M2REG-SPVP-2425-ASM-SET 4-MATH
Suggested solutions

1. (a) $\overrightarrow{OP} = \dfrac{3}{7}\mathbf{a} + \dfrac{4}{7}\mathbf{b}$ 1A

(b) (i) $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin \angle AOB$ 1M

$294 = (35)(14) \sin \angle AOB$

$\sin \angle AOB = \dfrac{3}{5}$

Since $\angle AOB$ is an acute angle, $\cos \angle AOB = \dfrac{\sqrt{5^2 - 3^2}}{5} = \dfrac{4}{5}$.

$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \angle AOB$ 1M

$= (35)(14)\left(\dfrac{4}{5}\right)$

$= 392$ 1A

(ii) $\overrightarrow{OP} \cdot \overrightarrow{OP} = \left(\dfrac{3}{7}\mathbf{a} + \dfrac{4}{7}\mathbf{b}\right) \cdot \left(\dfrac{3}{7}\mathbf{a} + \dfrac{4}{7}\mathbf{b}\right)$

$= \dfrac{9}{49}|\mathbf{a}|^2 + \dfrac{24}{49}\mathbf{a} \cdot \mathbf{b} + \dfrac{16}{49}|\mathbf{b}|^2$

$= \dfrac{9}{49}(35)^2 + \dfrac{24}{49}(392) + \dfrac{16}{49}(14)^2$ 1M

$= 481$

$\left|\overrightarrow{OP}\right| = \sqrt{481}$ 1A

2. (a) $\overrightarrow{AB} = -3\mathbf{i} + \mathbf{j}$ 1A

$\overrightarrow{AC} = -3\mathbf{i} + \mathbf{k}$

$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 1 & 0 \\ -3 & 0 & 1 \end{vmatrix}$ 1M

$= \mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$ 1A

(b) Let $\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$.

$\overrightarrow{OP} = \dfrac{\overrightarrow{OA} \cdot \mathbf{n}}{\mathbf{n} \cdot \mathbf{n}}(\mathbf{n})$ 1M+1M

$= \dfrac{3}{1^2 + 3^2 + 3^2}(\mathbf{i} + 3\mathbf{j} + 3\mathbf{k})$

$= \dfrac{3}{19}(\mathbf{i} + 3\mathbf{j} + 3\mathbf{k})$ 1A

3. (a) $\overrightarrow{AB} - \mathbf{i} - 2\mathbf{j} + \mathbf{k}$

$\overrightarrow{AC} = -2\mathbf{i} - 6\mathbf{j} + 6\mathbf{k}$

$\overrightarrow{AD} = (d+1)\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -2 & 1 \\ -2 & -6 & 6 \end{vmatrix}$$

$= -6\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$ 1A

Note that AD is perpendicular to $\overrightarrow{AB} \times \overrightarrow{AC}$.

$\overrightarrow{AD} \cdot (\overrightarrow{AB} \times \overrightarrow{AC}) = 0$ 1M

$-6(d+1) + 4(-4) + 2(5) = 0$

$d = -2$ 1A

(b) $\overrightarrow{DC} = -\mathbf{i} - 2\mathbf{j} + \mathbf{k} = \overrightarrow{AB}$ 1M

We have $DC // AB$ and $DC = AB$.

Thus, $ABCD$ is a parallelogram. 1

(c) $\overrightarrow{VD} = -9\mathbf{i} + 6\mathbf{j} + 3\mathbf{k}$

$= \dfrac{3}{2}(\overrightarrow{AB} \times \overrightarrow{AC})$

Thus, VD is perpendicular to the plane $ABCD$. 1

(d) Required volume $= \dfrac{1}{3}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|(VD)$

$= \dfrac{1}{3}\sqrt{6^2 + 4^2 + 2^2}\sqrt{9^2 + 6^2 + 3^2}$

$= 28$ 1A

4. (a) Let $AP : PB = r : (1-r)$.

$$\overrightarrow{OP} = (1-r)\overrightarrow{OA} + r\overrightarrow{OB}$$

$$= (1-r)(8\mathbf{j} + 8\mathbf{k}) + r(3\mathbf{i} - 4\mathbf{j} - \mathbf{k})$$ 1M

$3r\mathbf{i} + (-12r + 8)\mathbf{j} - 9r + 8\mathbf{k}$

Since $OP \perp OB$, $\overrightarrow{OP} \cdot \overrightarrow{OB} = 0$. 1M

$$3r(3) + (-12r+8)(-4) + (-9r+8)(-1) = 0$$

$$r = \frac{20}{33}$$

Put $r = \frac{20}{33}$.

$$\overrightarrow{OP} = 3\left(\frac{20}{33}\right)\mathbf{i} + \left[-12\left(\frac{20}{33}\right) + 8\right]\mathbf{j} + \left[-9\left(\frac{20}{33}\right) + 8\right]\mathbf{k}$$

$$= \frac{20}{11}\mathbf{i} + \frac{8}{11}\mathbf{j} + \frac{28}{11}\mathbf{k}$$ 1A

(b) Let $\overrightarrow{OQ} = s\overrightarrow{OP}$.

Then $\overrightarrow{OQ} = \frac{20s}{11}\mathbf{i} + \frac{8s}{11}\mathbf{j} + \frac{28s}{11}\mathbf{k}$.

$\overrightarrow{AQ} = \overrightarrow{OQ} - \overrightarrow{OA}$

$$= \frac{20s}{11}\mathbf{i} + \left(\frac{8s}{11} - 8\right)\mathbf{j} + \left(\frac{28s}{11} - 8\right)\mathbf{k}$$

$\overrightarrow{AB} = 3\mathbf{i} - 12\mathbf{j} - 9\mathbf{k}$

Note that $\angle QAB = \angle QOB = 90°$.

$$\overrightarrow{AQ} \cdot \overrightarrow{AB} = 0$$ 1M

$$\left(\frac{20s}{11}\right)(3) + \left(\frac{8s}{11} - 8\right)(-12) + \left(\frac{28s}{11} - 8\right)(-9) = 0$$

$$s = \frac{77}{12}$$

Thus, $\overrightarrow{OQ} = \frac{35}{3}\mathbf{i} + \frac{14}{3}\mathbf{j} + \frac{49}{3}\mathbf{k}$. 1A

(c) BQ is a diameter of C.

$\overrightarrow{OG} = \frac{1}{2}\overrightarrow{OQ} + \frac{1}{2}\overrightarrow{OB}$ 1A

$$= \frac{22}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{23}{3}\mathbf{k}$$ 1A

5. (a) $\mathbf{a}\cdot\mathbf{a}=|\mathbf{a}|^2=9$ 1A

$\mathbf{a}\cdot\mathbf{b}=|\mathbf{a}||\mathbf{b}|\cos\angle AOB$

$=(3)(4)\cos\dfrac{\pi}{3}$ 1M

$=6$ 1A

(b) $OC=3\cos\dfrac{\pi}{3}=\dfrac{3}{2}$

$\overrightarrow{OC}=\dfrac{\left(\frac{3}{2}\right)}{4}\overrightarrow{OB}$ 1M

$=\dfrac{3}{8}\mathbf{b}$ 1A

(c) $\overrightarrow{OH}=\dfrac{k}{k+1}\overrightarrow{OA}+\dfrac{1}{k+1}\overrightarrow{OC}$

$=\dfrac{k}{k+1}\mathbf{a}+\dfrac{3}{8(k+1)}\mathbf{b}$ 1A

H is the orthocentre of $\triangle OAB$.

$$\overrightarrow{OH}\cdot\overrightarrow{AB}=0$$

$$\left(\frac{k}{k+1}\mathbf{a}+\frac{3}{8(k+1)}\mathbf{b}\right)\cdot(\mathbf{b}-\mathbf{a})=0$$ 1M

$$-\frac{k}{k+1}|\mathbf{a}|^2+\frac{3}{8(k+1)}|\mathbf{b}|^2+\left(\frac{k}{k+1}-\frac{3}{8(k+1)}\right)\mathbf{a}\cdot\mathbf{b}=0$$

$$-\frac{9k}{k+1}+\frac{6}{k+1}+6\left(\frac{k}{k+1}-\frac{3}{8(k+1)}\right)=0$$

$$k=\frac{5}{4}$$ 1A

(d) (i) $\overrightarrow{OB}\cdot\overrightarrow{BP}=(OB)(BP)\cos(180^\circ-\angle OBP)$ 1M

$=-(OB)(BP)\cos\angle OBP$

$=-(OB)(BC)$

Thus, $\overrightarrow{OB}\cdot\overrightarrow{BP}$ is always a constant. 1

(ii) $\overrightarrow{OB}\cdot\overrightarrow{BP}=-(OB)(BC)$ 1M

$=-4\left(4-\dfrac{3}{2}\right)$

$=-10$ 1A

6. (a) (i) (1) $\overrightarrow{OW} = \dfrac{s}{s+t}\overrightarrow{OS} + \dfrac{t}{s+t}\overrightarrow{OT}$ 1A

(2) $\dfrac{\text{Area of } \triangle ITW}{\text{Area of } \triangle IWS} = \dfrac{TW}{WS} = \dfrac{s}{t}$

$$\text{Area of } \triangle ITW = (\text{area of } \triangle ITS) \times \frac{s}{s+t}$$ 1M

$$= \frac{su}{s+t}$$

$$\frac{UI}{IW} = \frac{\text{Area of } \triangle IUT}{\text{Area of } \triangle ITW}$$

$$= s \div \frac{su}{s+t}$$

$$= \frac{s+t}{u}$$

$UI : IW = (s+t) : u$ 1

(ii) Let r be the radius of the circumcircle of $\triangle STU$.

$$s = \frac{1}{2}r^2 \sin \angle TIU$$

$$= \frac{1}{2}r^2 \sin 2\alpha$$ 1M

Similarly, we have $t = \frac{1}{2}r^2 \sin 2\beta$ and $u = \frac{1}{2}r^2 \sin 2\gamma$.

Note that $UI : IW = (s+t) : u$.

$$\overrightarrow{OI} = \frac{s+t}{s+t+u}\overrightarrow{OW} + \frac{u}{s+t+u}\overrightarrow{OU}$$ 1M

$$= \frac{s+t}{s+t+u}\left(\frac{s}{s+t}\overrightarrow{OS} + \frac{t}{s+t}\overrightarrow{OT}\right) + \frac{u}{s+t+u}\overrightarrow{OU}$$

$$= \frac{s\overrightarrow{OS} + t\overrightarrow{OT} + u\overrightarrow{OU}}{s+t+u}$$

$$= \frac{\left(\frac{1}{2}r^2 \sin 2\alpha\right)\overrightarrow{OS} + \left(\frac{1}{2}r^2 \sin 2\beta\right)\overrightarrow{OT} + \left(\frac{1}{2}r^2 \sin 2\gamma\right)\overrightarrow{OU}}{\frac{1}{2}r^2 \sin 2\alpha + \frac{1}{2}r^2 \sin 2\beta + \frac{1}{2}r^2 \sin 2\gamma}$$

$$= \frac{(\sin 2\alpha)\overrightarrow{OS} + (\sin 2\beta)\overrightarrow{OT} + (\sin 2\gamma)\overrightarrow{OU}}{\sin 2\alpha + \sin 2\beta + \sin 2\gamma}$$ 1

(b) (i) $\overrightarrow{AB} = 4\mathbf{i}$

$AB = 4$

$\overrightarrow{AC} = 2\sqrt{2}\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$

$AC = \sqrt{(2\sqrt{2})^2 + 2^2 + 2^2} = 4 = AB$

Thus, $\triangle ABC$ is an isosceles triangle. 1A

(ii) $\overrightarrow{AB} \cdot \overrightarrow{AC} = (AB)(AC)\cos \angle BAC$ 1M

$$8\sqrt{2} = (4)(4)\cos \angle BAC$$

$$\cos \angle BAC = \frac{\sqrt{2}}{2}$$

$$\angle BAC = \frac{\pi}{4}$$ 1A

Since $AB = AC$, we have $\angle ABC = \angle ACB = \dfrac{1}{2}\left(\pi - \dfrac{\pi}{4}\right) = \dfrac{3\pi}{8} < \dfrac{\pi}{2}$.

$\triangle ABC$ is an acute-angled triangle.

$$\overrightarrow{OH} = \frac{\left(\sin\frac{\pi}{2}\right)\overrightarrow{OA} + \left(\sin\frac{3\pi}{4}\right)\overrightarrow{OB} + \left(\sin\frac{3\pi}{4}\right)\overrightarrow{OC}}{\sin\frac{\pi}{2} + \sin\frac{3\pi}{4} + \sin\frac{3\pi}{4}} \qquad \text{1M}$$

$$= \frac{(\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}) + \frac{\sqrt{2}}{2}(5\mathbf{i} - 5\mathbf{j} - 2\mathbf{k}) + \frac{\sqrt{2}}{2}(1 + 2\sqrt{2})\mathbf{i}}{1 + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}}$$

$$= \frac{3 + 3\sqrt{2}}{1 + \sqrt{2}}\mathbf{i} - \frac{2 + \sqrt{2}}{1 + \sqrt{2}}\mathbf{j} - \frac{2 + \sqrt{2}}{1 + \sqrt{2}}\mathbf{k}$$

$$= 3\mathbf{i} - \frac{\sqrt{2}(\sqrt{2} + 1)}{1 + \sqrt{2}}\mathbf{j} - \frac{\sqrt{2}(\sqrt{2} + 1)}{1 + \sqrt{2}}\mathbf{k}$$

$$= 3\mathbf{i} - \sqrt{2}\mathbf{j} - \sqrt{2}\mathbf{k} \qquad \text{1A}$$

(iii) $\overrightarrow{AH} = \overrightarrow{OH} - \overrightarrow{OA} = 2\mathbf{i} + (2 - \sqrt{2})\mathbf{j} + (2 - \sqrt{2})\mathbf{k}$

$$\overrightarrow{AH} \times \overrightarrow{AB} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 - \sqrt{2} & 2 - \sqrt{2} \\ 4 & 0 & 0 \end{vmatrix}$$

$$= 4(2 - \sqrt{2})\mathbf{j} - 4(2 - \sqrt{2})\mathbf{k} \qquad \text{1A}$$

Required distance

$$= AH \sin\angle HAB$$

$$= \frac{(AH)(AB)\sin\angle HAB}{AB}$$

$$= \frac{\left|\overrightarrow{AH} \times \overrightarrow{AB}\right|}{AB}$$

$$= \frac{\sqrt{[4(2 - \sqrt{2})]^2 + [-4(2 - \sqrt{2})]^2}}{4}$$

$$= \frac{\sqrt{32(2 - \sqrt{2})^2}}{4}$$

$$= 2\sqrt{2} - 2 \qquad \text{1A}$$

7. (a) $\overrightarrow{PA} = \mathbf{i} - 5\mathbf{j} + 2\mathbf{k}$ 1M

$\overrightarrow{PB} = -\mathbf{i} + (-1 - t)\mathbf{j} - \mathbf{k}$

$$PA = PB$$

$$\sqrt{1^2 + t^2 + 2^2} = \sqrt{1^2 + (-1 - t)^2 + 1^2}$$ 1M

$$t^2 + 5 = t^2 + 2t + 3$$

$$t = 1$$ 1A

(b) $\overrightarrow{CD} = 6\mathbf{i} - 3\mathbf{j} - 3\mathbf{k}$ 1A

$\overrightarrow{CE} = 4\mathbf{i} - 2\mathbf{j} - 2\mathbf{k} = \frac{2}{3}\overrightarrow{CD}$

Thus, C, D and E are collinear with $CE : ED = 2 : 1$. 1A+1A

(c) (i) $\overrightarrow{CG} = 2\mathbf{i} - 4\mathbf{j}$

$$\overrightarrow{CG} \times \overrightarrow{CD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -4 & 0 \\ 6 & -3 & -3 \end{vmatrix}$$

$$= 12\mathbf{i} + 6\mathbf{j} + 18\mathbf{k}$$ 1M

A required vector

$$= \frac{1}{\sqrt{12^2 + 6^2 + 18^2}}(12\mathbf{i} + 6\mathbf{j} + 18\mathbf{k})$$ 1M

$$= \frac{\sqrt{14}}{7}\mathbf{i} + \frac{\sqrt{14}}{14}\mathbf{j} + \frac{3\sqrt{14}}{14}\mathbf{k}$$ 1A

(ii) (1) $\overrightarrow{GM} = (x - 1)\mathbf{i} + (y + 2)\mathbf{j} + (z - 1)\mathbf{k}$

$$\overrightarrow{GM} \cdot \left(\frac{\sqrt{14}}{7}\mathbf{i} + \frac{\sqrt{14}}{14}\mathbf{j} + \frac{3\sqrt{14}}{14}\mathbf{k}\right) = 0$$ 1M

$$2(x - 1) + (y + 2) + 3(z - 1) = 0$$

$$2x + y + 3z = 3$$ 1

(2) When $x = 1$, $y = 1$ and $z = 0$, $2x + y + 3z = 2(1) + 1 + 0 = 3$. 1M

$\overrightarrow{GP}$ is a vector perpendicular to the vector found in (i).

Thus, P lies on Π. 1

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