

1. (a) $\sin^4 x + \cos^4 x = (\sin^2 x + \cos^2 x)^2 - 2 \sin^2 x \cos^2 x$
- $$= 1 - 2 \left(\frac{\sin 2x}{2} \right)^2 \quad 1\text{M}$$
- $$= 1 - \frac{1}{2} \left(\frac{1 - \cos 4x}{2} \right) \quad 1\text{M}$$
- $$= \frac{3}{4} + \frac{\cos 4x}{4} \quad 1\text{A}$$
- (b) $\int (\sin^4 x + \cos^4 x) dx = \int \left(\frac{3}{4} + \frac{\cos 4x}{4} \right) dx \quad 1\text{M}$
- $$= \frac{3x}{4} + \frac{\sin 4x}{16} + \text{常數} \quad 1\text{A}$$
2. $\int \frac{\left(\sqrt[3]{x} - \frac{2}{\sqrt{x}} \right)^3}{x^2} dx = \int \frac{x - 6x^{\frac{1}{6}} + 6x^{-\frac{2}{3}} - 8x^{-\frac{3}{2}}}{x^2} dx \quad 1\text{M}$
- $$= \int \left(\frac{1}{x} - 6x^{-\frac{11}{6}} + 6x^{-\frac{8}{3}} - 8x^{-\frac{7}{2}} \right) dx$$
- $$= \ln |x| + \frac{36}{5}x^{-\frac{5}{6}} - \frac{36}{5}x^{-\frac{5}{3}} + \frac{16}{5}x^{-\frac{5}{2}} + \text{常數} \quad 1\text{A}$$
3. $\int \frac{x^2 - 9}{x + 3} dx = \int \frac{(x + 3)(x - 3)}{x + 3} dx$
- $$= \int (x - 3) dx \quad 1\text{M}$$
- $$= \frac{1}{2}x^2 - 3x + \text{常數} \quad 1\text{A}$$
4. (a) $\left(\cos \frac{x}{2} + \sin \frac{x}{2} \right) \left(\cos \frac{x}{2} - \sin \frac{x}{2} \right) = \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}$
- $$= \cos \left(2 \times \frac{x}{2} \right)$$
- $$= \cos x \quad 1$$
- (b) $\int \left(\cos \frac{x}{2} + \sin \frac{x}{2} \right) \left(\cos \frac{x}{2} - \sin \frac{x}{2} \right) dx = \int \cos x dx \quad 1\text{M}$
- $$= \sin x + \text{常數} \quad 1\text{A}$$
5. $\int \sqrt[3]{4x + 3} dx = \int (4x + 3)^{\frac{1}{3}} dx$
- $$= \frac{3}{16} (4x + 3)^{\frac{4}{3}} + \text{常數} \quad 1\text{A}$$
6. (a) $\int \frac{\sqrt{x} - 3\sqrt[4]{x}}{\sqrt[3]{x}} dx = \int x^{\frac{1}{6}} dx - 3 \int x^{-\frac{1}{12}} dx \quad 1\text{M}$
- $$= \frac{6}{7}x^{\frac{7}{6}} - \frac{36}{11}x^{\frac{11}{12}} + \text{常數} \quad 1\text{A}$$

$$\begin{aligned}
 \text{(b)} \quad \int \frac{(2x+1)^2}{\sqrt[3]{x^2}} dx &= \int \frac{4x^2+4x+1}{\sqrt[3]{x^2}} dx \\
 &= 4 \int x^{\frac{4}{3}} dx + 4 \int x^{\frac{1}{3}} dx + \int x^{-\frac{2}{3}} dx & 1\text{M} \\
 &= \frac{12}{7}x^{\frac{7}{3}} + 3x^{\frac{4}{3}} + 3x^{\frac{1}{3}} + \text{常數} & 1\text{A}
 \end{aligned}$$

$$\begin{aligned}
 7. \quad \text{(a)} \quad \frac{d}{dx}(x^3 \ln x) &= x^3 \left(\frac{1}{x} \right) + (\ln x)(3x^2) \\
 &= x^2 + 3x^2 \ln x & 1\text{A}
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad \int (x^2 + 3x^2 \ln x) dx &= x^3 \ln x + \text{常數} & 1\text{M} \\
 \int x^2 dx + 3 \int x^2 \ln x dx &= x^3 \ln x + \text{常數} \\
 3 \int x^2 \ln x dx &= x^3 \ln x - \int x^2 dx + \text{常數} \\
 \int x^2 \ln x dx &= \frac{1}{3}x^3 \ln x - \frac{1}{3} \int x^2 dx + \text{常數} \\
 &= \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + \text{常數} & 1\text{A}
 \end{aligned}$$

$$\begin{aligned}
 8. \quad \text{(a)} \quad \frac{d}{dx}[(x^4-1)\ln(x^2+1)] &= 4x^3 \ln(x^2+1) + (x^4-1) \frac{2x}{x^2+1} & 1\text{M} \\
 &= 4x^3 \ln(x^2+1) + (x^2-1)(x^2+1) \frac{2x}{x^2+1} \\
 &= 2x^3 - 2x + 4x^3 \ln(x^2+1) & 1\text{A}
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad \int [2x^3 - 2x + 4x^3 \ln(x^2+1)] dx &= (x^4-1) \ln(x^2+1) + \text{常數} & 1\text{M} \\
 2 \int (x^3 - x) dx + 4 \int x^3 \ln(x^2+1) dx &= (x^4-1) \ln(x^2+1) + \text{常數} \\
 \int x^3 \ln(x^2+1) dx &= \frac{1}{4}(x^4-1) \ln(x^2+1) - \frac{1}{2} \int (x^3 - x) dx + \text{常數} \\
 &= \frac{1}{4}(x^4-1) \ln(x^2+1) - \frac{1}{8}x^4 + \frac{1}{4}x^2 + \text{常數} & 1\text{A}
 \end{aligned}$$

$$\begin{aligned}
 9. \quad \text{(a)} \quad \int \cos 4x \cos x dx &= \frac{1}{2} \int (\cos 5x + \cos 3x) dx & 1\text{M} \\
 &= \frac{\sin 5x}{10} + \frac{\sin 3x}{6} + \text{常數} & 1\text{A}
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad \frac{\sin 6x - \sin 2x}{\sin x} &= \frac{2 \cos 4x \sin 2x}{\sin x} & 1\text{M} \\
 &= \frac{2 \cos 4x (2 \sin x \cos x)}{\sin x} & 1\text{M} \\
 &= 4 \cos 4x \cos x & 1
 \end{aligned}$$

$$\begin{aligned}
\text{(c)} \quad \int \frac{\sin 6x}{\sin x} dx &= \int \left(\frac{\sin 6x - \sin 2x}{\sin x} + \frac{\sin 2x}{\sin x} \right) dx \\
&= 4 \int \cos 4x \cos x dx + \int \frac{\sin 2x}{\sin x} dx & 1M \\
&= 4 \int \cos 4x \cos x dx + 2 \int \cos x dx \\
&= 4 \left(\frac{\sin 5x}{10} + \frac{\sin 3x}{6} \right) + 2 \sin x + \text{常數} \\
&= 2 \sin x + \frac{2 \sin 5x}{5} + \frac{2 \sin 3x}{3} + \text{常數} & 1A
\end{aligned}$$

$$\begin{aligned}
10. \quad \text{(a)} \quad \int e^{\ln x^{-1}} dx &= \int x^{-1} dx & 1M \\
&= \ln |x| + \text{常數} & 1A
\end{aligned}$$

$$\begin{aligned}
\text{(b)} \quad \int 3^{2x} 7^x dx &= \int 63^x dx & 1M \\
&= \frac{1}{\ln 63} 63^x + \text{常數} & 1A
\end{aligned}$$

$$\begin{aligned}
11. \quad \int \left(x - \frac{2}{x} \right)^3 dx &= \int \left(x^3 - 6x + \frac{12}{x} - \frac{8}{x^3} \right) dx & 1M \\
&= \frac{x^4}{4} - 3x^2 + 12 \ln |x| + 4x^{-2} + \text{常數} & 1A
\end{aligned}$$

$$\begin{aligned}
12. \quad \int \frac{x^2(2x^2 + 3x - 3)}{\sqrt{x}} dx &= \int (2x^{\frac{7}{2}} + 3x^{\frac{5}{2}} - 3x^{\frac{3}{2}}) dx & 1M \\
&= \frac{4}{9} x^{\frac{9}{2}} + \frac{6}{7} x^{\frac{7}{2}} - \frac{6}{5} x^{\frac{5}{2}} + \text{常數} & 1A
\end{aligned}$$

$$\begin{aligned}
13. \quad y &= \int \left(x - \frac{1}{2x^2} \right)^2 dx \\
&= \int \left(x^2 - \frac{1}{x} + \frac{1}{4x^4} \right) dx & 1M \\
&= \frac{x^3}{3} - \ln |x| - \frac{1}{12} x^{-3} + \text{常數} & 1A
\end{aligned}$$

$$\begin{aligned}
14. \quad \int \frac{dx}{\sqrt[3]{(16x - 56)^4}} &= \int (16x - 56)^{-\frac{4}{3}} dx \\
&= -\frac{3}{16} (16x - 56)^{-\frac{1}{3}} + \text{常數} & 1A
\end{aligned}$$

$$\begin{aligned}
15. \int \frac{x^4 - 9}{x(x^2 - 3)} dx &= \int \frac{(x^2 - 3)(x^2 + 3)}{x(x^2 - 3)} dx \\
&= \int \frac{x^2 + 3}{x} dx \\
&= \int \left(x + \frac{3}{x} \right) dx && 1M \\
&= \frac{x^2}{2} + 3 \ln |x| + \text{常數} && 1A
\end{aligned}$$

$$\begin{aligned}
16. \int (\sqrt{x} + \sqrt[3]{x})^4 dx &= \int (x^{\frac{1}{2}} + x^{\frac{1}{3}})^4 dx \\
&= \int \left(x^2 + 4x^{\frac{11}{6}} + 6x^{\frac{5}{3}} + 4x^{\frac{3}{2}} + x^{\frac{4}{3}} \right) dx && 1M \\
&= \frac{1}{3}x^3 + \frac{24}{17}x^{\frac{17}{6}} + \frac{9}{4}x^{\frac{8}{3}} + \frac{8}{5}x^{\frac{5}{2}} + \frac{3}{7}x^{\frac{7}{3}} + \text{常數} && 1A
\end{aligned}$$

$$\begin{aligned}
17. y &= \int (2 + \sqrt{x})^2 dx \\
&= \int (4 + 4\sqrt{x} + x) dx && 1M \\
&= 4 \int dx + 4 \int x^{\frac{1}{2}} dx + \int x dx \\
&= 4x + \frac{8}{3}x^{\frac{3}{2}} + \frac{x^2}{2} + \text{常數} && 1A
\end{aligned}$$

$$\begin{aligned}
18. \int 2^{3-5x} dx &= \int e^{(3-5x) \ln 2} dx && 1M \\
&= \frac{1}{-5 \ln 2} e^{(3-5x) \ln 2} + \text{常數} \\
&= -\frac{1}{5 \ln 2} 2^{3-5x} + \text{常數} && 1A
\end{aligned}$$

$$\begin{aligned}
19. \int (2\sqrt{x} - 1)(\sqrt[3]{x} + 1) dx &= \int (2x^{\frac{1}{2}} - 1)(x^{\frac{1}{3}} + 1) dx \\
&= \int (2x^{\frac{5}{6}} + 2x^{\frac{1}{2}} - x^{\frac{1}{3}} - 1) dx && 1M \\
&= \frac{12}{11}x^{\frac{11}{6}} + \frac{4}{3}x^{\frac{3}{2}} - \frac{3}{4}x^{\frac{4}{3}} - x + \text{常數} && 1A
\end{aligned}$$

$$\begin{aligned}
20. \int \frac{\sqrt{x}(3x^3 - 2x + 1)}{x^3} dx &= \int \frac{3x^{\frac{7}{2}} - 2x^{\frac{3}{2}} + x^{\frac{1}{2}}}{x^3} dx \\
&= \int (3x^{\frac{1}{2}} - 2x^{-\frac{3}{2}} + x^{-\frac{5}{2}}) dx && 1M \\
&= 2x^{\frac{3}{2}} + 4x^{-\frac{1}{2}} - \frac{2}{3}x^{-\frac{3}{2}} + \text{常數} && 1A
\end{aligned}$$

$$\begin{aligned}
21. \quad (a) \quad & \int (e^{4x} + e^{-x})^5 dx \\
&= \int \left[\sum_{r=0}^5 \left(C_r^5 (e^{4x})^{5-r} (e^{-x})^r \right) \right] dx & 1M \\
&= \int (e^{20x} + 5e^{15x} + 10e^{10x} + 10e^{5x} + 5 + e^{-5x}) dx & 1A \\
&= \frac{e^{20x}}{20} + \frac{e^{15x}}{3} + e^{10x} + 2e^{5x} + 5x - \frac{1}{5e^{5x}} + \text{常數} & 1M+1A
\end{aligned}$$

(b) 設 $u = \ln x$ 。則 $du = \frac{1}{x} dx$ 。 1M

$$\begin{aligned}
\int \frac{\ln x}{x} dx &= \int u du \\
&= \frac{u^2}{2} + \text{常數} \\
&= \frac{(\ln x)^2}{2} + \text{常數} & 1A
\end{aligned}$$

22. 設 $u = x^3 - 2x + 3$ 。則 $du = (3x^2 - 2) dx$ 。 1M

$$\begin{aligned}
\int (6x^2 - 4)(x^3 - 2x + 3)^{\frac{1}{3}} dx &= 2 \int u^{\frac{1}{3}} du \\
&= \frac{3}{2} u^{\frac{4}{3}} + \text{常數} \\
&= \frac{3}{2} (x^3 - 2x + 3)^{\frac{4}{3}} + \text{常數} & 1A
\end{aligned}$$

23. 設 $u = x + x^2 + \frac{1}{x}$ 。則 $du = \left(1 + 2x - \frac{1}{x^2}\right) dx$ 。 1M

$$\begin{aligned}
\int \left(1 + 2x - \frac{1}{x^2}\right) \left(x + x^2 + \frac{1}{x}\right)^{-1} dx &= \int u^{-1} du \\
&= \ln |u| + \text{常數} \\
&= \ln \left| x + x^2 + \frac{1}{x} \right| + \text{常數} & 1A
\end{aligned}$$

24. 設 $u = \ln x$ 。則 $du = \frac{1}{x} dx$ 。 1M

$$\begin{aligned}
\int \frac{\sqrt{\ln \sqrt{x}}}{x} dx &= \int \frac{\sqrt{\frac{1}{2} \ln x}}{x} dx \\
&= \frac{1}{\sqrt{2}} \int u^{\frac{1}{2}} du & 1M \\
&= \frac{\sqrt{2}}{3} u^{\frac{3}{2}} + \text{常數} \\
&= \frac{\sqrt{2}}{3} (\ln x)^{\frac{3}{2}} + \text{常數}
\end{aligned}$$

$$25. \quad (a) \quad \int \cos^6 \frac{x}{2} \sin 2x \, dx = \int \left(\frac{1 + \cos x}{2} \right)^3 (2 \sin x \cos x) \, dx \quad 1M$$

設 $u = \cos x$ 。則 $du = -\sin x \, dx$ 。 1M

$$\begin{aligned} \int \cos^6 \frac{x}{2} \sin 2x \, dx &= -\frac{1}{4} \int (1+u)^3 u \, du \\ &= -\frac{1}{4} (u + 3u^2 + 3u^3 + u^4) \, du \quad 1A \\ &= -\frac{1}{4} \left(\frac{\cos^2 x}{2} + \cos^3 x + \frac{3 \cos^4 x}{4} + \frac{\cos^5 x}{5} \right) + \text{常數} \\ &= -\frac{\cos^2 x}{8} - \frac{\cos^3 x}{4} - \frac{3 \cos^4 x}{16} - \frac{\cos^5 x}{20} + \text{常數} \quad 1A \end{aligned}$$

(b) 設 $u = 2^x$ 。則 $du = 2^x \ln 2 \, dx$ 。 1M

$$\begin{aligned} \int 2^x \cos^6 2^{x-1} \sin 2^{x+1} \, dx &= \frac{1}{\ln 2} \int \cos^6 \frac{u}{2} \sin 2u \, du \\ &= \frac{1}{\ln 2} \left(\frac{\cos^2 u}{8} - \frac{\cos^3 u}{4} - \frac{3 \cos^4 u}{16} - \frac{\cos^5 u}{20} \right) + \text{常數} \quad 1M \\ &= -\frac{\cos^2 2^x}{8 \ln 2} - \frac{\cos^3 2^x}{4 \ln 2} - \frac{3 \cos^4 2^x}{16 \ln 2} - \frac{\cos^5 2^x}{20 \ln 2} + \text{常數} \quad 1A \end{aligned}$$

26. 設 $u = 2\sqrt{x} + 1$ 。則 $du = \frac{1}{\sqrt{x}} \, dx$ 。 1M

$$\begin{aligned} \int \frac{(2\sqrt{x} + 1)^5}{\sqrt{x}} \, dx &= \int u^5 \, du \\ &= \frac{u^6}{6} + \text{常數} \\ &= \frac{(2\sqrt{x} + 1)^6}{6} + \text{常數} \quad 1A \end{aligned}$$

27. 設 $u = \sqrt{x} + 1$ 。則 $du = \frac{1}{2\sqrt{x}} \, dx$ 。 1M

$$\begin{aligned} \int \frac{\sin(\sqrt{x} + 1)}{\sqrt{x}} \, dx &= 2 \int \sin u \, du \\ &= -2 \cos u + \text{常數} \\ &= -2 \cos(\sqrt{x} + 1) + \text{常數} \quad 1A \end{aligned}$$

28. 設 $u = e^x + 1$ 。則 $du = e^x dx$ 。 1M

$$\begin{aligned}\int \frac{e^x}{e^{2x} + 2e^x + 1} dx &= \int \frac{e^x}{(e^x + 1)^2} dx \\ &= \int \frac{du}{u^2} & 1A \\ &= -\frac{1}{u} + \text{常數} \\ &= -\frac{1}{e^x + 1} + \text{常數} & 1A\end{aligned}$$

29. 設 $u = 3 + \sin(\ln x)$ 。則 $du = \frac{\cos(\ln x)}{x} dx$ 。 1M

$$\begin{aligned}\int \frac{\cos(\ln x)}{x[3 + \sin(\ln x)]} dx &= \int \frac{du}{u} & 1A \\ &= \ln |3 + \sin(\ln x)| + \text{常數} \\ &= \ln(3 + \sin(\ln x)) + \text{常數} & 1A\end{aligned}$$

30. 設 $u = \log_2 x$ 。則 $du = \frac{1}{x \ln 2} dx$ 。 1M

$$\begin{aligned}\int \frac{3(\log_2 x)^2 + 2 \log_2 x - 4}{x} dx &= \int (\ln 2)(3u^2 + 2u - 4) du & 1A \\ &= (\ln 2)(u^3 + u^2 - 4u) + \text{常數} \\ &= (\ln 2)[(\log_2 x)^3 + (\log_2 x)^2 - 4 \log_2 x] + \text{常數} & 1A\end{aligned}$$

31. 設 $u = 1 + \frac{3}{x^2}$ 。則 $du = -\frac{6}{x^3} dx$ 。 1M

$$\begin{aligned}\int \frac{\sqrt{x^2 + 3}}{x^4} dx &= -\frac{1}{6} \int \left(\frac{\sqrt{x^2 + 3}}{x} \right) \left(-\frac{6}{x^3} \right) dx \\ &= -\frac{1}{6} \int \sqrt{1 + \frac{3}{x^2}} \left(-\frac{6}{x^3} \right) dx & 1M \\ &= -\frac{1}{6} \int \sqrt{u} du & 1A \\ &= -\frac{1}{9} u^{\frac{3}{2}} + \text{常數} \\ &= -\frac{1}{9} \left(1 + \frac{3}{x^2} \right)^{\frac{3}{2}} + \text{常數} & 1A\end{aligned}$$

32. 設 $u = x^2 - 3x + 4$ 。則 $du = (2x - 3) dx$ 。 1M

$$\begin{aligned}
\int (2x-3)(x^2-3x+4)^8 dx &= \int u^8 du \\
&= \frac{u^9}{9} + \text{常數} \\
&= \frac{1}{9}(x^2-3x+4)^9 + \text{常數}
\end{aligned}$$

1A

33. (a) $(1+ax)^7 = \sum_{r=0}^7 C_r^7 a^r x^r$ 1M

$$\frac{\beta_1}{\beta_3} = \frac{C_1^7 a}{C_3^7 a^3} = \frac{1}{45}$$

1M

$$\frac{7}{35a^2} = \frac{1}{45}$$

$$a^2 = 9$$

$$a = 3 \quad \text{或} \quad -3 \quad (\text{捨去})$$

1A

(b) $\int \sum_{r=0}^7 \beta_r x^r dx = \int (1+3x)^7 dx$ 1M

$$= \frac{(1+3x)^8}{24} + \text{常數}$$

1A

34. 設 $u = \sqrt{x} - 1$ 。則 $du = \frac{1}{2\sqrt{x}} dx$ 。 1M

$$\int \sqrt{\sqrt{x}-1} dx = \int 2\sqrt{x}\sqrt{\sqrt{x}-1} \cdot \frac{1}{2\sqrt{x}} dx$$

$$= \int 2(u+1)\sqrt{u} du$$

$$= \int (2u^{\frac{3}{2}} + 2u^{\frac{1}{2}}) du$$

1M

$$= \frac{4}{5}u^{\frac{5}{2}} + \frac{4}{3}u^{\frac{3}{2}} + \text{常數}$$

$$= \frac{4}{5}(\sqrt{x}-1)^{\frac{5}{2}} + \frac{4}{3}(\sqrt{x}-1)^{\frac{3}{2}} + \text{常數}$$

1A

35. (a) $P + \frac{Q}{x+4} + \frac{R}{2x-5} = \frac{P(x+4)(2x-5) + Q(2x-5) + R(x+4)}{(x+4)(2x-5)}$

$$= \frac{2Px^2 + (3P+2Q+R)x + (-20P-5Q+4R)}{(x+4)(2x-5)}$$

可得 $2P = 4$ 、 $3P + 2Q + R = 11$ 及 $-20P - 5Q + 4R = -59$ 。 1M

求解後，可得 $P = 2$ 、 $Q = 3$ 及 $R = -1$ 。 1A+1A

(b) $\int \frac{4x^2 + 11x - 59}{2x^2 + 3x - 20} dx = \int \frac{4x^2 + 11x - 59}{(x+4)(2x-5)} dx$

$$= \int \left(2 + \frac{3}{x+4} - \frac{1}{2x-5} \right) dx$$

1M

$$= 2x + 3 \ln|x+4| - \frac{1}{2} \ln|2x-5| + \text{常數}$$

1A

36. 設 $u = e^x$ 。則 $du = e^x dx$ 。 1M

$$\begin{aligned}\int e^x 5^{e^x} dx &= \int 5^u du \\ &= \int e^{u \ln 5} du & 1M \\ &= \frac{1}{\ln 5} e^{u \ln 5} + \text{常數} \\ &= \frac{1}{\ln 5} 5^{e^x} + \text{常數} & 1A\end{aligned}$$

37. 設 $u = \sin x$ 。則 $du = \cos x dx$ 。 1M

$$\begin{aligned}\int \frac{\cos^5 x}{\sin^6 x} dx &= \int \frac{(1-u^2)^2}{u^6} du \\ &= \int (u^{-6} - 2u^{-4} + u^{-2}) du & 1A \\ &= -\frac{1}{5u^5} + \frac{2}{3u^3} - \frac{1}{u} + \text{常數} \\ &= -\frac{1}{5 \sin^5 x} + \frac{2}{3 \sin^3 x} - \frac{1}{\sin x} + \text{常數} & 1A\end{aligned}$$

38. 設 $u = 2e^x + 1$ 。則 $du = 2e^x dx$ 。 1M

$$\begin{aligned}\int e^{2x} \sqrt{2e^x + 1} dx &= \int \frac{1}{2} \left(\frac{u-1}{2} \right) \sqrt{u} du \\ &= \int \left(\frac{1}{4} u^{\frac{3}{2}} - \frac{1}{4} u^{\frac{1}{2}} \right) du & 1M \\ &= \frac{1}{10} u^{\frac{5}{2}} - \frac{1}{6} u^{\frac{3}{2}} + \text{常數} \\ &= \frac{1}{10} (2e^x + 1)^{\frac{5}{2}} - \frac{1}{6} (2e^x + 1)^{\frac{3}{2}} + \text{常數} & 1A\end{aligned}$$

39. 設 $u = e^x + e^{-x}$ 。則 $du = (e^x - e^{-x}) dx$ 。 1M

$$\begin{aligned}\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx &= \int \frac{du}{u} \\ &= \ln |u| + \text{常數} \\ &= \ln |e^x + e^{-x}| + \text{常數} & 1A\end{aligned}$$

40. 設 $u = 1 - e^{-5x}$ 。則 $du = 5e^{-5x} dx$ 。 1M

$$\begin{aligned}\int \frac{dx}{e^{5x} - 1} &= \int \frac{e^{-5x}}{1 - e^{-5x}} dx \\ &= \frac{1}{5} \int \frac{du}{u} \\ &= \frac{1}{5} \ln |u| + \text{常數} \\ &= \frac{1}{5} \ln |1 - e^{-5x}| + \text{常數}\end{aligned}$$

1M
1A

另一答案

設 $u = e^{5x} - 1$ 。則 $du = 5e^{5x} dx$ 。 1M

$$\begin{aligned}\int \frac{dx}{e^{5x} - 1} &= \int \frac{1 - e^{5x} + e^{5x}}{e^{5x} - 1} dx \\ &= \int \frac{1 - e^{5x}}{e^{5x} - 1} dx + \int \frac{e^{5x}}{e^{5x} - 1} dx \\ &= - \int dx + \frac{1}{5} \int \frac{du}{u} \\ &= -x + \frac{1}{5} \ln |u| + \text{常數} \\ &= -x + \frac{1}{5} \ln |e^{5x} - 1| + \text{常數}\end{aligned}$$

1M
1A

41. (a) $A + B = \int \frac{\sin \theta}{\sin \theta + \cos \theta} d\theta + \int \frac{\cos \theta}{\sin \theta + \cos \theta} d\theta$

$$\begin{aligned}&= \int \frac{\sin \theta + \cos \theta}{\sin \theta + \cos \theta} d\theta \\ &= \int d\theta \\ &= \theta + \text{常數}\end{aligned}$$

1A

設 $u = \sin \theta + \cos \theta$ 。則 $du = (\cos \theta - \sin \theta) d\theta$ 。 1M

$$\begin{aligned}B - A &= \int \frac{\cos \theta}{\sin \theta + \cos \theta} d\theta - \int \frac{\sin \theta}{\sin \theta + \cos \theta} d\theta \\ &= \int \frac{\cos \theta - \sin \theta}{\sin \theta + \cos \theta} d\theta \\ &= \int \frac{du}{u} \\ &= \ln |u| + \text{常數} \\ &= \ln |\sin \theta + \cos \theta| + \text{常數}\end{aligned}$$

1A

(b) 設 $x = \sin \theta$ 。則 $dx = \cos \theta d\theta$ 。

$$\int \frac{1}{x + \sqrt{1-x^2}} dx = \int \frac{\cos \theta}{\sin \theta + \sqrt{1-\sin^2 \theta}} d\theta \quad 1M$$

$$= \int \frac{\cos \theta}{\sin \theta + \cos \theta} d\theta$$

$$= \frac{(A+B) + (B-A)}{2}$$

$$= \frac{\theta}{2} + \frac{1}{2} \ln |\sin \theta + \cos \theta| + \text{常數} \quad 1M$$

$$= \frac{1}{2} \sin^{-1} x + \frac{1}{2} \ln |x + \sqrt{1-x^2}| + \text{常數} \quad 1A$$

42. (a) 設 $u = 25 + x^2$ 。則 $du = 2x dx$ 。 1M

$$\int \frac{x}{25+x^2} dx = \frac{1}{2} \int \frac{1}{u} du \quad 1A$$

$$= \frac{1}{2} \ln |u| + \text{常數}$$

$$= \frac{1}{2} \ln(25+x^2) + \text{常數} \quad 1A$$

(b) 設 $x = 5 \tan \theta$ 。則 $dx = 5 \sec^2 \theta d\theta$ 。 1M

$$\int \frac{x^2}{25+x^2} dx = \int \frac{25 \tan^2 \theta}{25+25 \tan^2 \theta} \cdot 5 \sec^2 \theta d\theta$$

$$= 5 \int \tan^2 \theta d\theta \quad 1A$$

$$= 5 \int (\sec^2 \theta - 1) d\theta \quad 1M$$

$$= 5 \tan \theta - 5\theta + \text{常數}$$

$$= x - 5 \tan^{-1} \frac{x}{5} + \text{常數} \quad 1A$$

43. (a) $x^2 + 6x + 10 = (x+3)^2 + 1^2$ 1A

(b) 設 $x+3 = \tan \theta$ 。則 $dx = \sec^2 \theta d\theta$ 。 1M

$$\int \frac{1}{x^2+6x+10} dx = \int \frac{\sec^2 \theta}{\tan^2 \theta + 1} d\theta$$

$$= \int d\theta \quad 1A$$

$$= \theta + \text{常數}$$

$$= \tan^{-1}(x+3) + \text{常數} \quad 1A$$

44. (a) 設 $x = 3 \tan \theta$ 。則 $dx = 3 \sec^2 \theta d\theta$ 。 1M

$$\begin{aligned} \int \frac{1}{x^2 + 9} dx &= \int \frac{3 \sec^2 \theta}{9 \tan^2 \theta + 9} d\theta \\ &= \frac{1}{3} \int d\theta & 1A \\ &= \frac{1}{3} \theta + \text{常數} \\ &= \frac{1}{3} \tan^{-1} \frac{x}{3} + \text{常數} & 1A \end{aligned}$$

- (b) 設 $u = x^2 + 9$ 。則 $du = 2x dx$ 。 1M

$$\begin{aligned} \int \frac{2x + 1}{x^2 + 9} dx &= \int \frac{2x}{x^2 + 9} dx + \int \frac{1}{x^2 + 9} dx & 1M \\ &= \int \frac{du}{u} + \frac{1}{3} \tan^{-1} \frac{x}{3} \\ &= \ln |u| + \frac{1}{3} \tan^{-1} \frac{x}{3} + \text{常數} \\ &= \ln(x^2 + 9) + \frac{1}{3} \tan^{-1} \frac{x}{3} + \text{常數} & 1A \end{aligned}$$

45. (a) $6x - x^2 - 8 = 1^2 - (x - 3)^2$ 1A

- (b) 設 $x - 3 = \sin \theta$ 。則 $dx = \cos \theta d\theta$ 。 1M

$$\begin{aligned} \int \sqrt{6x - x^2 - 8} dx &= \int \sqrt{1 - \sin^2 \theta} \cos \theta d\theta \\ &= \int \cos^2 \theta d\theta & 1A \\ &= \frac{1}{2} \int (1 + \cos 2\theta) d\theta & 1M \\ &= \frac{\theta}{2} + \frac{\sin 2\theta}{4} + \text{常數} \\ &= \frac{\theta}{2} + \frac{1}{2} \sin \theta \cos \theta + \text{常數} \\ &= \frac{1}{2} \sin^{-1}(x - 3) + \frac{1}{2}(x - 3)\sqrt{6 - x^2 - 8} + \text{常數} & 1A \end{aligned}$$

46. (a) $x^2 + 2x \sin \frac{5\pi}{12} + 1 = x^2 + 2x \sin \frac{5\pi}{12} + \sin^2 \frac{5\pi}{12} + \cos^2 \frac{5\pi}{12}$
 $= \left(x + \sin \frac{5\pi}{12}\right)^2 + \cos^2 \frac{5\pi}{12}$ 1A

(b) 設 $x + \sin \frac{5\pi}{12} = \cos \frac{5\pi}{12} \tan \theta$ 。則 $dx = \cos \frac{5\pi}{12} \sec^2 \theta d\theta$ 。 1M

$$\int \frac{\cos \frac{5\pi}{12}}{x^2 + 2x \sin \frac{5\pi}{12} + 1} dx = \int \frac{\cos \frac{5\pi}{12}}{\cos^2 \frac{5\pi}{12} \tan^2 \theta + \cos^2 \frac{5\pi}{12}} \cdot \cos \frac{5\pi}{12} \sec^2 \theta d\theta$$

$$= \int d\theta$$
 1A

$$= \theta + \text{常數}$$

$$= \tan^{-1} \left(\frac{x + \sin \frac{5\pi}{12}}{\cos \frac{5\pi}{12}} \right) + \text{常數}$$
 1A

$$= \tan^{-1} \left(\frac{x \csc \frac{5\pi}{12} + 1}{\cot \frac{5\pi}{12}} \right) + \text{常數}$$

$$= \tan^{-1} \frac{x(\sqrt{6} - \sqrt{2}) + 1}{2 - \sqrt{3}} + \text{常數}$$
 1A

47. (a) $f(x) = 4^2 - (x - 3)^2$ 1A

(b) 設 $x - 3 = 4 \sin \theta$ 。則 $dx = 4 \cos \theta d\theta$ 。 1M

$$\int \frac{1}{\sqrt{7 + 6x - x^2}} dx = \int \frac{4 \cos \theta}{\sqrt{16 - 16 \sin^2 \theta}} d\theta$$

$$= \int d\theta$$
 1A

$$= \theta + \text{常數}$$

$$= \sin^{-1} \frac{x - 3}{4} + \text{常數}$$
 1A

48. (a) $x^2 + 2x \cos \frac{7\pi}{12} + 1 = x^2 + 2x \cos \frac{7\pi}{12} + \sin^2 \frac{7\pi}{12} + \cos^2 \frac{7\pi}{12}$

$$= \left(x + \cos \frac{7\pi}{12} \right)^2 + \sin^2 \frac{7\pi}{12}$$
 1A

(b) 設 $x + \cos \frac{7\pi}{12} = \sin \frac{7\pi}{12} \tan \theta$ 。則 $dx = \sin \frac{7\pi}{12} \sec^2 \theta d\theta$ 。 1M

$$\int \frac{1}{x^2 + 2x \cos \frac{7\pi}{12} + 1} dx = \int \frac{\sin \frac{7\pi}{12} \sec^2 \theta}{\sin^2 \frac{7\pi}{12} \tan^2 \theta + \sin^2 \frac{7\pi}{12}} d\theta$$

$$= \csc \frac{7\pi}{12} \int d\theta$$
 1A

$$= \csc \frac{7\pi}{12} \theta + \text{常數}$$

$$= \csc \frac{7\pi}{12} \tan^{-1} \left(x \csc \frac{7\pi}{12} + \cot \frac{7\pi}{12} \right) + \text{常數}$$
 1A

$$= (\sqrt{6} - \sqrt{2}) \tan^{-1} \left[(\sqrt{6} - \sqrt{2})x + \sqrt{3} - 2 \right] + \text{常數}$$
 1A

49. 設 $\sqrt{2}x = \sin \theta$ 。則 $\sqrt{2} dx = \cos \theta d\theta$ 。 1M

$$\begin{aligned} \int \frac{x^2}{\sqrt{1-2x^2}} dx &= \int \frac{\frac{1}{2} \sin^2 \theta}{\sqrt{1-\sin^2 \theta}} \cdot \frac{\cos \theta}{\sqrt{2}} d\theta \\ &= \frac{\sqrt{2}}{4} \int \sin^2 \theta d\theta \end{aligned}$$

1A

$$= \frac{\sqrt{2}}{8} \int (1 - \cos 2\theta) d\theta$$

1M

$$= \frac{\sqrt{2}}{8} \theta - \frac{\sqrt{2}}{16} \sin 2\theta + \text{常數}$$

$$= \frac{\sqrt{2}}{8} \theta - \frac{\sqrt{2}}{8} \sin \theta \cos \theta + \text{常數}$$

$$= \frac{\sqrt{2}}{8} \sin^{-1}(\sqrt{2}x) - \frac{1}{4} x \sqrt{1-2x^2} + \text{常數}$$

1A

50. $x^2 + 8x + 17 = (x+4)^2 + 1$

設 $x+4 = \tan \theta$ 。則 $dx = \sec^2 \theta d\theta$ 。 1M

$$\begin{aligned} \int \frac{x}{x^2 + 8x + 17} dx &= \int \frac{(\tan \theta - 4) \sec^2 \theta}{\tan^2 \theta + 1} d\theta \\ &= \int (\tan \theta - 4) d\theta \end{aligned}$$

1A

設 $u = \cos \theta$ 。則 $du = -\sin \theta d\theta$ 。 1M

$$\begin{aligned} \int \frac{x}{x^2 + 8x + 17} dx &= - \int \frac{du}{u} - \int 4 d\theta \\ &= -\ln |\cos \theta| - 4\theta + \text{常數} \\ &= -\ln \left| \frac{1}{\sqrt{x^2 + 8x + 17}} \right| - 4 \tan^{-1}(x+4) + \text{常數} \\ &= \ln \sqrt{x^2 + 8x + 17} - 4 \tan^{-1}(x+4) + \text{常數} \end{aligned}$$

1A

51. (a) $4x - x^2 = 4 - (x-2)^2$ 1A

(b) 設 $x-2 = 2 \sin \theta$ 。則 $dx = 2 \cos \theta d\theta$ 。 1M

$$\begin{aligned} \int \frac{dx}{\sqrt{4x-x^2}} &= \int \frac{2 \cos \theta}{\sqrt{4-4 \sin^2 \theta}} d\theta \\ &= \int \frac{\cos \theta}{\cos \theta} d\theta \\ &= \int d\theta \end{aligned}$$

1A

$$= \theta + \text{常數}$$

$$= \sin^{-1} \frac{x-2}{2} + \text{常數}$$

1A

52. 設 $\ln x = \sin \theta$ 。則 $\frac{1}{x} dx = \cos \theta d\theta$ 。 1M

$$\int \frac{\sqrt{1 - (\ln x)^2}}{x} dx = \int \sqrt{1 - \sin^2 \theta} \cos \theta d\theta$$

$$= \int \cos^2 \theta d\theta$$

1A

$$= \frac{1}{2} \int (1 + \cos 2\theta) d\theta$$

1M

$$= \frac{\theta}{2} + \frac{\sin 2\theta}{4} + \text{常數}$$

$$= \frac{\theta}{2} + \frac{\sin \theta \cos \theta}{2} + \text{常數}$$

$$= \frac{1}{2} \sin^{-1}(\ln x) + \frac{1}{2} (\ln x) \sqrt{1 - (\ln x)^2} + \text{常數}$$

1A

53. (a) $\sin 3\theta = \sin(2\theta + \theta)$

$$= \sin 2\theta \cos \theta + \cos 2\theta \sin \theta$$

1M

$$= (2 \sin \theta \cos \theta) \cos \theta + (2 \cos^2 \theta - 1) \sin \theta$$

$$= \sin \theta (4 \cos^2 \theta - 1)$$

1

(b) 設 $u = \cos \theta$ 。則 $du = -\sin \theta d\theta$ 。 1M

$$\int \frac{\sin 3\theta}{1 + \cos^2 \theta} d\theta = \int \frac{\sin \theta (4 \cos^2 \theta - 1)}{1 + \cos^2 \theta} d\theta$$

$$= \int \frac{1 - 4u^2}{1 + u^2} du$$

$$= \int \left(\frac{5}{1 + u^2} - 4 \right) du$$

1M

設 $u = \tan \phi$ 。則 $du = \sec^2 \phi d\phi$ 。 1M

$$\int \frac{\sin 3\theta}{1 + \cos^2 \theta} d\theta = \int \frac{5 \sec^2 \phi}{1 + \tan^2 \phi} d\phi - 4u$$

$$= 5 \int d\phi - 4u$$

$$= 5\phi - 4u + \text{常數}$$

$$= 5 \tan^{-1}(\cos \theta) - 4 \cos \theta + \text{常數}$$

1A

54. 設 $x = e \sin \theta$ 。則 $dx = e \cos \theta d\theta$ 。 1M

$$\int x^2 \sqrt{e^2 - x^2} dx = \int (e \sin \theta)^2 \sqrt{e^2 - e^2 \sin^2 \theta} e \cos \theta d\theta$$

$$= e^4 \int \sin^2 \theta \cos^2 \theta d\theta$$

1A

$$= \frac{e^4}{4} \int (2 \sin \theta \cos \theta)^2 d\theta$$

$$= \frac{e^4}{4} \int \sin^2 2\theta d\theta$$

1M

$$= \frac{e^4}{8} \int (1 - \cos 4\theta) d\theta$$

1M

$$= \frac{e^4 \theta}{8} - \frac{e^4 \sin 4\theta}{32} + \text{常數}$$

$$= \frac{e^4 \theta}{8} - \frac{e^4}{16} \sin 2\theta \cos 2\theta + \text{常數}$$

$$= \frac{e^4 \theta}{8} - \frac{e^4}{8} \sin \theta \cos \theta (1 - 2 \sin^2 \theta) + \text{常數}$$

$$= \frac{e^4 \theta}{8} - \frac{e^4}{8} \left(\frac{x}{e} \right) \left(\frac{\sqrt{e^2 - x^2}}{e} \right) \left[1 - 2 \left(\frac{x}{e} \right)^2 \right] + \text{常數}$$

$$= \frac{e^4}{8} \sin^{-1} \frac{x}{e} - \frac{1}{8} x (e^2 - 2x^2) \sqrt{e^2 - x^2} + \text{常數}$$

1A

55. 設 $x = 5 \sin \theta$ 。則 $dx = 5 \cos \theta d\theta$ 。 1M

$$\int \frac{1}{(25 - x^2)^{\frac{5}{2}}} dx = \int \frac{5 \cos \theta}{(25 - 25 \sin^2 \theta)^{\frac{5}{2}}} d\theta$$

$$= \frac{1}{625} \int \sec^4 \theta d\theta$$

1A

設 $u = \tan \theta$ 。則 $du = \sec^2 \theta d\theta$ 。 1M

$$\int \frac{1}{(25 - x^2)^{\frac{5}{2}}} dx = \frac{1}{625} \int (u^2 + 1) du$$

$$= \frac{1}{1875} u^3 + \frac{1}{625} u + \text{常數}$$

$$= \frac{1}{1875} \tan^3 \theta + \frac{1}{625} \tan \theta + \text{常數}$$

$$= \frac{x^3}{1875(25 - x^2)^{\frac{3}{2}}} + \frac{x}{625\sqrt{25 - x^2}} + \text{常數}$$

1A

$$56. \quad (a) \quad \frac{\sec^2 \theta}{\sec^2 \theta + 2 \tan^2 \theta} = \frac{\frac{1}{\cos^2 \theta}}{\frac{1}{\cos^2 \theta} + 2 \cdot \frac{\sin^2 \theta}{\cos^2 \theta}} \times \frac{\cos^2 \theta}{\cos^2 \theta}$$

$$= \frac{1}{1 + 2 \sin^2 \theta}$$

1

(b) 設 $t = \tan x$ 。則 $dt = \sec^2 x \, dx$ 。 1M

$$\begin{aligned} \int \frac{dx}{1 + 2 \sin^2 x} &= \int \frac{\sec^2 x}{\sec^2 x + 2 \tan^2 x} dx \\ &= \int \frac{dt}{(t^2 + 1) + 2t^2} \\ &= \int \frac{dt}{3t^2 + 1} \end{aligned} \quad 1A$$

設 $\sqrt{3}t = \tan \theta$ 。則 $\sqrt{3} \, dt = \sec^2 \theta \, d\theta$ 。 1M

$$\begin{aligned} \int \frac{dx}{1 + 2 \sin^2 x} &= \frac{1}{\sqrt{3}} \int \frac{\sec^2 \theta}{\tan^2 \theta + 1} d\theta \\ &= \frac{\sqrt{3}}{3} \int d\theta \\ &= \frac{\sqrt{3}}{3} \theta + \text{常數} \\ &= \frac{\sqrt{3}}{3} \tan^{-1}(\sqrt{3} \tan x) + \text{常數} \end{aligned} \quad 1A$$

57. 設 $x = 2 \tan \theta$ 。則 $dx = 2 \sec^2 \theta \, d\theta$ 。 1M

$$\begin{aligned} \int \frac{x^4}{x^2 + 4} dx &= \frac{1}{2} \int \frac{(2 \tan \theta)^4 \cdot 2 \sec^2 \theta}{4 \tan^2 \theta + 4} d\theta \\ &= 8 \int \tan^4 \theta \, d\theta \\ &= 8 \int (\sec^2 \theta - 1)^2 d\theta \\ &= 8 \int \sec^4 \theta \, d\theta - 16 \int \sec^2 \theta \, d\theta + 8 \int d\theta \end{aligned} \quad 1M$$

設 $u = \tan \theta$ 。則 $du = \sec^2 \theta \, d\theta$ 。 1M

$$\begin{aligned} \int \frac{x^4}{x^2 + 4} dx &= 8 \int (u^2 + 1) du - 16 \tan \theta + 8\theta \\ &= \frac{8u^3}{3} + 8u - 16 \tan \theta + 8\theta + \text{常數} \\ &= \frac{8 \tan^3 \theta}{3} - 8 \tan \theta + 8\theta + \text{常數} \\ &= \frac{x^3}{3} - 4x + 8 \tan^{-1} \frac{x}{2} + \text{常數} \end{aligned} \quad 1A$$

$$\begin{aligned} 58. \quad (a) \quad x^2 - 4x \cos \frac{\pi}{12} + 4 &= x^2 - 4x \cos \frac{\pi}{12} + 4 \cos^2 \frac{\pi}{12} + 4 \sin^2 \frac{\pi}{12} \\ &= \left(x - 2 \cos \frac{\pi}{12} \right)^2 + 4 \sin^2 \frac{\pi}{12} \end{aligned} \quad 1A$$

(b) 設 $x - 2 \cos \frac{\pi}{12} = 2 \sin \frac{\pi}{12} \tan \theta$ 。則 $dx = 2 \sin \frac{\pi}{12} \sec^2 \theta d\theta$ 。 1M

$$\begin{aligned} \int \frac{\sin \frac{\pi}{12}}{x^2 - 4 \cos \frac{\pi}{12} x + 4} dx &= \int \frac{\sin \frac{\pi}{12}}{(x - 2 \cos \frac{\pi}{12})^2 + 4 \sin^2 \frac{\pi}{12}} dx \\ &= \int \frac{\sin \frac{\pi}{12}}{4 \sin^2 \frac{\pi}{12} \tan^2 \theta + 4 \sin^2 \frac{\pi}{12}} \cdot 2 \sin \frac{\pi}{12} \sec^2 \theta d\theta \\ &= \frac{1}{2} \int d\theta \end{aligned} \quad 1A$$

$$\begin{aligned} &= \frac{1}{2} \theta + \text{常數} \\ &= \frac{1}{2} \tan^{-1} \left(\frac{x}{2} \csc \frac{\pi}{12} - \cot \frac{\pi}{12} \right) + \text{常數} \\ &= \frac{1}{2} \tan^{-1} \left(\frac{\sqrt{6} + \sqrt{2}}{2} x - 2 - \sqrt{3} \right) + \text{常數} \end{aligned} \quad 1A$$

59. 設 $e^x = \tan \theta$ 。則 $e^x dx = \sec^2 \theta d\theta$ 。 1M

$$\begin{aligned} \int \frac{e^x dx}{(1 + e^{2x})^2} &= \int \frac{\sec^2 \theta}{(1 + \tan^2 \theta)^2} d\theta \\ &= \int \cos^2 \theta d\theta \end{aligned} \quad 1A$$

$$= \frac{1}{2} \int (1 + \cos 2\theta) d\theta \quad 1M$$

$$\begin{aligned} &= \frac{\theta}{2} + \frac{\sin 2\theta}{4} + \text{常數} \\ &= \frac{\theta}{2} + \frac{1}{2} \sin \theta \cos \theta + \text{常數} \\ &= \frac{1}{2} \tan^{-1} e^x + \frac{1}{2} \left(\frac{e^x}{\sqrt{1 + e^{2x}}} \right) \left(\frac{1}{\sqrt{1 + e^{2x}}} \right) + \text{常數} \\ &= \frac{1}{2} \tan^{-1} e^x + \frac{e^x}{2(1 + e^{2x})} + \text{常數} \end{aligned} \quad 1A$$

60. 設 $x = 7 \sin \theta$ 。則 $dx = 7 \cos \theta d\theta$ 。 1M

$$\begin{aligned} \int \sqrt{49 - x^2} dx &= \int \sqrt{49 - 49 \sin^2 \theta} 7 \cos \theta d\theta \\ &= 49 \int \cos^2 \theta d\theta \end{aligned} \quad 1A$$

$$= \frac{49}{2} \int (1 + \cos 2\theta) d\theta \quad 1M$$

$$\begin{aligned} &= \frac{49\theta}{2} + \frac{49}{4} \sin 2\theta + \text{常數} \\ &= \frac{49\theta}{2} + \frac{49}{2} \sin \theta \cos \theta + \text{常數} \\ &= \frac{49}{2} \sin^{-1} \frac{x}{7} + \frac{x}{2} \sqrt{49 - x^2} + \text{常數} \end{aligned} \quad 1A$$

$$61. \quad (a) \quad \int x \cos 2x \, dx = \frac{x \sin 2x}{2} - \frac{1}{2} \int \sin 2x \, dx \quad 1M$$

$$= \frac{x \sin 2x}{2} + \frac{\cos 2x}{4} + \text{常數} \quad 1A$$

$$(b) \quad \int x \sin^2 x \, dx = \frac{1}{2} \int x(1 - \cos 2x) \, dx \quad 1M$$

$$= \frac{1}{2} \int x \, dx - \frac{1}{2} \int x \cos 2x \, dx$$

$$= \frac{x^2}{4} - \frac{x \sin 2x}{4} - \frac{\cos 2x}{8} + \text{常數} \quad 1A$$

$$62. \quad (a) \quad \int e^x \sin x \, dx = e^x \sin x - \int e^x \cos x \, dx \quad 1M$$

$$= e^x \sin x - e^x \cos x - \int e^x \sin x \, dx \quad 1M$$

$$2 \int e^x \sin x \, dx = e^x \sin x - e^x \cos x + \text{常數}$$

$$\int e^x \sin x \, dx = \frac{e^x \sin x}{2} - \frac{e^x \cos x}{2} + \text{常數} \quad 1A$$

$$(b) \quad \text{設 } u = \frac{\pi}{2} - x \circ \text{則 } du = -dx \circ \quad 1M$$

$$\int \frac{\cos x}{e^x} \, dx = - \int \frac{\cos(\frac{\pi}{2} - u)}{e^{\frac{\pi}{2} - u}} \, du$$

$$= - \frac{1}{e^{\frac{\pi}{2}}} \int e^u \sin u \, du$$

$$= - \frac{1}{e^{\frac{\pi}{2}}} \left(\frac{e^u \sin u}{2} - \frac{e^u \cos u}{2} \right) + \text{常數} \quad 1M$$

$$= - \frac{1}{2e^{\frac{\pi}{2}}} \left[e^{\frac{\pi}{2} - x} \sin \left(\frac{\pi}{2} - x \right) - e^{\frac{\pi}{2} - x} \cos \left(\frac{\pi}{2} - x \right) \right] + \text{常數} \quad 1A$$

$$= - \frac{1}{2} (e^{-x} \cos x - e^{-x} \sin x) + \text{常數}$$

$$= \frac{\sin x - \cos x}{2e^x} + \text{常數} \quad 1A$$

$$63. \quad \text{設 } u = \sqrt{x} \circ \text{則 } du = \frac{1}{2\sqrt{x}} \, dx \circ \quad 1M$$

$$\int e^{\sqrt{x}} \, dx = 2 \int \sqrt{x} \cdot \frac{1}{2\sqrt{x}} e^{\sqrt{x}} \, dx$$

$$= 2 \int u e^u \, du$$

$$= 2ue^u - 2 \int e^u \, du \quad 1M$$

$$= 2ue^u - 2e^u + \text{常數}$$

$$= 2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + \text{常數} \quad 1A$$

$$64. \int (x^2 + e^x)e^x dx = x^2 e^x - 2 \int x e^x dx + \int e^{2x} dx \quad 1M$$

$$= x^2 e^x - 2x e^x + 2 \int e^x dx + \int e^{2x} dx \quad 1M$$

$$= x^2 e^x - 2x e^x + 2e^x + \frac{1}{2} e^{2x} + \text{常數} \quad 1A$$

$$65. \int (2x^2 + 3)e^x dx = (2x^2 + 3)e^x - 4 \int x e^x dx \quad 1M$$

$$= (2x^2 + 3)e^x - 4x e^x + 4 \int e^x dx \quad 1M$$

$$= (2x^2 - 4x + 7)e^x + \text{常數} \quad 1A$$

$$66. \int \ln(1-x) dx = x \ln(1-x) + \int \frac{x}{1-x} dx \quad 1M$$

$$= x \ln(1-x) + \int \frac{1-(1-x)}{1-x} dx$$

$$= x \ln(1-x) + \int \frac{dx}{1-x} - \int dx \quad 1M$$

$$= x \ln(1-x) - \ln|1-x| - x + \text{常數} \quad 1A$$

$$67. (a) \int e^x \sec^2 x dx = e^x \tan x - \int \frac{d}{dx}(e^x) \tan x dx$$

$$= e^x \tan x - \int e^x \tan x dx \quad 1$$

$$(b) \int e^x \sec^2 x dx = e^x \tan x - \int e^x \tan x dx$$

$$\int e^x (\tan^2 x + 1) dx = e^x \tan x - \int e^x \tan x dx \quad 1M$$

$$\int e^x (\tan^2 x + 1) dx + \int e^x \tan x dx = e^x \tan x + \text{常數}$$

$$\int e^x (\tan^2 x + \tan x + 1) dx = e^x \tan x + \text{常數} \quad 1A$$

$$68. \int e^{mx} \sin nx dx = \frac{1}{m} e^{mx} \sin nx - \frac{n}{m} \int e^{mx} \cos nx dx \quad 1M$$

$$= \frac{1}{m} e^{mx} \sin nx - \frac{n}{m^2} e^{mx} \cos nx - \frac{n^2}{m^2} \int e^{mx} \sin nx dx \quad 1M$$

$$\left(1 + \frac{n^2}{m^2}\right) \int e^{mx} \sin nx dx = \frac{1}{m} e^{mx} \sin nx - \frac{n}{m^2} e^{mx} \cos nx + \text{常數}$$

$$\int e^{mx} \sin nx dx = \frac{e^{mx}}{m^2 + n^2} (m \sin nx - n \cos nx) + \text{常數} \quad 1A$$

69. 設 $u = \sqrt{2x}$ 。則 $du = \frac{\sqrt{2}}{2}x^{-\frac{1}{2}}dx = \frac{1}{u}dx$ 。 1M

$$\int \sin \sqrt{2x} dx = \int u \sin u du$$

$$= -u \cos u + \int \cos u du$$
 1M

$$= -u \cos u + \sin u + \text{常數}$$

$$= -\sqrt{2x} \cos \sqrt{2x} + \sin \sqrt{2x} + \text{常數}$$
 1A

70. $\int e^{-3x} \cos 4x dx = -\frac{1}{3}e^{-3x} \cos 4x - \frac{4}{3} \int e^{-3x} \sin 4x dx$ 1M

$$= -\frac{1}{3}e^{-3x} \cos 4x + \frac{4}{9}e^{-3x} \sin 4x - \frac{16}{9} \int e^{-3x} \cos 4x dx$$
 1M

$$\frac{25}{9} \int e^{-3x} \cos 4x dx = -\frac{1}{3}e^{-3x} \cos 4x + \frac{4}{9}e^{-3x} \sin 4x + \text{常數}$$

$$\int e^{-3x} \cos 4x dx = -\frac{3}{25}e^{-3x} \cos 4x + \frac{4}{25}e^{-3x} \sin 4x + \text{常數}$$
 1A

71. $\int x \cdot 7^x dx = \int x e^{x \ln 7} dx$ 1M

$$= \frac{1}{\ln 7} x e^{x \ln 7} - \frac{1}{\ln 7} \int e^{x \ln 7} dx$$
 1M

$$= \frac{1}{\ln 7} x e^{x \ln 7} - \frac{1}{(\ln 7)^2} e^{x \ln 7} + \text{常數}$$

$$= \frac{1}{\ln 7} x \cdot 7^x - \frac{1}{(\ln 7)^2} 7^x + \text{常數}$$
 1A

72. $\int (\ln x)^2 dx = x(\ln x)^2 - 2 \int \ln x dx$ 1M

$$= x(\ln x)^2 - 2x \ln x + 2 \int dx$$
 1M

$$= x(\ln x)^2 - 2x \ln x + 2x + \text{常數}$$
 1A

73. (a) 設 $u = \frac{\sqrt{x}}{2}$ 。則 $du = \frac{1}{4\sqrt{x}}dx = \frac{1}{8u}dx$ 。 1M

$$\int e^{\frac{\sqrt{x}}{2}} dx = 8 \int u e^u du$$

$$= 8u e^u - 8 \int e^u du$$
 1M

$$= 8u e^u - 8e^u + \text{常數}$$

$$= 4\sqrt{x} e^{\frac{\sqrt{x}}{2}} - 8e^{\frac{\sqrt{x}}{2}} + \text{常數}$$
 1A

(b) 設 $w = \frac{\sqrt{x}}{2}$ 。則 $dw = \frac{1}{8w} dx$ 。

$$\begin{aligned}\int \sqrt{x} e^{\frac{\sqrt{x}}{2}} dx &= 16 \int w^2 e^w dw \\ &= 16w^2 e^w - 32 \int w e^w dw\end{aligned}\quad 1M$$

$$= 16w^2 e^w - 32we^w + 32 \int e^w dw \quad 1M$$

$$= 16w^2 e^w - 32we^w + 32e^w + \text{常數}$$

$$= 4xe^{\frac{\sqrt{x}}{2}} - 16\sqrt{x}e^{\frac{\sqrt{x}}{2}} + 32e^{\frac{\sqrt{x}}{2}} + \text{常數} \quad 1A$$

$$74. \int x \cos 2x \cos 4x dx = \frac{1}{2} \int x(\cos 6x + \cos 2x) dx \quad 1M$$

$$= \frac{1}{2} \int x \cos 6x dx + \frac{1}{2} \int x \cos 2x dx$$

$$= \frac{1}{12} x \sin 6x - \frac{1}{12} \int \sin 6x dx + \frac{1}{4} x \sin 2x - \frac{1}{4} \int \sin 2x dx \quad 1M$$

$$= \frac{1}{12} x \sin 6x + \frac{1}{72} \cos 6x + \frac{1}{4} x \sin 2x + \frac{1}{8} \cos 2x + \text{常數} \quad 1A$$

$$75. \int e^x \sin(x+1) dx = e^x \sin(x+1) - \int e^x \cos(x+1) dx \quad 1M$$

$$= e^x [\sin(x+1) - \cos(x+1)] - \int e^x \sin(x+1) dx \quad 1M$$

$$2 \int e^x \sin(x+1) dx = e^x [\sin(x+1) - \cos(x+1)] + \text{常數}$$

$$\int e^x \sin(x+1) dx = \frac{1}{2} e^x [\sin(x+1) - \cos(x+1)] + \text{常數} \quad 1A$$

$$76. \int x^5 (\ln x)^2 dx = \frac{1}{6} x^6 (\ln x)^2 - \frac{1}{3} \int x^5 \ln x dx \quad 1M$$

$$= \frac{1}{6} x^6 (\ln x)^2 - \frac{1}{18} x^6 \ln x + \frac{1}{18} \int x^5 dx \quad 1M$$

$$= \frac{1}{6} x^6 (\ln x)^2 - \frac{1}{18} x^6 \ln x + \frac{1}{108} x^6 + \text{常數} \quad 1A$$

$$77. \int e^{2x} \sin 5x dx = \frac{1}{2} e^{2x} \sin 5x - \frac{5}{2} \int e^{2x} \cos 5x dx \quad 1M$$

$$= \frac{1}{2} e^{2x} \sin 5x - \frac{5}{4} e^{2x} \cos 5x - \frac{25}{4} \int e^{2x} \sin 5x dx \quad 1M$$

$$\frac{29}{4} \int e^{2x} \sin 5x dx = \frac{1}{2} e^{2x} \sin 5x - \frac{5}{4} e^{2x} \cos 5x + \text{常數}$$

$$\int e^{2x} \sin 5x dx = \frac{e^{2x}}{29} (2 \sin 5x - 5 \cos 5x) + \text{常數} \quad 1A$$

$$78. \int \cos x \ln(\sin x) dx = \sin x \ln(\sin x) - \int \cos x dx \quad 1M$$

$$= \sin x \ln(\sin x) - \sin x + \text{常數} \quad 1A$$

$$79. \int \ln(\sqrt{x^2+1}-x) dx$$

$$= x \ln(\sqrt{x^2+1}-x) - \int \frac{x}{\sqrt{x^2+1}-x} \cdot \left(\frac{2x}{2\sqrt{x^2+1}} - 1 \right) dx \quad 1M+1M$$

$$= x \ln(\sqrt{x^2+1}-x) - \int \frac{x}{\sqrt{x^2+1}-x} \cdot \frac{x-\sqrt{x^2+1}}{\sqrt{x^2+1}} dx$$

$$= x \ln(\sqrt{x^2+1}-x) + \int \frac{x}{\sqrt{x^2+1}} dx$$

$$\text{設 } u = x^2 + 1 \circ \text{ 則 } du = 2x dx \circ \quad 1M$$

$$\int \ln(\sqrt{x^2+1}-x) dx$$

$$= x \ln(\sqrt{x^2+1}-x) + \frac{1}{2} \int \frac{du}{\sqrt{u}}$$

$$= x \ln(\sqrt{x^2+1}-x) + \sqrt{u} + \text{常數}$$

$$= x \ln(\sqrt{x^2+1}-x) + \sqrt{x^2+1} + \text{常數} \quad 1A$$

$$80. \int x^2 \cos(\ln x) dx = x^3 \sin(\ln x) - 3 \int x^2 \sin(\ln x) dx \quad 1M$$

$$= x^3 \sin(\ln x) + 3 \int x^3 \cdot \left[-\frac{\sin(\ln x)}{x} \right] dx$$

$$= x^3 \sin(\ln x) + 3x^3 \cos(\ln x) - 9 \int x^2 \cos(\ln x) dx \quad 1M$$

$$10 \int x^2 \cos(\ln x) dx = x^3 \sin(\ln x) + 3x^3 \cos(\ln x) + \text{常數}$$

$$\int x^2 \cos(\ln x) dx = \frac{1}{10} x^3 \sin(\ln x) + \frac{3}{10} x^3 \cos(\ln x) + \text{常數} \quad 1A$$

$$81. (a) \frac{dy}{dx} = kx^2 - 18x + 23$$

$$\frac{d^2y}{dx^2} = 2kx - 18$$

(3, 33) 為該曲線的拐點。

$$2k(3) - 18 = 0 \quad 1M$$

$$k = 3 \quad 1A$$

$$(b) y = \int (3x^2 - 18x + 23) dx \quad 1M$$

$$= x^3 - 9x^2 + 23x + C$$

該曲線通過 $(3, 33)$ 。

$$3^3 - 9(3)^2 + 23(3) + C = 33 \quad 1M$$

$$C + 18$$

該曲線的方程為 $y = x^3 - 9x^2 + 23x + 18$ 。 1A

82. (a) 設 $x - 1 = 6 \sin \theta$ 。則 $dx = 6 \cos \theta d\theta$ 。 1M

$$\begin{aligned} \int \frac{1}{\sqrt{35 + 2x - x^2}} dx &= \int \frac{1}{\sqrt{36 - (x - 1)^2}} dx \\ &= \int \frac{6 \cos \theta}{\sqrt{36 - 36 \sin^2 \theta}} d\theta \\ &= \int d\theta \\ &= \theta + \text{常數} \\ &= \sin^{-1} \frac{x - 1}{6} + \text{常數} \end{aligned} \quad \begin{array}{l} 1M \\ 1A \end{array}$$

(b) 設 $u = 35 + 2x - x^2$ 。則 $du = (2 - 2x) dx$ 。 1M

$$\begin{aligned} y &= \int \frac{-2x + 3}{\sqrt{35 + 2x - x^2}} dx \\ &= \int \frac{-2x + 2}{\sqrt{35 + 2x - x^2}} dx + \int \frac{1}{\sqrt{35 + 2x - x^2}} dx \\ &= \int u^{-\frac{1}{2}} du + \sin^{-1} \frac{x - 1}{6} + C_1 \\ &= 2u^{\frac{1}{2}} + \sin^{-1} \frac{x - 1}{6} + C \\ &= 2\sqrt{35 + 2x - x^2} + \sin^{-1} \frac{x - 1}{6} + C \end{aligned} \quad 1M$$

G 通過點 $(4, 6\sqrt{3})$ 。

$$6\sqrt{3} = 2\sqrt{35 + 8 - 16} + \sin^{-1} \frac{4 - 1}{6} + C \quad 1M$$

$$C = -\frac{\pi}{6}$$

代 $x = 1$ 。

$$\begin{aligned} y &= 2\sqrt{35 + 2 - 1} + \sin^{-1} \frac{1 - 1}{6} - \frac{\pi}{6} \\ &= 12 - \frac{\pi}{6} \end{aligned}$$

因此， G 通過 $(1, 12 - \frac{\pi}{6})$ 。 1A

83. (a) $f(x) = \int (2x - 4) dx$
 $= x^2 - 4x + C$ 1M

Γ 通過 $(5, 15)$ 。

$$15 = 5^2 - 4(5) + C \quad 1M$$

$$C = 10$$

所求方程為 $y = x^2 - 4x + 10$ 。 1A

(b) (i) 設 (a, b) 為 P 的坐標。

$$\frac{b+1}{a+1} = 2a - 4 \quad 1M$$

$$\frac{(a^2 - 4a + 10) + 1}{a + 1} = 2a - 4$$

$$a^2 - 4a + 11 = 2a^2 - 2a - 4$$

$$0 = a^2 + 2a - 15$$

$$a = 3 \quad \text{或} \quad -5$$

留意 Γ 的斜率為正數，可得 $a = 3$ 。

P 的坐標為 $(3, 7)$ 。 1A

(ii) L 的斜率 $= f'(3) = 2$

所求直線的斜率 $= -\frac{1}{2}$ 1M

所求方程為

$$y - 7 = -\frac{1}{2}(x - 3) \quad 1M$$

$$x + 2y - 17 = 0 \quad 1A$$

84. (a) 設 $u = 1 + x^3$ 。則 $du = 3x^2 dx$ 。 1M

$$\begin{aligned} \int \frac{x^5}{\sqrt{1+x^3}} dx &= \frac{1}{3} \int \frac{u-1}{\sqrt{u}} du \\ &= \frac{1}{3} \int (u^{\frac{1}{2}} - u^{-\frac{1}{2}}) du \\ &= \frac{2}{9} u^{\frac{3}{2}} - \frac{2}{3} u^{\frac{1}{2}} + \text{常數} \\ &= \frac{2}{9} (1+x^3)^{\frac{3}{2}} - \frac{2}{3} (1+x^3)^{\frac{1}{2}} + \text{常數} \end{aligned} \quad 1A$$

$$(b) \quad y = \int \frac{9x^5}{2\sqrt{1+x^3}} dx \quad 1M$$

$$= \frac{9}{2} \left[\frac{2}{9} (1+x^3)^{\frac{3}{2}} - \frac{2}{3} (1+x^3)^{\frac{1}{2}} \right] + C \quad 1M$$

$$= (1+x^3)^{\frac{3}{2}} - 3\sqrt{1+x^3} + C$$

$$3 = 1 - 3 + C \quad 1M$$

$$C = 5$$

所求方程為 $y = (1+x^3)^{\frac{3}{2}} - 3\sqrt{1+x^3} + 5$ 。 1A

85. (a) (i) 設 r cm 為倒置圓錐的底半徑。

$$\frac{r}{12} = \frac{24-h}{24}$$

$$r = \frac{24-h}{2}$$

考慮該容器的容量。

$$V = \frac{1}{3}\pi \left(\frac{24-h}{2}\right)^2 h$$

$$= \frac{\pi}{12}(h^3 - 48h^2 + 576h)$$

(ii) $\frac{dV}{dh} = \frac{\pi}{12}(3h^2 - 96h + 576)$

$$= \frac{\pi}{4}(h-8)(h-24)$$

當 $\frac{dV}{dh} = 0$ 時，可得 $h = 8$ 或 24 （捨去）。

h	$0 < h < 8$	$8 < h < 24$
$\frac{dV}{dh}$	+	-

當 V 為極大時， $h = 8$ 。

(b) (i) $\frac{dW}{dt} = 60e^{-\frac{t}{30}} - 55e^{-\frac{t}{25}}$

$$W = \int \left(60e^{-\frac{t}{30}} - 55e^{-\frac{t}{25}}\right) dt$$

$$= -1800e^{-\frac{t}{30}} + 1375e^{-\frac{t}{25}} + C$$

當 $t = 0$ 時， $W = 0$ 。

$$0 = -1800e^0 + 1375e^0 + C$$

$$C = 425$$

因此， $W = -1800e^{-\frac{t}{30}} + 1375e^{-\frac{t}{25}} + 425$ 。

- (ii) 該容器的容量

$$= \frac{\pi}{12}[8^3 - 48(8)^2 + 576(8)]$$

$$= \frac{512\pi}{3} \text{ cm}^3$$

當 $\frac{dW}{dt} = 0$ 。

$$60e^{-\frac{t}{30}} - 55e^{-\frac{t}{25}} = 0$$

$$60e^{-\frac{t}{30}} = 55e^{-\frac{t}{25}}$$

$$e^{\frac{t}{150}} = \frac{11}{12}$$

$$\frac{t}{150} = \ln \frac{11}{12}$$

$$t = 150 \ln \frac{11}{12} \text{ (捨去)}$$

當 $t = 0$ 時， $W = 0$ 。

$$\lim_{t \rightarrow \infty} W = \lim_{t \rightarrow \infty} (-1800e^{-\frac{t}{30}} + 1375e^{-\frac{t}{25}} + 425) \quad 1M$$

$$= -0 + 0 + 425$$

$$= 425$$

$$< \frac{512\pi}{3}$$

水不會溢出。

同意該宣稱。

1A

86. (a) 留意 $x^2 + 6x + 34 = (x + 3)^2 + 25$ 。

設 $x + 3 = 5 \tan \theta$ 。則 $dx = 5 \sec^2 \theta d\theta$ 。

1M

$$\int \frac{1}{x^2 + 6x + 34} dx = \int \frac{5 \sec^2 \theta}{25 \tan^2 \theta + 25} d\theta \quad 1M$$

$$= \frac{1}{5} \int d\theta$$

$$= \frac{\theta}{5} + \text{常數}$$

$$= \frac{1}{5} \tan^{-1} \frac{x+3}{5} + \text{常數}$$

1

(b) 設 $u = x^2 + 6x + 34$ 。則 $du = (2x + 6) dx$ 。

1M

$$y = \int \frac{2x+5}{x^2+6x+34} dx$$

$$= \int \frac{2x+6}{x^2+6x+34} dx - \int \frac{1}{x^2+6x+34} dx$$

$$= \int \frac{du}{u} - \frac{1}{5} \tan^{-1} \frac{x+3}{5}$$

$$= \ln |u| - \frac{1}{5} \tan^{-1} \frac{x+3}{5} + C$$

$$= \ln(x^2 + 6x + 34) - \frac{1}{5} \tan^{-1} \frac{x+3}{5} + C$$

1M

G 通過點 $(-8, \ln 2)$ 。

$$\ln 2 = \ln(64 - 48 + 34) - \frac{1}{5} \tan^{-1} \frac{-8+3}{5} + C$$

1M

$$\ln 2 = \ln 50 - \frac{1}{5} \left(-\frac{\pi}{4} \right) + C$$

$$C = -2 \ln 5 - \frac{\pi}{20}$$

代 $x = -3$ 。

$$y = \ln(9 - 18 + 34) - 0 - 2 \ln 5 - \frac{\pi}{20}$$

$$= -\frac{\pi}{20}$$

G 通過點 $\left(-3, -\frac{\pi}{20}\right)$ 。

1A

87. (a) L 的斜率 $= \cos \frac{\pi}{3} \sin^2 \frac{\pi}{2} = \frac{3}{8}$

1M

L 的方程為

$$y - \frac{\sqrt{3}}{4} = \frac{3}{8} \left(x - \frac{\pi}{3} \right)$$

$$3x - 8y + 2\sqrt{3} - \pi = 0 \quad 1A$$

(b) 設 $u = \sin x$ 。則 $du = \cos x \, dx$ 。 1M

$$\begin{aligned} y &= \int \cos x \sin^2 x \, dx \\ &= \int u^2 \, du \\ &= \frac{u^3}{3} + C \\ &= \frac{1}{3} \sin^3 x + C \end{aligned}$$

Γ 通過 P 。

$$\begin{aligned} \frac{\sqrt{3}}{4} &= \frac{1}{3} \sin^3 \frac{\pi}{3} + C \\ C &= \frac{\sqrt{3}}{8} \end{aligned}$$

Γ 的方程為 $y = \frac{1}{3} \sin^3 x + \frac{\sqrt{3}}{8}$ 。 1A

(c) $\frac{d^2y}{dx^2} = 2 \sin x \cos^2 x - \sin^3 x$ 1M

$$= 2 \sin x (2 \cos^2 x - \sin^2 x)$$

當 $\frac{d^2y}{dx^2} = 0$ 時， $\sin x = 0$ 或 $\tan x = \pm\sqrt{2}$ 。

因此，可得 $x = \tan^{-1} \sqrt{2}$ 或 $x = \pi - \tan^{-1} \sqrt{2}$ 。

x	$0 < x < \tan^{-1} \sqrt{2}$	$\tan^{-1} \sqrt{2} < x < \pi - \tan^{-1} \sqrt{2}$	$\pi - \tan^{-1} \sqrt{2} < x < \pi$
$\frac{d^2y}{dx^2}$	+	-	+

1M

所求數目為 2。

1A

88. (a) $h'(x) = \frac{2x^2 - 8x + 18}{x}$

$$= \frac{2}{x} [(x^2 - 4x + 4) + 9]$$

$$= \frac{2}{x} [(x - 2)^2 + 5]$$

$$> 0 \quad (\text{對所有 } x > 0)$$

1M

因此， $h(x)$ 為遞增函數。

1A

$$\begin{aligned} \text{(b) (i) } y &= \int \frac{2x^2 - 8x + 18}{x} dx \\ &= \int \left(2x - 8 + \frac{18}{x} \right) dx \end{aligned} \quad 1M$$

$$= x^2 - 8x + 18 \ln x + C$$

當 $x = 1$ 時, $y = 5$ 。

$$5 = 1^2 - 8(1) + 18 \ln 1 + C \quad 1M$$

$$C = 12$$

H 的方程為 $y = x^2 - 8x + 18 \ln x + 12$ 。 1A

$$\text{(ii) } h''(x) = 2 - \frac{18}{x^2} \quad 1A$$

$$= \frac{2(x+3)(x-3)}{x^2}$$

當 $h''(x) = 0$ 時, $x = 3$ 或 -3 (捨去)。

x	$0 < x < 3$	$x > 3$
$h''(x)$	$-$	$+$

1A

因此, $(3, 18 \ln 3)$ 為 H 的拐點。 1A

89. (a) 設 $x = 4 \sin \theta$ 。則 $dx = 4 \cos \theta d\theta$ 。 1M

$$\begin{aligned} y &= \int \frac{2 - kx}{\sqrt{16 - x^2}} dx \\ &= \int \frac{2 - 4k \sin \theta}{\sqrt{16 - 16 \sin^2 \theta}} (4 \cos \theta) d\theta \quad 1M \\ &= \int (2 - 4k \sin \theta) d\theta \end{aligned}$$

$$= 2\theta - 4k \cos \theta + C$$

$$= 2 \sin^{-1} \frac{x}{4} + k \sqrt{16 - x^2} + C$$

Γ 通過原點。

$$0 = 2 \sin^{-1} \frac{0}{4} + k \sqrt{16 - 0^2} + C$$

$$C = -4k$$

因此, $y = 2 \sin^{-1} \frac{x}{4} + k \sqrt{16 - x^2} - 4k$ 。 1A

(b) (i) 當 $\frac{dy}{dx} = 0$, $x = \frac{2}{k}$ 。 1M

由於 $-4 < \frac{2}{k} < 4$, 可得 $k < -\frac{1}{2}$ 或 $k > \frac{1}{2}$ 。 1A

$$\begin{aligned} \text{(ii) } \frac{d^2y}{dx^2} &= \frac{-k \sqrt{16 - x^2} - (2 - kx) \left(\frac{1}{2\sqrt{16 - x^2}} \right) (-2x)}{16 - x^2} \quad 1M \\ &= \frac{2x - 16k}{(16 - x^2)^{\frac{3}{2}}} \end{aligned}$$

利用 (b)(i), 對 $-4 < x < 4$, 可得 $2x - 16k < 2x - 8$ 或 $2x - 16k > 2x + 8$ 。

在區間 $(-4, 4)$ 上, $\frac{d^2y}{dx^2}$ 可為正數或負數。

不同意該宣稱。

1A

90. (a) $y = \int x \ln x^4 dx$

$$= 4 \int x \ln x dx$$

$$= 2x^2 \ln x - 2 \int x dx$$

1M

$$= 2x^2 \ln x - x^2 + C$$

1A

Γ 通過點 $P(e^2, 3e^4 + 7)$ 。

$$3e^4 + 7 = 2(e^2)^2 \ln e^2 - (e^2)^2 + C$$

1M

$$C = 7$$

所求方程為 $y = 2x^2 \ln x - x^2 + 7$ 。

1A

(b) $f'(x) = 4x \ln x$

$$f''(x) = 4(\ln x + 1)$$

1A

當 $f''(x) = 0$ 時， $x = \frac{1}{e}$ 。

x	$0 < x < \frac{1}{e}$	$x > \frac{1}{e}$
$f''(x)$	-	+

1M

Γ 的拐點為 $\left(\frac{1}{e}, -3e^{-2} + 7\right)$ 。

1A

91. (a) 切線斜率 $= f'(5) = 5e$

1A

所求方程為

$$y - 13 = 5e(x - 5)$$

$$y = 5ex - 25e + 13$$

1A

(b) $f(x) = \int xe^{\frac{x}{5}} dx$

$$= 5xe^{\frac{x}{5}} - 5 \int e^{\frac{x}{5}} dx$$

1M

$$= 5xe^{\frac{x}{5}} - 25e^{\frac{x}{5}} + C$$

Γ 通過 $(5, 13)$ 。

$$13 = 5(5)e^1 - 25e^1 + C$$

1M

$$C = 13$$

所求方程為 $y = 5xe^{\frac{x}{5}} - 25e^{\frac{x}{5}} + 13$ 。

1A

(c) $f''(x) = \frac{1}{5}xe^{\frac{x}{5}} + e^{\frac{x}{5}}$

1M

$$= \frac{1}{5}e^{\frac{x}{5}}(x + 5)$$

當 $f''(x) = 0$ 時， $x = -5$ 。

x	$x < -5$	$x > -5$
$f''(x)$	-	+

1M

Γ 的拐點為 $\left(-5, -\frac{50}{e} + 13\right)$ 。

1A

92. (a) $\int \frac{\ln x}{x^2} dx = -\frac{\ln x}{x} + \int \frac{1}{x^2} dx$
 $= -\frac{\ln x}{x} - \frac{1}{x} + C_1$

1M+1A

1A

(b) $y = \int \frac{e - \ln x}{x^2} dx$

1M

$$= \int \frac{e}{x^2} dx - \int \frac{\ln x}{x^2} dx$$

$$= -\frac{e}{x} + \frac{\ln x}{x} + \frac{1}{x} + C_2$$

1M

C 通過 $(1, 1)$ 。

$$1 = -e + 0 + 1 + C_2$$

1M

$$C_2 = e$$

所求方程為 $y = -\frac{e}{x} + \frac{\ln x}{x} + \frac{1}{x} + e$ 。

1A

93. (a) 設 $u = \ln x$ 。則 $du = \frac{1}{x} dx$ 。

1M

$$\int \frac{(\ln x)^2}{x} dx = \int u^2 du$$

1M

$$= \frac{u^3}{3} + C$$

$$= \frac{(\ln x)^3}{3} + C$$

1A

(b) $y = \int \frac{(\ln x^3)^2}{x} dx$

$$= \int \frac{(3 \ln x)^2}{x} dx$$

1M

$$= 9 \int \frac{(\ln x)^2}{x} dx$$

$$= 3(\ln x)^3 + C$$

Γ 通過點 $(1, 0)$ 。

$$0 = 3(\ln 1)^3 + C$$

1M

$$C = 0$$

Γ 的方程為 $y = 3(\ln x)^3$ 。

1A

94. (a) $y = \int (-e^{-x} + 1) dx$
 $= e^{-x} + x + C$ 1A
 該曲線通過點 $(-1, e - 2)$ 。
 $e - 2 = e - 1 + C$ 1M
 $C = -1$
 所求方程為 $y = e^{-x} + x - 1$ 。 1A
- (b) 當 $x = 0$ 時, $\frac{dy}{dx} = -e^0 + 1 = 0$ 。 1M
 所求方程為
 $y - 0 = 0(x - 0)$
 $y = 0$ 1A
95. (a) $y = \int \frac{x+1}{x+3} dx$ 1M
 $= \int \frac{x+3-2}{x+3} dx$
 $= \int \left(1 - \frac{2}{x+3}\right) dx$
 $= x - 2 \ln|x+3| + C$ 1A
 Γ 通過 $(e-3, 2e-5)$ 。
 $2e - 5 = (e - 3) - 2 \ln e + C$
 $C = e$
 所求方程為 $x - 2 \ln|x+3| + e$ 。 1A
- (b) $\frac{d^2y}{dx^2} = \frac{d}{dx} \left(1 - \frac{2}{x+3}\right)$
 $= \frac{2}{(x+3)^2}$ 1A
 > 0
 Γ 沒有拐點。 1A
96. (a) $x = \int (-7t + a) dt$
 $= -\frac{7}{2}t^2 + at + C$ 1M
 當 $t = 0$ 時, $x = 6280$ 。
 $6280 = 0 + 0 + C$ 1M
 $C = 6280$

當 $t = 4$ 時, $x = \frac{6280}{2} = 3140$ 。

$$3140 = -\frac{7}{2}(4)^2 + a(4) + 6280$$

$$a = -771$$

由此, 可得 $x = -\frac{7}{2}t^2 - 771t + 6280$ 。
於 2:00 p.m., $t = 6$ 。

$$x = -\frac{7}{2}(6)^2 - 771(6) + 6280$$

$$= 1528$$

$$> 1500$$

不同意該宣稱。

1A

(b) 設 $y^\circ\text{C}$ 為當 $t = 6$ 時, 該容器內的溫度。

$$191(y^2 + 7y) = 1528$$

1M

$$y^2 + 7y - 8 = 0$$

$$y = -8 \quad \text{或} \quad 1 \quad (\text{捨去})$$

$$\frac{dx}{dt} = 191(2T + 7)\frac{dT}{dt}$$

1M

當 $t = 6$ 及 $T = -8$,

$$\frac{dx}{dt} = 191(2(-8) + 7)\frac{dT}{dt}$$

$$-7(6) - 771 = -1719\frac{dT}{dt}$$

$$\frac{dT}{dt} = \frac{271}{573}$$

所求變率為 $\frac{271}{573}^\circ\text{C/h}$ 。

1A

97. (a) $\int 3x^2 \ln x \, dx = x^3 \ln x - \int x^2 \, dx$

1M+1A

$$= x^3 \ln x - \frac{x^3}{3} + C$$

1A

(b) $y = \int 3x^2 \ln x \, dx$

1M

$$= x^3 \ln x - \frac{x^3}{3} + C$$

該曲線通過點 (e, e^3) 。

$$e^3 = e^3 \ln e - \frac{e^3}{3} + C$$

1M

$$C = \frac{e^3}{3}$$

所求方程為 $y = x^3 \ln x - \frac{x^3}{3} + \frac{e^3}{3}$ 。

1A

98. $y = \int x \cos 2x \, dx$ 1M

$$= \frac{x \sin 2x}{2} - \frac{1}{2} \int \sin 2x \, dx$$
 1M+1A

$$= \frac{x \sin 2x}{2} + \frac{\cos 2x}{4} + C$$
 1A

該曲線通過點 $(\frac{\pi}{4}, 0)$ 。

$$0 = \frac{\frac{\pi}{4} \sin \frac{\pi}{2}}{2} + \frac{\cos \frac{\pi}{2}}{4} + C$$
 1M

$$C = -\frac{\pi}{8}$$

所求方程為 $y = \frac{x \sin 2x}{2} + \frac{\cos 2x}{4} - \frac{\pi}{8}$ 。 1A

99. (a) 設 $u = x^2 + 4$ 。則 $du = 2x \, dx$ 。 1M

$$\int \frac{20x^3}{\sqrt[3]{x^2+4}} \, dx = 10 \int \frac{u-4}{u^{\frac{1}{3}}} \, du$$
 1M

$$= 10 \int (u^{\frac{2}{3}} - 4u^{-\frac{1}{3}}) \, du$$

$$= 6u^{\frac{5}{3}} - 60u^{\frac{2}{3}} + C$$

$$= 6(x^2+4)^{\frac{5}{3}} - 60(x^2+4)^{\frac{2}{3}} + C$$
 1A

(b) $y = \int \frac{x^3}{\sqrt[3]{x^2+4}} \, dx$ 1M

$$= \frac{1}{20} \int \frac{20x^3}{\sqrt[3]{x^2+4}} \, dx$$

$$= \frac{3}{10} (x^2+4)^{\frac{5}{3}} - 3(x^2+4)^{\frac{2}{3}} + C$$

Γ 通過點 $(2, 1)$ 。

$$1 = \frac{3}{10} (2^2+4)^{\frac{5}{3}} - 3(2^2+4)^{\frac{2}{3}} + C$$
 1M

$$C = 68$$

所求方程為 $y = \frac{3}{10} (x^2+4)^{\frac{5}{3}} - 3(x^2+4)^{\frac{2}{3}} + \frac{17}{5}$ 。 1A

100. (a) $f'(1) = \frac{1}{1^2 - 2(1) + 2} - \frac{1}{1} = 0$ 1M

所求方程為 $y = 0$ 。 1A

(b) 設 $x - 1 = \tan \theta$ 。則 $dx = \sec^2 \theta d\theta$ 。 1M

$$\begin{aligned} f(x) &= \int \left(\frac{1}{x^2 - 2x + 2} - \frac{1}{x} \right) dx \\ &= \int \frac{1}{(x-1)^2 + 1} dx - \int \frac{1}{x} dx \\ &= \int \frac{\sec^2 \theta}{\tan^2 \theta + 1} d\theta - \ln |x| \\ &= \int d\theta - \ln x \\ &= \theta - \ln x + C_1 \\ &= \tan^{-1}(x-1) - \ln x + C_1 \end{aligned}$$

C 通過點 $(1, 0)$ 。

$$0 = \tan^{-1}(1-1) - \ln 1 + C_1 \quad 1M$$

$$C_1 = 0$$

所求方程為 $y = \tan^{-1}(x-1) - \ln x$ 。 1A

$$\begin{aligned} \text{(c) } f'(x) &= \frac{1}{x^2 - 2x + 2} - \frac{1}{x} \\ &= \frac{-x^2 + 3x - 2}{x(x^2 - 2x + 2)} \\ &= -\frac{(x-1)(x-2)}{x(x^2 - 2x + 2)} \end{aligned}$$

當 $f'(x) = 0$ ，可得 $x = 1$ 或 2 。 1M

x	$0 < x < 1$	$1 < x < 2$	$x > 2$
$f'(x)$	—	+	—

1M

C 的極大點為 $\left(2, \frac{\pi}{4} - \ln 2\right)$ 。 1A