M

$$\ln x^e \leq x$$

$$e^x \geq x^e$$
 1