1. (a) $f_n(x) = \frac{-1}{n}x^2 + 2x$

$= \frac{-1}{n}(x^2 - 2nx + n^2) + n$ 1M

$= \frac{-1}{n}(x - n)^2 + n$

The coordinates of V_n are $(n,\ n)$.

Thus, V_n lies on $y = x$. 1

(b) (i) 2 1A

(ii) The coordinates of A_2 and V_2 are $(4,\ 0)$ and $(2,\ 2)$ respectively.

Required area

$= (\text{area of } \triangle OA_2V_2) - (\text{area of } \triangle OA_1V_1)$

$= \frac{(4)(2)}{2} - \frac{(2)(1)}{2}$ 1M

$= 3$ 1A

(c) Area of $\triangle A_3V_3V_4A_4$

$= \frac{(8)(4)}{2} - \frac{(6)(3)}{2}$

$= 7$

The first term and the common difference of the sequence are 3 and 4 respectively. 1M

Let p be the number of trapeziums required.

$\frac{p}{2}[2(3) + (p-1)4] > 500$ 1M

$2p^2 + p - 500 > 0$

$p > 15.6 \quad \text{or} \quad p < -16.1$ (rejected)

It takes 16 trapeziums to get the required sum of areas.

Since $n = 2p - 1$, the least value of n is 31. 1A

2. (a) $(2x^2 - 3) - (11x + 1) = (3x + 1) - (2x^2 - 3)$ 1M

$4x^2 - 14x - 8 = 0$

$x = 4 \quad \text{or} \quad -\frac{1}{2}$ (rejected)

We have $A(2) = 45$, $A(6) = 29$ and $A(10) = 13$.

Common difference $= \frac{A(6) - A(2)}{4} = -4$

We have $A(1) = 45 - (-4) = 49$ and $A(n) = 49 + (n-1)(-4) = -4n + 53$.

$A(1) + A(2) + A(3) + \ldots + A(n)$

$= [49 + (-4n + 53)]\frac{n}{2}$ 1M

$= -2n^2 + 51n$ 1A

(b) $$B(k+1)B(k+2)\dots B(4k+1) < 0.09^{-149}$$

$$\log_{0.3} B(k+1) + \log_{0.3} B(k+2) + \dots + \log_{0.3} B(4k+1) > -149 \log_{0.3} 0.09 \qquad \text{1M}$$

$$A(k+1) + A(k+2) + \dots + A(4k+1) > -298$$

$$[A(1) + A(2) + \dots + A(4k+1)] - [A(1) + A(2) + \dots + A(k)] > -298$$

$$[-2(4k+1)^2 + 51(4k+1)] - [-2k^2 + 51k] > -298$$

$$-30k^2 + 137k + 347 > 0$$

$$-1.81 < k < 6.38 \qquad \text{1M}$$

The maximum value of k is 6. 1A

3. (a) $a_3 = a_1 - 3$

$a_1 = 66$ 1A

$a_4 = a_2 + 2$

$a_4 = 9$ 1A

(b) Sequence $a_1, a_3, a_5, \dots$ has first term 66 and common difference -3.

Sequence $a_2, a_4, a_6, \dots$ has first term 7 and common difference 2.

$$a_1 + a_2 + a_3 + \dots + a_{2m-1} > 2251$$

$$(a_1 + a_3 + a_5 + \dots + a_{2m-1}) + (a_2 + a_4 + a_6 + \dots + a_{2m-2}) > 2251$$

$$\frac{m}{2}[66 + (66 + (m-1)(-3))] + \frac{m-1}{2}[7 + (7 + (m-2)2)] > 2251 \qquad \text{1M+1M}$$

$$m(-3m + 135) + (m-1)(2m+10) > 4502$$

$$-m^2 + 143m - 4512 > 0$$

$$47 < m < 96$$

The smallest value of m is 48. 1A

(c) $$b_1 b_2 b_3 \dots b_k < 10^{30}$$

$$1.2^{a_2 + a_4 + a_6 + \dots + a_{2k}} < 10^{30} \qquad \text{1M}$$

$$(a_2 + a_4 + a_6 + \dots + a_{2k}) \log 1.2 < 30 \qquad \text{1M}$$

$$\frac{k}{2}[7 + 7 + (k-1)2] \log 1.2 < 20$$

$$k^2 + 6k - \frac{30}{\log 1.2} < 0$$

$$-22.7 < k < 16.7$$

The greatest value of k is 16. 1A

4. (a) $a - 16 = 4 - a$ 1M

$a = 10$ 1A

$$\frac{1(16) + 2(10) + 3(4) + 4(b)}{16 + 10 + 4 + b} = 1.75$$

$b = 2$ 1A

(b) 1.5 1A

5. (a) $A_{n+1} - A_n = \log_2(a \times 2^n) - \log_2(a \times 2^{n-1})$

$$= (\log_2 a + n) - (\log_2 a + (n-1))$$

$$= 1$$

$A_1, A_2, A_3, \ldots, A_n$ is an arithmetic sequence. 1M

$$A_1 + A_2 + \ldots + A_{10} = 75$$

$$[2\log_2 a + (10-1)(1)]\frac{10}{2} = 75 \quad \text{1M}$$

$$\log_2 a = 3$$

$$a = 2^3$$

$$a = 8 \quad \text{1A}$$

(b) $A_1 + A_2 + \ldots + A_n > 100$

$$3 + 4 + \ldots + (n+2) > 100$$

$$[3 + (n+2)]\frac{n}{2} > 100$$

$$\frac{n^2}{2} + \frac{5n}{2} - 100 > 0 \quad \text{1M}$$

$$n < -16.9 \quad \text{or} \quad n > 11.9 \quad \text{1M}$$

The least value of n is 12.

6. (a) We have $\alpha_n + \beta_n = \sqrt{\log_2 3^n}$ and $\alpha_n\beta_n = \log_4 \sqrt{3^n}$. 1A

$$\alpha_n^2 + \beta_n^2 = (\alpha_n + \beta_n)^2 - 2\alpha_n\beta_n$$

$$= \log_2 3^n - 2\log_4 \sqrt{3^n}$$

$$= \frac{n\log 3}{\log 2} - \frac{2 \times \frac{n}{2} \times \log 3}{2\log 2} \quad \text{1M}$$

$$= \frac{n\log 3}{2\log 2} \quad \text{1A}$$

(b)

$$\left(\frac{1}{8}\right)^{\alpha_5^2+\beta_5^2} \cdot \left(\frac{1}{8}\right)^{\alpha_6^2+\beta_6^2} \cdot \left(\frac{1}{8}\right)^{\alpha_7^2+\beta_7^2} \cdot \ldots \cdot \left(\frac{1}{8}\right)^{\alpha_k^2+\beta_k^2} > \frac{1}{27^{5k+13}}$$

$$(\alpha_5^2 + \beta_5^2)\log\frac{1}{8} + (\alpha_6^2 + \beta_6^2)\log\frac{1}{8} + \ldots + (\alpha_k^2 + \beta_k^2)\log\frac{1}{8} > (5k+13)\log\frac{1}{27} \quad \text{1M}$$

$$-3\log 2 \cdot \left(\frac{5\log 3}{2\log 2} + \frac{6\log 3}{2\log 2} + \ldots + \frac{k\log 3}{2\log 2}\right) > -3(5k+13)\log 3$$

$$5 + 6 + \ldots + k < 10k + 26$$

$$\frac{(k+5)(k-4)}{2} < 10k + 26 \quad \text{1M}$$

$$\frac{k^2}{2} - \frac{19k}{2} - 36 < 0$$

$$-3.24 < k < 22.2 \quad \text{1M}$$

Note that $k \geq 5$, we have $5 \leq k < 22.2$.

The maximum value of k is 22. 1A

7. (a) Required sum

$= 25 + (25 + \log 9) + \ldots + [25 + (n-1)\log 9]$

$= [25 + (25 + 2(n-1)\log 3)]\frac{n}{2}$ 1M

$= n(n-1)\log 3 + 25n$ 1A

(b) $6[k(k-1)\log 3 + 25k] < 4.5 \times 100^2$ 1M

$k^2 \log 3 + (25 - \log 3)k - 7500 < 0$

$-153.7 < k < 102.3$ 1A

The greatest value of k is 102. 1A

8. (a) α and β are roots of the equation $x^2 - 128x + 2 = 0$.

We have $\alpha + \beta = 128$ and $\alpha\beta = 2$. 1A

(b) 1st term $= \log\alpha + \log\beta = \log\alpha\beta = \log 2$ 1M

3rd term $= \log(\alpha + \beta) = \log 128 = 7\log 2$

Common difference $= \dfrac{7\log 2 - \log 2}{2} = 3\log 2$ 1M

$\frac{n}{2}[2\log 2 + (n-1)(3\log 2)] < 2025$ 1M

$(3\log 2)n^2 - (\log 2)n - 4050 < 0$

$-66.8 < n < 67.1$ 1M

The greatest value of n is 67. 1A

9. (a) Let a and d be the first term and the common difference respectively.

$$\begin{cases} a + 7d = 35 \\ a + 12d = 55 \end{cases}$$ 1M

Solving, we have $a = 7$ and $d = 4$.

Thus, $A(1) = 7$. 1A

(b) $\log_{81}[G(1)G(2)G(3)\ldots G(k)] < 666$

$\log_{81} G(1) + \log_{81} G(2) + \ldots + \log_{81} G(k) < 666$ 1M

$\dfrac{\log_3 G(1)}{\log_3 81} + \dfrac{\log_3 G(2)}{\log_3 81} + \ldots + \dfrac{\log_3 G(k)}{\log_3 81} < 666$ 1M

$\frac{1}{4}[A(1) + A(2) + \ldots + A(k)] < 666$

$\frac{1}{4} \times \frac{k}{2}[2(7) + (k-1)4] < 666$ 1M

$\dfrac{k^2}{4} + \dfrac{5k}{4} - 666 < 0$

$-37.8 < k < 35.3$ 1M

The greatest value of k is 35. 1A

10. (a) We have $\alpha_n + \beta_n = \sqrt{\log_3 2^n}$ and $\alpha_n \beta_n = \log_9 \sqrt{2^n}$. 1A

$$\alpha_n^2 + \beta_n^2 = (\alpha_n + \beta_n)^2 - 2\alpha_n\beta_n \quad \text{1M}$$
$$= \log_3 2^n - 2\log_9 \sqrt{2^n}$$
$$= \log_3 2^n - \frac{2 \times \frac{n}{2} \log_3 2^n}{\log_3 9} \quad \text{1M}$$
$$= n \log_3 2 - \frac{n}{2} \log_3 2$$
$$= \frac{n \log_3 2}{2} \quad \text{1A}$$

(b) $\alpha_1^2 + \beta_1^2 + \alpha_2^2 + \beta_2^2 + \ldots + \alpha_k^2 + \beta_k^2 = \log_3 32^k$

$$\frac{\log_3 2}{2} + \frac{2\log_3 2}{2} + \ldots + \frac{k \log_3 2}{2} = 5k \log_3 2 \quad \text{1M}$$
$$\frac{k(k+1)}{2} \times \frac{\log_3 2}{2} = 5k \log_3 2 \quad \text{1M}$$
$$\frac{k(k+1)}{4} = 5k$$
$$\frac{k^2}{4} - \frac{19k}{4} = 0$$
$$k = 19 \quad \text{or} \quad 0 \text{ (rejected)} \quad \text{1A}$$

11. (a)
$$\frac{A(3)}{A(6)} = 9 \div \frac{1}{3}$$
$$\frac{\alpha}{\beta^3} \div \frac{\alpha}{\beta^6} = 27 \quad \text{1M}$$
$$\beta^3 = 27$$
$$\beta = 3 \quad \text{1A}$$

We have $\alpha = 243$. 1A

(b)
$$B(n) = \log_9 \frac{243}{3^n}$$
$$= \log_9 243 - n \log_9 3 \quad \text{1M}$$
$$= \frac{5-n}{2}$$
$$B(1) + B(2) + B(3) + \ldots + B(n) > 0$$
$$\frac{n}{2}\left[2 + \frac{5-n}{2}\right] > 0 \quad \text{1M}$$
$$-\frac{n^2}{4} + \frac{9n}{4} > 0$$
$$0 < n < 9 \quad \text{1M}$$

The greatest value of n is 8. 1A

12. (a) $S(n) = P(1) + P(2) + \ldots + P(n)$

$$= -1 + 3 + \ldots + (4n - 5)$$

$$= [-1 + (4n - 5)]\frac{n}{2} \quad \text{1M}$$

$$= 2n^2 - 3n \quad \text{1A}$$

(b)

$$Q(1)Q(2)\ldots Q(m) < 10^{16\,000}$$

$$P(1)\log 100 + P(2)\log 100 + \ldots + P(m)\log 100 < 16\,000 \quad \text{1M}$$

$$2(2n^2 - 3n) < 16\,000$$

$$2n^2 - 3n - 8000 < 0$$

$$-62.5 < m < 64 \quad \text{1M}$$

The greatest value of m is 63. 1A

13. (a) Let d be the common difference.

$$\begin{cases} A(1) + 3d = 16 \\ A(1) + 9d = 28 \end{cases} \quad \text{1M}$$

Solving, we have $A(1) = 10$ and $d = 2$. 1A

(b)

$$\log_4[G(1)G(2)G(3)\ldots G(k)] < 2024$$

$$\frac{\log_2[G(1)G(2)G(3)\ldots G(k)]}{\log_2 4} < 2024 \quad \text{1M}$$

$$\frac{A(1) + A(2) + A(3) + \ldots + A(k)}{2} < 2024 \quad \text{1M}$$

$$\frac{1}{2}\cdot[10 + (10 + (k - 1)2)]\frac{k}{2} < 2024 \quad \text{1M}$$

$$\frac{k^2}{2} + \frac{9k}{2} - 2024 < 0$$

$$-68.3 < k < 59.3 \quad \text{1M}$$

The greatest value of k is 59. 1A

14. (a) Let d be the common difference of the sequence.

$$-321 + (23 - 1)d = -123 \quad \text{1M}$$

$$d = 9 \quad \text{1A}$$

The common difference of the sequence is 9.

(b)

$$\frac{[2(-321) + (n - 1)9]n}{2} > 0 \quad \text{1M}$$

$$(9n - 651)n > 0$$

$$n < 0 \quad \text{or} \quad n > \frac{217}{3} \quad \text{1A}$$

Required value is 73. 1A

15. (a) Let d be the common difference.

$$\begin{cases} A(1)+2d=-14 \\ A(1)+9d=7 \end{cases}$$ 1M

Solving, we have $A(1)=-20$ and $d=3$. 1A

(b) $$G(1)G(2)G(3)\ldots G(k) < 100\,000$$

$$3^{A(1)+A(2)+A(3)+\ldots+A(k)} < 100\,000$$ 1M

$$[A(1)+A(2)+A(3)+\ldots+A(k)]\log 3 < \log 100\,000$$ 1M

$$[-20+(-20+(k-1)3)]\frac{k}{2}\cdot\log 3 < 5$$ 1M

$$\frac{3(\log 3)k^2}{2}-\frac{43(\log 3)k}{2}-5<0$$

$$-0.472 < k < 14.8$$ 1M

The greatest value of k is 14. 1A

16. (a) Let d be the common difference of the arithmetic sequence.

$$\begin{cases} A(1)+3d=13 \\ A(1)+16d=52 \end{cases}$$ 1M

Solving, we have $A(1)=4$ and $d=3$. 1A

(b) (i) $A(n)=4+(n-1)3=3n+1$

$G(1)G(2)\ldots G(n)$

$=10^{A(1)}10^{A(2)}\ldots 10^{A(n)}$

$=10^{A(1)+A(2)+\ldots+A(n)}$ 1M

$=10^{[4+(3n+1)]\frac{n}{2}}$ 1M

$=10^{\frac{3n^2+5n}{2}}$ 1A

(ii) $\log(G(1)G(2)\ldots G(k)) < 2000$

$$\log 10^{\frac{3k^2+5k}{2}} < 2000$$

$$\frac{3k^2+5k}{2} < 2000$$

$$\frac{3k^2}{2}+\frac{5k}{2}-2000<0$$

$$-37.4 < k < 35.7$$ 1M

The greatest value of k is 35. 1A

17. Let n be the number of cans of food in the bottom layer.

$$n + (n-1) + (n-2) + \ldots + 1 \geq 74$$

$$\frac{(n+1)n}{2} \geq 74 \qquad \text{1M}$$

$$n^2 + n - 148 \geq 0$$

$$n \geq 11.7 \quad \text{or} \quad n \leq -12.7 \text{ (rejected)} \qquad \text{1A}$$

There are 12 cans of food in the bottom layer.
Note that $(12 + 11 + 10 + \ldots + 1) - 74 = 4$.
Consider taking 4 cans of food from a full stack of 12 layers of cans with 1 can of food on top layer and 12 cans of food in the bottom layer. 1M
There are 10 layers. 1A

18. (a)

$$A_1 + A_2 + A_3 + \ldots + A_{10} = 75$$

$$\log_2 a + \log_2 2a + \log_2 2^2 a + \ldots + \log_2 2^9 a = 75$$

$$\log_2 a + (\log_2 a + 1) + (\log_2 a + 2) + \ldots + (\log_2 a + 9) = 75 \qquad \text{1M}$$

$$10 \log_2 a + \frac{(1+9)9}{2} = 75 \qquad \text{1M}$$

$$\log_2 a = 3$$

$$a = 2^3$$

$$a = 8 \qquad \text{1A}$$

(b)

$$A_1 + A_2 + A_3 + \ldots + A_n > 100$$

$$3 + 4 + 5 + \ldots + [3 + (n-1)1] > 100$$

$$\frac{[3 + (n+2)]n}{2} > 100$$

$$\frac{n^2}{2} + \frac{5n}{2} - 100 > 0 \qquad \text{1M}$$

$$n < -16.9 \quad \text{or} \quad n > 11.9 \qquad \text{1M}$$

The smallest value of n is 12. 1A

19. (a) Let the common difference be d.

$$2014 + 15d = 1729 \qquad \text{1M}$$

$$d = -19 \qquad \text{1A}$$

(b) $S(n) = \frac{n}{2}[2(2014) + (n-1)(-19)] < 0$ 1M

$$-19n^2 + 4047n < 0$$

$$n < 0 \text{ (rejected)} \quad \text{or} \quad n > 213 \qquad \text{1A}$$

The least value of n is 214. 1A

20. (a) $x^2 - 8x + 7 = k$

$x^2 - 8x + (7 - k) = 0$

We have $\alpha + \beta = 8$ and $\alpha\beta = 7 - k$. 1M

$(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$

$= 8^2 - 4(7 - k)$

$= 4k + 36$ 1A

(b) $(4k + 36) - 6k = 122 - (4k + 36)$

$k = 25$ 1A

We have $A(1) = 150$ and $A(2) = 136$.

$A(n) = 150 + (n - 1)(-14)$

$= -14n + 164$

$$\log_2[G(1)G(2)G(3)\ldots G(m)] > 2000$$

$$\frac{\log_{16}[G(1)G(2)G(3)\ldots G(m)]}{\log_{16} 2} > 2000$$ 1M

$$\log_{16} G(1) + \log_{16} G(2) + \log_{16} G(3) + \ldots + \log_{16} G(m) > 500$$

$$A(1) + A(2) + A(3) + \ldots + A(m) > 500$$

$$\frac{[150 + (-14m + 164)]m}{2} > 500$$ 1M

$$-7m^2 + 157m - 500 > 0$$

$$3.84 < m < 18.6$$ 1M

The greatest value of m is 18. 1A

21. (a) (i) $W(2) - W(1) = 364\,650 - 363\,900$

$1000a^2 - 1000a = 750$ 1M

$4a^2 - 4a - 3 = 0$

$a = 1.5$ or -0.5 (rejected) 1A

$k = W(1) - 1000a = 362\,400$ 1A

(ii) $W(n) = 362\,400 + 1000(1.5)^n$

$$\frac{W(n) - W(n-1)}{W(n-1)} \times 100\% > 40\%$$

$$\frac{1000(1.5)^n - 1000(1.5)^{n-1}}{362\,400 + 1000(1.5)^{n-1}} > 0.4$$ 1M

$$1000(1.5)^n - 1000(1.5)^{n-1} > 144\,960 + 400(1.5)^{n-1}$$

$$(1500 - 1000 - 400)(1.5)^{n-1} > 144\,960$$

$$1.5^{n-1} > 1449.6$$

$$(n - 1)\log 1.5 > \log 1449.6$$ 1M

$$n > 18.95$$

Starting from the 19th year, the weight of waste that P needs to handle will increase by more than 40% annually. 1A

(b) (i) Required weight

$= (k + 1000a) + (k + 1000a^2) + \ldots + (k + 1000a^n)$

$= nk + 1000(a + a^2 + \ldots + a^n)$

$= nk + 1000 \times \dfrac{a(a^n - 1)}{a - 1}$ 1M

$= [362\,400n + 3000(1.5^n - 1)]$ tonnes 1A

(ii) Required weight

$= 362\,400(18) + 3000(1.5^{18} - 1)$

$\approx 10\,953\,876$ tonnes 1A

(c) $W(19) = 362\,400 + 1000(1.5)^{19} \approx 2\,579\,238$

Upper limit of the weight of waste in the 19th year

$= 700\,000 + (19 - 1)100\,000$ 1M

$= 2\,500\,000$ tonnes

$< 2\,579\,238$ tonnes

The weight of waste exceeds the limit that P can handle in the 19th year.

$2\,579\,238 \times 40\% \approx 1\,031\,695$

$> 100\,000$ 1M

The weight of waste exceeds the limit that P can handle in every subsequent year after the 19th year.

The claim is agreed. 1A

22. $360\,000(1 + r\%)^{5-2} = 622\,080$ 1M

$(1 + r\%)^3 = 1.728$

$1 + r\% = 1.2$

$r = 20$ 1A

$A(1) = \dfrac{360\,000}{1.2} = 300\,000$

$A(n) = 300\,000(1.2)^{n-1}$

$A(1) + A(2) + \ldots + A(k) > 9\,500\,000$

$\dfrac{300\,000(1.2^k - 1)}{1.2 - 1} > 9\,500\,000$ 1M

$1.2^k > \dfrac{22}{3}$

$k \log 1.2 > \log \dfrac{22}{3}$ 1M

$k > 10.9$

The minimum value of k is 11. 1A

23. (a) Let r be the common ratio of the geometric sequence.

$$\log_3 \beta \times r^2 = \log_\beta 81 \quad \text{1M}$$

$$\frac{\log \beta}{\log 3} \times r^2 = \frac{4 \log 3}{\log \beta} \quad \text{1M}$$

$$r^2 = \frac{4(\log 3)^2}{(\log \beta)^2}$$

$$r = \frac{2 \log 3}{\log \beta} \quad \text{or} \quad \frac{-2 \log 3}{\log \beta} \text{ (rejected)} \quad \text{1A}$$

(b)

$$\frac{\log_3 \beta}{1 - \frac{2 \log 3}{\log \beta}} = 9 \quad \text{1M}$$

$$\frac{\log \beta}{\log 3} = 9 - \frac{18 \log 3}{\log \beta}$$

$$\left(\frac{\log \beta}{\log 3}\right)^2 - 9\left(\frac{\log \beta}{\log 3}\right) + 18 = 0$$

$$\frac{\log \beta}{\log 3} = 3 \quad \text{or} \quad 6$$

$$\beta = 27 \quad \text{or} \quad 729 \quad \text{1A}$$

24. (a) Let a and r be the first term and the common ratio of the geometric sequence respectively.

We have $a + ar = 1$ and $\dfrac{a(r^4 - 1)}{r - 1} = 5$. 1M

$$\frac{1}{1 + r} \times \frac{r^4 - 1}{r - 1} = 5$$

$$\frac{(r^2 + 1)(r^2 - 1)}{r^2 - 1} = 5$$

$$r^2 = 4$$

$$r = 2 \quad \text{or} \quad -2 \text{ (rejected)}$$

First term $= \dfrac{1}{1 + r} = \dfrac{1}{3}$ 1A

(b)

$$\frac{\frac{1}{3}(2^n - 1)}{2 - 1} > 5^{30} \quad \text{1M}$$

$$2^n > 3(5^{30}) + 1$$

$$n \log 2 > \log[3(5^{30}) + 1] \quad \text{1M}$$

$$n > 71.2$$

The least value of n is 72. 1A

25. (a) $(x + 2)(x - 2) = 8(x - 1)$

$$x^2 - 8x + 4 = 0$$

Thus, $p = 8$ and $q = 4$. 1A+1A

(b) Common ratio $= \dfrac{\log 8}{\log 4} = \dfrac{3\log 2}{2\log 2} = \dfrac{3}{2}$. 1M

$$(\log 4)\left(\frac{3}{2}\right)^{\alpha} + (\log 4)\left(\frac{3}{2}\right)^{2\alpha} < \log 2^{2020} \quad \text{1M}$$

$$2(1.5)^{\alpha} + 2(1.5)^{2\alpha} < 2020$$

$$(1.5)^{2\alpha} + 1.5^{\alpha} - 1010 < 0$$

$$\frac{-1-\sqrt{4041}}{2} < 1.5^{\alpha} < \frac{-1+\sqrt{4041}}{2}$$

Since $1.5^{\alpha} > 0$,

$$0 < 1.5^{\alpha} < \frac{-1+\sqrt{4041}}{2}$$

$$\alpha \log 1.5 < \log \frac{-1+\sqrt{4041}}{2} \quad \text{1M}$$

$$\alpha < 8.49$$

The greatest value of α is 8. 1A

26. (a) Required number

$= C_1^5 C_1^3 \times 2! \times 6!$ 1M

$= 21\,600$ 1A

(b) (i) Required probability

$= 0.3 + (0.7)(0.6)(0.3) + [(0.7)(0.6)]^2(0.3) + \ldots$ 1M

$= \dfrac{0.3}{1-0.42}$ 1M

$= \dfrac{15}{29}$ 1A

(ii) Required probability

$= \dfrac{15}{29} \times \dfrac{C_4^6 \times 4! \times 3!}{8!} + \left(1 - \dfrac{15}{29}\right) \times \dfrac{C_2^6 \times 2! \times 5!}{8!}$ 1M+1M

$= \dfrac{115}{1624}$ 1A

27. (a) $8(\alpha+\beta) - \dfrac{1}{\alpha} = \dfrac{1}{\beta} - 8(\alpha+\beta)$ 1M

$$16(\alpha+\beta) = \frac{\alpha+\beta}{\alpha\beta}$$

$$16 = \frac{1}{\alpha\beta}$$

$$\log_4 16 = \log_4 \frac{1}{\alpha\beta}$$

$$2 = -\log_4 \alpha - \log_4 \beta \quad \text{1M}$$

$$\log_4 \beta = -2 - \log_4 \alpha \quad \text{1A}$$

(b) $\log_{\alpha\beta}\dfrac{\alpha}{\beta} \div \log_4 \dfrac{4}{\alpha} = \log_4 \dfrac{4}{\beta} \div \log_{\alpha\beta}\dfrac{\alpha}{\beta}$ 1M

$\left(\dfrac{\log_4 \alpha - \log_4 \beta}{\log_4 \alpha\beta}\right)^2 = (1 - \log_4 \beta)(1 - \log_4 \alpha)$ 1M

$\left(\dfrac{\log_4 \alpha - \log_4 \beta}{-2}\right)^2 = (1 - \log_4 \beta)(1 - \log_4 \alpha)$

Let $u = \log_4 \alpha$.

$\dfrac{[u - (-2 - u)]^2}{4} = [1 - (-2 - u)](1 - u)$ 1M

$(u + 1)^2 = (3 + u)(1 - u)$

$2u^2 + 4u - 2 = 0$

$u = \dfrac{-4 \pm \sqrt{4^2 - 4(2)(-2)}}{2(2)}$

$u = -1 - \sqrt{2}$ or $-1 + \sqrt{2}$ (rejected)

Common ratio $= \log_{\alpha\beta}\dfrac{\alpha}{\beta} \div \log_4 \dfrac{4}{\alpha}$

$= \dfrac{\log_4 \alpha - \log_4 \beta}{\log_4 \alpha\beta} \div (1 - \log_4 \alpha)$

$= \dfrac{u - (-2 - u)}{-2} \div (1 - u)$

$= -(u + 1)(1 - u)$

$= -(-\sqrt{2}) \div (2 + \sqrt{2})$ 1M

$= \dfrac{\sqrt{2}}{2 + \sqrt{2}} \times \dfrac{2 - \sqrt{2}}{2 - \sqrt{2}}$

$= \sqrt{2} - 1$

$= \log_4 \beta$ 1

28. (a) (i) $\dfrac{m(1 - k\%)^2}{m(1 - k\%)} = \dfrac{3\,739\,770}{4\,617\,000}$ 1M

$1 - k\% = 0.81$

$k = 19$ 1A

$m = \dfrac{4\,617\,000}{1 - k\%} = 5\,700\,000$ 1A

(ii) Total revenue

$< 5\,700\,000 + 5\,700\,000(1 - 19\%) + 5\,700\,000(1 - 19\%)^2 + \dots$

$= \dfrac{5\,700\,000}{1 - 0.81}$ 1M

$= \$30\,000\,000$

The total revenue of Company P will not exceed \$30 000 000. 1A

(b) (i) Total revenue

$= 5\,700\,000 + 5\,700\,000(1-10\%) + \ldots + 5\,700\,000(1-10\%)^{n-1}$

$= \dfrac{5\,700\,000(1-0.9^n)}{1-0.9}$ 1M

$= \$5\,700\,000(1-0.9^n)$ 1A

(ii) $5\,700\,000(1-0.9^n) > 5\,700\,000 + 5\,700\,000(0.81) + \ldots + 5\,700\,000(0.81)^{n+2}$ 1M

$5\,700\,000(1-0.9^n) > \dfrac{5\,700\,000(1-0.81^{n+3})}{1-0.81}$ 1A

$1.9(1-0.9^n) > 1-0.81^{n+3}$

$(0.81^3)(0.9^n)^2 - 1.9(0.9^n) + 0.9 > 0$ 1M

$0.9^n < 0.562$ or $0.9^n > 3.01$ (rejected)

$n \log 0.9 < \log 0.562$ 1M

$n > 5.47$

Kent will stop running Company P in the 6th year since the establishment of Company Q. 1A

29. (a) We have $\alpha + \beta = -c$ and $\alpha\beta = -7$. 1A

$\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$ 1M

$= c^2 + 14$ 1A

(b) $\dfrac{c^2}{25} = \dfrac{c^2+14}{c^2}$ 1M

$c^4 - 25c^2 - 350 = 0$

$c^2 = 35$ or -10 (rejected)

Common ratio $= \dfrac{35}{25} = \dfrac{7}{5}$

$25 + 25\left(\dfrac{7}{5}\right) + 25\left(\dfrac{7}{5}\right)^2 + \ldots + 25\left(\dfrac{7}{5}\right)^{n-1} > 3 \times 10^7$

$\dfrac{25\left[\left(\frac{7}{5}\right)^n - 1\right]}{\frac{7}{5}-1} > 3 \times 10^7$ 1M

$\left(\dfrac{7}{5}\right)^n > 4.8 \times 10^5$

$n \log \dfrac{7}{5} > \log\left(4.8 \times 10^5\right)$ 1M

$n > 38.9$

The least value of n is 39. 1A

30. (a) Let a and r be the first term and the common ratio of the geometric sequence respectively.

$$a + ar^2 = ar \times \frac{10}{3} \qquad \text{1M}$$

$$r^2 - \frac{10r}{3} + 1 = 0$$

$$r = \frac{1}{3} \quad \text{or} \quad 3 \text{ (rejected)} \qquad \text{1A}$$

The sum to infinity of the geometric sequence is 2322.

$$\frac{a}{1 - \frac{1}{3}} = 2322$$

$$a = 1548$$

The first term of the sequence is 1548. 1A

(b) $$1548\left(\frac{1}{3}\right)^{n-1} + 1548\left(\frac{1}{3}\right)^{n} + \ldots + 1548\left(\frac{1}{3}\right)^{2n-1} > 1$$

$$\frac{1548\left(\frac{1}{3}\right)^{n-1}\left[1 - \left(\frac{1}{3}\right)^{n+1}\right]}{1 - \frac{1}{3}} > 1 \qquad \text{1M+1A}$$

$$2322\left(\frac{1}{3}\right)^{n-1} - 2322\left(\frac{1}{3}\right)^{2n} > 1$$

$$-2322\left(\frac{1}{3}\right)^{2n} + 6966\left(\frac{1}{3}\right)^{n} - 1 > 0$$

$$1.44 \times 10^{-4} < \left(\frac{1}{3}\right)^{n} < 3.00$$

$$\log 1.44 \times 10^{-4} < n \log \frac{1}{3} < \log 3.00 \qquad \text{1M}$$

$$-1.00 < n < 8.05$$

The greatest value of n is 8. 1A

31. (a) Let a and r be the first term and common ratio of the sequence respectively.

$$\frac{ar^5}{ar^2} = \frac{3072}{48} \qquad \text{1M}$$

$$r^3 = 64$$

$$r = 4$$

$$a = 3$$

We have $A(n) = 3(4^{n-1})$. 1A

(b) $B(n) = \log[3(4^{n-1})]$

$= \log 3 + (n-1)\log 4$ 1M

$$B(1) + B(2) + B(3) + \ldots + B(k) > 2023$$
$$k \log 3 + [1 + 2 + 3 + \ldots + (k-1)] \log 4 > 2023$$
$$k \log 3 + \frac{k(k-1)}{2} \cdot \log 4 > 2023 \qquad \text{1M}$$
$$k^2 \log 2 + (\log 3 - \log 2)k - 2023 > 0$$
$$k > 81.7 \quad \text{or} \quad k < -82.3 \text{ (rejected)} \qquad \text{1M}$$

The least value of k is 82. 1A

32. (a) Let a and r be the first term and the common ratio of the geometric sequence respectively.

$$\frac{ar^5}{ar^2} = \frac{1}{27} \qquad \text{1M}$$
$$r^3 = \frac{1}{27}$$
$$r = \frac{1}{3}$$

First term $= a = \frac{27}{r^2} = 243$ 1A

(b)
$$\log_9(G(1)G(2)\ldots G(n)) < -100$$
$$\log_9 243 + \log_9\left(243 \times \frac{1}{3}\right) + \ldots + \log_9\left[243 \times \left(\frac{1}{3}\right)^{n-1}\right] < -100 \qquad \text{1M+1M}$$
$$\frac{5}{2} + 2 + \ldots + \frac{6-n}{2} < -100$$
$$\left(\frac{5}{2} + \frac{6-n}{2}\right)\frac{n}{2} < -100 \qquad \text{1M}$$
$$-\frac{n^2}{4} + \frac{11n}{4} + 100 < 0$$
$$n < -15.2 \quad \text{or} \quad n > 26.2 \qquad \text{1M}$$

The least value of n is 27. 1A

33. (a)
$$\frac{\log_{8m} k}{\log_{m^2} k} = \frac{\log_{128m} k}{\log_{8m} k} \qquad \text{1M}$$
$$\left(\frac{\log k}{3\log 2 + \log m}\right)^2 = \frac{\log k}{2\log m} \times \frac{\log k}{7\log 2 + \log m} \qquad \text{1M}$$
$$14(\log 2)(\log m) + 2(\log m)^2 = 9(\log 2)^2 + 6(\log 2)(\log m) + (\log m)^2$$
$$(\log m)^2 + 8(\log 2)(\log m) - 9(\log 2)^2 = 0$$
$$(\log m - \log 2)(\log m + 9\log 2) = 0 \qquad \text{1M}$$
$$\log m = \log 2 \quad \text{or} \quad -9\log 2 \text{ (rejected)}$$
$$m = 2 \qquad \text{1A}$$

(b) $\log_{m^2} k - \log_k (128m) = \log_m \dfrac{k}{2} - \log_{m^2} k$ 1M

$$2\log_4 k = \log_2 \frac{k}{2} + \log_k 256 \quad 1\text{M}$$

$$2 \times \frac{\log k}{2\log 2} = \frac{\log k - \log 2}{\log 2} + \frac{8\log 2}{\log k} \quad 1\text{M}$$

Let $u = \dfrac{\log k}{\log 2}$.

$$u = (u-1) + \frac{8}{u}$$

$$\frac{8}{u} = 1$$

$$u = 8$$

$$\log k = \log 2^8$$

$$k = 256$$

Common difference

$= \log_4 256 - \log_{256}(128 \times 2)$

$= 3$ 1A

34. (a) Required amount

$= 3.24 \times 10^{10}[1 + (1+5\%) + (1+5\%)^2 + \ldots + (1+5\%)^{14}]$ 1M

$= \dfrac{3.24 \times 10^{10}(1.05^{15} - 1)}{1.05 - 1}$ 1M

$\approx \$6.99 \times 10^{11}$ 1A

(b) $3.24 \times 10^{10}(1 + 1.05 + 1.05^2 + \ldots + 1.05^{n-1}) > 10^{12}$ 1M

$$\frac{3.24 \times 10^{10}(1.05^n - 1)}{1.05 - 1} > 10^{12}$$

$$1.05^n > \frac{206}{81}$$

$$n \log 1.05 > \log \frac{206}{81} \quad 1\text{M}$$

$$n > 19.1$$

The least value of n is 20. 1A

35. (a) $A_n = 100\,000 \times 1.0404^n + 100\,000 \times 1.0201 \times 1.0404^{n-1} + \ldots + 100\,000 \times 1.0201^{n-1} \times 1.0404$ 1M

$= 10^6 \times \dfrac{1.0404^n \left[1 - \left(\frac{1.0201}{1.0404}\right)^n\right]}{1 - \frac{1.0201}{1.0404}}$ 1M

$= \dfrac{10\,404}{203} \times 10^6 (1.0404^n - 1.0201^n)$ 1A

(b) $10\,353\,510 \times (1 + r\%) = 10\,666\,186$

$$1 + r\% = 1.0302$$

$$r = 3.02$$

$$W = \frac{10\,353\,510}{1 + r\%} = 1.005 \times 10^7 \quad \text{1M}$$

$$\frac{10\,404}{203} \times 10^6(1.0404^m - 1.0201^m) > 1.005 \times 10^7(1.0302)^{m-1}$$

$$\frac{10\,404}{203} \times 1.0404^m - 10.05 \times 1.0302^{m-1} - \frac{10\,404}{203} \times 1.0201^m > 0$$

$$\frac{10\,404}{203} \times \frac{1.0404^m}{1.0201^m} - \frac{10.05}{1.0302} \times \frac{1.0302^m}{1.0201^m} - \frac{10\,404}{203} > 0$$

$$\frac{10\,404}{203} \times \left(\frac{102}{101}\right)^{2m} - \frac{10.05}{1.0302} \times \left(\frac{102}{101}\right)^m - \frac{10\,404}{203} > 0$$

$$\left(\frac{102}{101}\right)^m < -0.910 \text{ (rejected)} \quad \text{or} \quad \left(\frac{102}{101}\right)^m > 1.10$$

$$m \log \frac{102}{101} > \log 1.10 \quad \text{1M}$$

$$m > 9.65$$

The least value of m is 10. 1A

36. (a) Let a and r be the first term and the common ratio respectively.

$$\frac{ar^4}{ar} = \frac{625}{320} \quad \text{1M}$$

$$r^3 = \frac{125}{64}$$

$$r = 1.25$$

First term $= \dfrac{320}{r} = 256$ 1A

(b)

$$\frac{256(1.25)^n(1.25^n - 1)}{1.25 - 1} > 8 \times 10^{12} \quad \text{1M}$$

$$256(1.25^{2n}) - 256(1.25^n) - 2 \times 10^{12} > 0$$

$$1.25^n > 88\,400 \quad \text{or} \quad 1.25^n < -88\,400 \text{ (rejected)} \quad \text{1M}$$

$$n \log 1.25 > \log 88\,400 \quad \text{1M}$$

$$n > 51.04$$

The least value of n is 52. 1A

37. (a) $f(x) = x^2 - (2k + 6)x + k^2 + 7k + 7$

$= [x^2 - 2(k + 3)x + (k + 3)^2 - (k + 3)^2] + k^2 + 7k + 7$ 1M

$= [x - (k + 3)]^2 + k - 2$

The coordinates of P are $(k + 3,\ k - 2)$. 1A

(b) Enlarge along the y-axis to 3 times the original. 1A

Translate leftwards by 1 unit. 1A

(c) (i) The coordinates of Q are $(k+2,\ 3k-6)$. 1M

$$\frac{11k-2}{b_2}=\frac{b_2}{b_1}$$ 1M

$$\frac{11k-2}{3k-6}=\frac{3k-6}{k-2}$$

$$11k-2=3(3k-6)$$

$$k=-8$$

The coordinates of P and Q are $(-5,\ -10)$ and $(-6,\ -30)$ respectively. 1A

(ii) (1) Let $(r,\ 0)$ be the coordinates of R.

$$\frac{-10+30}{-5+6}\times\frac{0+10}{r+5}=-1$$ 1M

$$r=-205$$

The coordinates of R are $(-205,\ 0)$. 1A

(2) The coordinates of S are $(-10,\ -8)$.

Slope of $RS=\dfrac{15-0}{-10+205}=\dfrac{1}{13}$ 1M

Slope of $QS=\dfrac{15+30}{-10+6}=-\dfrac{45}{4}\neq -13$

$\angle QSR\neq 90^\circ$

We have $\angle QPR\neq\angle QSR$. 1M

$PQRS$ is not a cyclic quadrilateral.

The claim is disagreed. 1A

38. (a) $2048+2048r+2048r^2=5408$ 1M

$2048r^2+2048r-3360=0$

$$r=\frac{7}{8}\quad\text{or}\quad-\frac{15}{8}$$ 1A

(b) We have $r=\dfrac{7}{8}$.

$$\frac{2048\left[1-\left(\frac{7}{8}\right)^n\right]}{1-\frac{7}{8}}>16\,270$$ 1M

$$\left(\frac{7}{8}\right)^n<\frac{57}{8192}$$

$$n\log\frac{7}{8}<\log\frac{57}{8192}$$ 1M

$$n>37.2$$

The least value of n is 38. 1A

39. (a) Let a and r be the first term and the common ratio of the geometric sequence respectively.

$$\frac{ar^4}{ar} = \frac{2304}{972} \qquad \text{1M}$$

$$r^3 = \frac{64}{27}$$

$$r = \frac{4}{3}$$

First term $= \dfrac{972}{r} = 729$ 1A

(b) $729 + 729 \times \dfrac{4}{3} + \ldots + 729 \times \left(\dfrac{4}{3}\right)^{n-1} < 5 \times 10^{12}$

$$\frac{729\left[\left(\frac{4}{3}\right)^n - 1\right]}{\frac{4}{3} - 1} < 5 \times 10^{12} \qquad \text{1M}$$

$$\left(\frac{4}{3}\right)^n < \frac{5 \times 10^{12}}{2187} + 1$$

$$n \log \frac{4}{3} < \log\left(\frac{5 \times 10^{12}}{2187} + 1\right) \qquad \text{1M}$$

$$n < 74.9$$

The greatest value of n is 74. 1A

40. (a) We have $\alpha + \beta = \dfrac{c+3}{6}$ and $\alpha\beta = \dfrac{4c-2}{6} = \dfrac{2c-1}{3}$.

$$\frac{7}{\alpha + 6} = \frac{\beta + 8}{7} \qquad \text{1M}$$

$$49 = \alpha\beta + 8\alpha + 6\beta + 48$$

$$49 = \frac{2c-1}{3} + 6\left(\frac{c+3}{6}\right) + 2\alpha + 48 \qquad \text{1M}$$

$$\alpha = -\frac{5c+5}{6} \qquad \text{1}$$

(b) $6\left(-\dfrac{5c+5}{6}\right)^2 - (c+3)\left(-\dfrac{5c+5}{6}\right) + 4c - 2 = 0$ 1M

$$5c^2 + \frac{47c}{3} + \frac{14}{3} = 0$$

$$c = -\frac{14}{5} \quad \text{or} \quad -\frac{1}{3}$$

Note that when $c = -\dfrac{1}{3}$, the sum to infinity of the geometric sequence does not exist.

Thus, $c = -\dfrac{14}{5}$. 1A

$$\frac{\frac{15}{2}\left(\frac{14}{15}\right)^{n-1}\left[1-\left(\frac{14}{15}\right)^{n+3}\right]}{1-\frac{14}{15}} > 37 \qquad \text{1M}$$

$$-\left(\frac{14}{15}\right)^{2n+2}+\left(\frac{14}{15}\right)^{n-1}-\frac{74}{225} > 0$$

$$-\frac{196}{225}\left(\frac{14}{15}\right)^{2n}+\frac{15}{14}\left(\frac{14}{15}\right)^{n}-\frac{74}{225} > 0$$

$$0.590 < \left(\frac{14}{15}\right)^{n} < 0.640$$

$$\log 0.590 < n\log\frac{14}{15} < \log 0.640 \qquad \text{1M}$$

$$6.46 < n < 7.66$$

Thus, $n = 7$. 1A

41. $\boxed{\text{D}}$

Let d be the common difference of the sequence.
$(x_1,\ x_4,\ x_7) = (x_1,\ x_1+3d,\ x_1+6d)$
$(2x_3,\ 2x_6,\ 2x_9) = (2x_1+4d,\ 2x_1+10d,\ 2x_1+16d)$
The new numbers can be obtained by multiplying each number by 2 and then add $4d$.
Thus, $v_2 = 2^2 v_1 = 4v_1$ and $\dfrac{v_1}{v_2} = \dfrac{1}{4}$.

42. $\boxed{\text{B}}$

Let the first term and the common difference be a and d respectively.
Note that the new group is formed by adding d to each of the number in the original group.

I. ✗. We have $x_2 = x_1 + d$.
Take $d = -1$, then we have $x_1 > x_2$.

II. ✓.

III. ✗. We have $z_1 = z_2$.

43. $\boxed{\text{C}}$

Let a and d be the first term and the common difference respectively.

$$\begin{cases} a+(a+d)=3 \\ (a+3d)+(a+4d)=-27 \end{cases}$$

Solving, we have $a = 4$ and $d = -5$.

$$\begin{aligned} a_2 + a_3 &= (a+d)+(a+2d) \\ &= -7 \end{aligned}$$

44. $\boxed{\text{A}}$

Common difference $= \dfrac{-7+13}{2} = 3$

I. ✓. 11th term $= -13 + (11-1)(3) = 17$

II. ✓. The first two odd terms are -13 and -7.

Sum of first 50 odd terms

$= \dfrac{[-13 + (-13 + (50-1)6)]50}{2}$

$= 6700$

III. ✗. Let the number of terms be n.

$$\begin{aligned} -13 + (n-1)(3) &< 200 \\ n &< 72 \end{aligned}$$

There are 71 terms in the sequence smaller than 200.

45. $\boxed{\text{D}}$

Let d be the common difference of the original arithmetic sequence.

I. ✓.

Note that $2b - 2a = 2(b-a) = 2d$ and $2c - 2b = 2(c-b) = 2d$.

II. ✓.

We have $b - c = a - b = -d$.

III. ✓.

We have $(b+x) - (a+x) = b - a = d$ and $(c+x) - (b+x) = c - b = d$.

46. $\boxed{\text{D}}$

Note that $\log 9 - \log 3 = \log 27 - \log 9 = \log 81 - \log 27 = \log 3$.

$\log 3, \log 9, \log 27, \log 81$ is an arithmetic sequence.

47. $\boxed{\text{A}}$

I. ✓.

Common difference is $1 - m$.

II. ✓.

Common difference is $\log m$.

III. ✗.

Take $m = 2$, the sequence becomes 8, 16, 32, 64.

Note that $16 - 8 \neq 32 - 16$.

It is not an arithmetic sequence.

48. $\boxed{\text{C}}$

I. ✓.
1st term $= 5(1)^2 + 1 = 6$

II. ✗.
Let $T(n)$ be the nth term of the sequence.

$$\begin{aligned} T(n) &= (5n^2 + n) - [5(n-1)^2 + (n-1)] \\ &= 10n - 4 \end{aligned}$$

When $T(n) = 31$, $n = 3.5$ (rejected).
Thus, 31 is not a term of the sequence.

III. ✓.
Note that $T(n) = 10n - 4$.

$$\begin{aligned} T(n+1) - T(n) &= [10(n+1) - 4] - (10n - 4) \\ &= 10 \end{aligned}$$

The sequence is an arithmetic sequence with common difference 10.

49. $\boxed{\text{C}}$

Common difference $= \log 2a - \log a = \log 2$

$$\begin{aligned} \text{Required sum} &= [2\log a + (n-1)\log 2]\frac{n}{2} \\ &= n\log a + \frac{n(n-1)}{2}\log 2 \end{aligned}$$

50. $\boxed{\text{A}}$

Let a and d be the first term and the common difference respectively.

$$\begin{cases} a + 25d = 224 \\ (a + 41d) - (a + 33d) = -256 \end{cases}$$

Solving, we have $a = 1024$ and $d = -32$.

$$\begin{aligned} A(1) + A(2) + A(3) + \ldots + A(k) &> 5500 \\ \frac{k}{2}[1024 + 1024 + (k-1)(-32)] &> 5500 \\ -16k^2 + 1040k - 5500 &> 0 \\ 5.81 < k &< 59.2 \end{aligned}$$

The greatest value of k is 59.

51. $\boxed{\text{B}}$

The new set of numbers are obtained through the following steps:

(1) Multiply by 4.

(2) Add 3.

I. ✗.
Note that $u_1 = T(50)$ and $u_2 = 4T(50) + 3$.
It is possible that $T(50) = -2$, such that $u_1 = -2$ and $u_2 = -5 < u_1$.

II. ✓.

III. ✗.
Suppose the common difference of the sequence is 0.
We have $w_1 = w_2 = 0$.

52. $\boxed{\text{A}}$

Let a and d be the first term and the common difference respectively.

$$\begin{cases} (a + 17d) + (a + 19d) = 92 \\ (a + 199d) + 300 = a + 99d \end{cases}$$

Solving, we have $a = 100$ and $d = -3$.

I. ✓.

II. ✓.
We have $x_{2018} = 100 + 2017(-3) = -5951$.

$$\begin{aligned} x_1 + x_2 + x_3 + \ldots + x_{2018} &= (100 - 5951)\frac{2018}{2} \\ &= -5\,903\,659 \\ &< -5.9 \times 10^6 \end{aligned}$$

III. ✓.

$$\begin{aligned} 100 + (n - 1)(-3) &> 0 \\ n &< \frac{103}{3} \end{aligned}$$

x_{33} is the smallest positive term.

53. $\boxed{\text{C}}$

Common difference = $\log 10 - \log 2 = \log 5$

$$\begin{aligned} \text{Required sum} &= \frac{[\log 2 + (\log 2 + 6\log 5)](7)}{2} \\ &= 7(\log 2 + 3\log 5) \\ &= 7\log\left(2 \times 5^3\right) \\ &= 7\log 250 \end{aligned}$$

54. $\boxed{\text{D}}$

I. ✓.

Let $S(n) = a_1 + a_2 + a_3 + \ldots + a_n$. Then $S(n) = nb_n = 3n^2 - 8n$.

$$\begin{aligned} a_n &= S(n) - S(n-1) \\ &= (3n^2 - 8n) - [3(n-1)^2 - 8(n-1)] \\ &= 6n - 11 \\ a_{n+1} - a_n &= [6(n+1) - 11] - (6n - 11) \\ &= 6 \end{aligned}$$

It is an arithmetic sequence with common difference 6.

II. ✓.

$$\begin{aligned} 3n - 8 &= 79 \\ n &= 29 \end{aligned}$$

It is the 29th term of the sequence.

III. ✓.

Note that $b_2 - b_1 = (-2) - (-5) = 3$.

$$\begin{aligned} b_1 + b_2 + b_3 + \ldots + b_n &= [(-5) + ((-5) + (n-1)(3))]\frac{n}{2} \\ &= \frac{3n^2 - 13n}{2} \end{aligned}$$

55. $\boxed{\text{C}}$

Let D be the common difference.

I. ✓.

$c - 5d = (a + 2D) - 5(a + 3D) = -4a - 13D$

$9a - 13b = 9a - 13(a + D) = -4a - 13D = c - 5d$

II. ✓.

$a + d = a + (a + 3D) = 2a + 3D$

$b + c = (a + D) + (a + 2D) = 2a + 3D$

III. ✗.

Take $a = 1$, $b = 2$, $c = 3$ and $d = 4$ such that a, b, c, d is an arithmetic sequence.

We have $ad = 4$ and $bc = 6 \neq ad$.

56. $\boxed{\text{A}}$

I. ✓.

Let $T(n)$ be the nth term of the sequence.

$$\begin{aligned} T(n) &= (3n^2 + 8n) - [3(n-1)^2 + 8(n-1)] \\ &= 6n + 5 \end{aligned}$$

When $T(n) = 95$, we have $n = 15$.
95 is the 15th term of the sequence.

II. ✓.

Note that $T(n) = 6n + 5$.

$$\begin{aligned} T(n+1) - T(n) &= [6(n+1) + 5] - (6n + 5) \\ &= 6 \end{aligned}$$

The sequence is an arithmetic sequence with common difference 6.

III. ✗.

Note that $a_1 = 11$ and $a_{2019} = 12\,119$.

$$\begin{aligned} a_1 + a_3 + a_5 + \ldots + a_{2019} &= (11 + 12\,119)\frac{1010}{2} \\ &= 6\,125\,650 \\ &> 7 \times 10^5 \end{aligned}$$

57. $\boxed{\text{A}}$

Let d be the common difference.

I. ✓.

We have $z - x = 2d$ and $y - w = 2d = z - x$.

II. ✗.

Take $w = 1, x = 2, y = 3, z = 4$ such that w, x, y, z is an arithmetic sequence.
We have $\frac{z}{x} = 2$ and $\frac{y}{w} = 3 \neq \frac{z}{x}$.

III. ✗.

Take $w = -1, x = -2, x = -3, z = -4$ such that w, x, y, z is an arithmetic sequence.
We have $w > x > y > z$.

58. $\boxed{\text{A}}$

I. ✓.

Common difference is 5.

II. ✓.

Common difference is $5\sqrt{a}$.

III. ✗.

Take $a = 4$, the sequence becomes $2^{20}, 2^{25}, 2^{30}, 2^{35}$.
Note that $2^{25} - 2^{20} \neq 2^{30} - 2^{25}$.
It is not an arithmetic sequence.

59. $\boxed{\text{C}}$

Let a and d be the first term and the common difference of the arithmetic sequence respectively.

$$\begin{cases} (a+d)+(a+3d)=242 \\ (a+4d)+(a+6d)=224 \end{cases}$$

Solving, we have $a = 127$ and $d = -3$.

$$\begin{aligned} \text{Required sum} &= [127+127+(10-1)(-3)]\frac{10}{2} \\ &= 1135 \end{aligned}$$

60. $\boxed{\text{D}}$

Let d be the common difference.

$$\begin{aligned} 3+19d &= 10(3+d) \\ d &= 3 \end{aligned}$$

I. ✓.

$$\begin{aligned} \text{Required sum} &= \frac{2n}{2}[3+3+(2n-1)(3)] \\ &= 3n(1+2n) \end{aligned}$$

II. ✓.

$a_1 = 3$, $a_4 = 12$, $a_{16} = 48$ is a geometric sequence with common ratio 4.

III. ✓.

$$\begin{aligned} &(a_2+a_4+a_6+\ldots+a_{50})-(a_1+a_3+a_5+\ldots+a_{49}) \\ &= (a_2-a_1)+(a_4-a_3)+(a_6-a_5)+\ldots+(a_{50}-a_{49}) \\ &= 3\times 25 \\ &= 75 \\ &> 0 \end{aligned}$$

Thus, $a_2+a_4+a_6+\ldots+a_{50} > a_1+a_3+a_5+\ldots+a_{49}$.

61. $\boxed{\text{D}}$

$$\begin{aligned} \frac{18}{x-16} &= \frac{x-1}{18} \\ 324 &= (x-1)(x-16) \\ 0 &= x^2-17x-308 \\ x &= 28 \quad \text{or} \quad -11 \end{aligned}$$

Let r be the common ratio of the geometric sequence.

When $x = -11$, $r^2 = \dfrac{18}{-11-16} < 0$. The case is rejected.

When $x = 28$, $r^2 = \dfrac{18}{28-16} = \dfrac{3}{2}$.

We have $a_8 = a_6 \times r^2 = (28-1)\times\dfrac{3}{2} = \dfrac{81}{2}$.

62. $\boxed{\text{C}}$

Let $S(n) = 2n^2 - 5n$ and $T(n)$ be the nth term of the sequence.

I. ✓.

2014th term $= S(2014) - S(2013) = 8049$

II. ✗.

$T(1) = S(1) = -3$

$T(2) = S(2) - S(1) = 1$

$T(3) = S(3) - S(2) = 5$

Note that $\dfrac{T(2)}{T(1)} \neq \dfrac{T(3)}{T(2)}$.

The sequence is not a geometric sequence.

III. ✓.

$$\begin{aligned} T(n) &= S(n) - S(n-1) \\ &= 2[n^2 - (n-1)^2] - 5[n - (n-1)] \\ &= 4n - 7 \\ &= -3 + (n-1)4 \end{aligned}$$

It is an arithmetic sequence with first term -3 and common difference 4.

The only negative term is -3.

63. $\boxed{\text{C}}$

I. ✓.

The common ratio is $\log x$.

II. ✗.

Note that $\dfrac{\log x^2}{\log x} = 2$ and $\dfrac{\log x^3}{\log x^2} = \dfrac{3\log x}{2\log x} = \dfrac{3}{2} \neq 2$.

III. ✓.

$$\frac{\log_4 x}{\log_2 x} = \frac{\log x}{2\log 2} \div \frac{\log x}{\log 2} = \frac{1}{2}$$

$$\frac{\log_{16} x}{\log_4 x} = \frac{\log x}{4\log 2} \div \frac{\log x}{2\log 2} = \frac{1}{2}$$

The common ratio is $\dfrac{1}{2}$.

64. $\boxed{\text{D}}$

Let first term and common ratio be a and r respectively.

$$\begin{aligned} \frac{ar^7}{ar^3} &= \frac{4}{243} \div \frac{1}{12} \\ r^4 &= \frac{16}{81} \\ r &= \pm\frac{2}{3} \end{aligned}$$

When $r = \frac{2}{3}$, $a = \frac{1}{12} \div r^3 = \frac{9}{32}$ and $S(\infty) = \frac{a}{1-r} = \frac{27}{32} > \frac{1}{2}$ (rejected).
When $r = -\frac{2}{3}$, $a = \frac{1}{12} \div r^3 = -\frac{9}{32}$.

I. ✗.

II. ✓. $a_1 + a_2 + \ldots + a_{10} = \frac{a(r^9 - 1)}{r - 1} \approx -0.173 < -\frac{1}{10}$.

III. ✓. $a_2 + a_4 + \ldots = \frac{ar}{1 - r^2} = \frac{27}{80}$.

65. $\boxed{\text{C}}$

I. ✓. We have $\frac{7^b}{7^a} = 7^{b-a}$ and $\frac{7^c}{7^b} = 7^{c-b} = 7^{b-a}$.
Thus, $7^a, 7^b, 7^c$ form a geometric sequence.

II. ✓. We have $(a + c) - (a + b) = c - b$ and $(b + c) - (a + c) = b - a = c - b$.
Thus, $a + b, a + c, b + c$ form an arithmetic sequence.

66. $\boxed{\text{D}}$

The last 5 terms of the sequence can be obtained by multiplying 2^7 to each term of the first five terms.
Required variance $= 10 \times (2^7)^2 = 163\,840$.

67. $\boxed{\text{A}}$

Let r be the common ratio of the geometric sequence.
Note that $0 < \frac{T(2)}{T(1)} = r < 1$.
The second group of numbers can be obtained by multiplying r^{40} to each number in the first group.

I. ✓.
$x_2 = r^{40}x_1 < x_1$

II. ✓.
$y_2 = r^{40}y_1 < y_1$

III. ✗.
$z_2 = (r^{40})^2 z_1 < z_1$

68. $\boxed{\text{D}}$

Let the first term and common ratio be a and r respectively.

$$\begin{aligned} \frac{ar^7}{ar^2} &= \frac{6}{192} \\ r^5 &= \frac{1}{32} \\ r &= \frac{1}{2} \end{aligned}$$

I. ✗.

II. ✓. $a = \dfrac{192}{r^2} = 768.$

$$
\begin{aligned}
768\left(\frac{1}{2}\right)^{n-1} &> 10^{-2} \\
(n-1)\log\frac{1}{2} &> \log\frac{1}{76\,800} \\
n &< 17.2
\end{aligned}
$$

17 terms are greater than 10^{-2}.

III. ✓. $\text{Sum} = \dfrac{768\left(1-\left(\frac{1}{2}\right)^{13}\right)}{1-\frac{1}{2}} \approx 1535.8$

> 1535

69. $\boxed{\text{B}}$

I. ✓. $\text{General term} = (1-2^{-n}) - (1-2^{-(n-1)})$

$$
\begin{aligned}
&= 2^{-n}(-1+2) \\
&= 2^{-n}
\end{aligned}
$$

We have $2^{-n} < 1$ for all positive integers n.

II. ✗. The nth term $= \dfrac{1}{2^n}$ is a rational number for all positive integers n.

III. ✓. $\log T_{n+1} - \log T_n = \log 2^{-n-1} - \log 2^{-n}$

$$
\begin{aligned}
&= (-n-1)\log 2 + n\log 2 \\
&= -\log 2 = \text{constant}
\end{aligned}
$$

Thus, it is an arithmetic sequence.

70. $\boxed{\text{D}}$

Let the first term and common difference be a and d respectively.
We have $d = \dfrac{12-8}{2} = 2$ and $a = 8 - 3d = 2$.

I. ✓. $x_{n+1} - x_n = 2 > 0$

II. ✓. $x_{92} + x_{100} = (a+91d) + (a+99d) = 2(a+95d) = 2x_{96}$

III. ✓. $2^{-x_1} + 2^{-x_2} + 2^{-x_3} + \ldots + 2^{-x_n}$

$$
\begin{aligned}
&= \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \ldots + \frac{1}{2^{2n}} \\
&< \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \ldots \\
&= \frac{\frac{1}{4}}{1-\frac{1}{4}} \\
&= \frac{1}{3} < \frac{1}{2}
\end{aligned}
$$

71. $\boxed{\text{D}}$

$$\begin{aligned}\text{Variance} &= 10 \times (2^{10})^2 \\ &= 10\,485\,760\end{aligned}$$

72. $\boxed{\text{A}}$

$$\begin{aligned}\text{General term} &= (2^{n+1} - 2) - (2^n - 2) \\ &= 2^n(2 - 1) \\ &= 2^n\end{aligned}$$

I. ✓. $\dfrac{T(n+1)}{T(n)} = \dfrac{2^{n+1}}{2^n} = 2 = \text{constant}.$

II. ✗. Second term $= 2^2 = 4 \neq 6$.

III. ✗.

73. $\boxed{\text{A}}$

Required probability

$$\begin{aligned}&= \left(\frac{5}{10}\right)\left(\frac{3}{10}\right) + \left(\frac{5}{10}\right)^3\left(\frac{3}{10}\right) + \left(\frac{5}{10}\right)^5\left(\frac{3}{10}\right) + \dots \\ &= \frac{\frac{3}{20}}{1 - \frac{1}{4}} \\ &= \frac{1}{5}\end{aligned}$$

74. $\boxed{\text{D}}$

Let a and r be the first term and the common ratio of the geometric sequence respectively.

$$\begin{aligned}\frac{ar^5}{ar} &= \frac{2}{27} \div \frac{3}{8} \\ r^4 &= \frac{16}{81} \\ r &= \pm\frac{2}{3}\end{aligned}$$

When $r = \dfrac{2}{3}$, $a = \dfrac{3}{8} \div r = \dfrac{9}{16}$.

Sum to infinity $= \dfrac{a}{1-r} = \dfrac{27}{16} > 1$

Rejected.

When $r = \dfrac{-2}{3}$, $a = \dfrac{3}{8} \div r = -\dfrac{9}{16}$.

Sum to infinity $= \dfrac{a}{1-r} = -\dfrac{27}{80}$

I. ✗.

$$-\frac{9}{16}\left(\frac{-2}{3}\right)^{n-1} = \frac{1}{4}$$
$$\left(-\frac{2}{3}\right)^{n-1} = -\frac{4}{9}$$
$$\left(-\frac{2}{3}\right)^{n-1} = -\left(\frac{2}{3}\right)^2$$

Note that when $n - 1 = 2$, L.H.S. gives positive number.

The equation has no roots.

Thus, $\frac{1}{4}$ is not a term of the sequence.

II. ✓.

$a_1 - a_2 + a_3 - a_4 + \ldots + a_9 - a_{10}$

$$= \frac{-\frac{9}{16}\left[1 - \left(\frac{2}{3}\right)^{10}\right]}{1 - \frac{2}{3}}$$

≈ -1.66

< -1

III. ✓.

$a_2 + a_4 + a_6 + \ldots$

$$= \frac{-\frac{9}{16} \times \frac{-2}{3}}{1 - \left(-\frac{2}{3}\right)^2}$$

$$= \frac{27}{40}$$

$$\frac{1}{20a_6} = 1 \div \left[20\left(-\frac{9}{16}\right)\left(-\frac{2}{3}\right)^5\right]$$
$$= \frac{27}{40}$$
$$= a_2 + a_4 + a_6 + \ldots$$

75. $\boxed{\text{C}}$

a, b, c is an arithmetic sequence.

$$b - a = c - b$$
$$2b = a + c$$
$$2b = 24 - b$$
$$b = 8$$

b, c, d is a geometric sequence.

$$\frac{c}{b} = \frac{d}{c}$$
$$c^2 = bd$$
$$c^2 = \frac{1728}{c}$$
$$c = 12$$

We have $a = 24 - b - c = 4$ and $d = \dfrac{1728}{bc} = 18$.
Thus, $ad = 4 \times 18 = 72$.

76. $\boxed{\text{B}}$

Required probability

$$= \left(\frac{5}{10}\right)\frac{3}{10} + \left(\frac{5}{10}\right)^3 \frac{3}{10} + \left(\frac{5}{10}\right)^5 \frac{3}{10} + \ldots$$
$$= \frac{\frac{3}{20}}{1 - \left(\frac{1}{2}\right)^2}$$
$$= \frac{1}{5}$$

77. $\boxed{\text{D}}$

Probability of getting two equal numbers in one throw

$$= \frac{6}{36}$$
$$= \frac{1}{6}$$

Required probability

$$= \frac{5}{6} \times \frac{1}{36} + \left(\frac{5}{6}\right)^3 \times \frac{1}{36} + \left(\frac{5}{6}\right)^5 \times \frac{1}{36} + \ldots$$
$$= \frac{\frac{5}{6} \times \frac{1}{36}}{1 - \left(\frac{5}{6}\right)^2}$$
$$= \frac{5}{66}$$

78. $\boxed{\text{C}}$

Required sum $= 1024 + 256 + \ldots$

$$= \frac{1024}{1 - \frac{1}{4}}$$
$$= \frac{4096}{3}$$

79. $\boxed{\text{C}}$

$$\frac{5a}{1-a} = 4 + 2a$$
$$5a = (1-a)(4+2a)$$
$$0 = -2a^2 - 7a + 4$$
$$a = -4 \text{ (rejected)} \quad \text{or} \quad \frac{1}{2}$$

80. $\boxed{\text{B}}$

Let r be the common ratio of the geometric sequence.

I. ✓.
Common ratio $= r^3$

II. ✗.
Take $x = 1$, $y = 2$ and $z = 4$ such that x, y, z is a geometric sequence.
$$\frac{3^y}{3^x} = \frac{3^2}{3^1} = 3$$
$$\frac{3^z}{3^y} = \frac{3^4}{3^2} = 9 \neq 3$$
The sequence is not a geometric sequence.

III. ✓.
$$\log y^2 - \log x^2 = 2\log\frac{y}{x} = 2\log r$$
$$\log z^2 - \log y^2 = 2\log\frac{z}{y} = 2\log r$$
The common difference is $2\log r$.

END OF PAPER