REG-CP1B-2425-ASM-SET 3-MATH
Suggested solutions

Conventional Questions

1. (a) $(4m,\ 3m)$ 1A

(b) (i) The locus of P is the angle bisector of $\angle SOT$.
Note that the locus of P passes through the centre of C. 1M
Required equation is

$$y - 0 = \frac{3m - 0}{4m - 0}(x - 0)$$
$$y = \frac{3x}{4}$$

We have $g(x) = \frac{3x}{4}$. 1A

(ii) Note that $OC = \sqrt{(4m)^2 + (3m)^2} = 5m$ and $OC : OE = 5 : (5 + 3)$.

$$\frac{4m}{32} = \frac{5}{3 + 5}$$ 1M
$$m = 5$$ 1A

The centroid of $\triangle UST$ lies on OS.
Thus, S is the mid-point of UT.
We have $OU = OT = 4m = 20$. 1M
The graph of $y = g(x + r)$ can be obtained by translating the graph of $y = g(x)$ leftwards by 20 units.
Thus, $r = 20$. 1A

2. (a) Centre $(-4,\ 3)$ and radius $= \sqrt{4^2 + 3^2 - 5} = 2\sqrt{5}$ 1A

(b) $$(2y - k)^2 + y^2 + 8(2y - k) - 6y + 5 = 0$$ 1M
$$(4 + 1)y^2 + (-4k + 16 - 6)y + (k^2 - 8k + 5) = 0$$
$$5y^2 + (-4k + 10)y + (k^2 - 8k + 5) = 0$$

y-coordinate of $M = \frac{1}{2}\left(-\frac{-4k + 10}{5}\right) = \frac{2k - 5}{5}$ 1M

When $y = \frac{2k - 5}{5}$, $x = 2\left(\frac{2k - 5}{5}\right) - k = \frac{-k - 10}{5}$.

Required coordinates are $\left(\frac{-k - 10}{5},\ \frac{2k - 5}{5}\right)$. 1A

(c) If M lies above the x-axis, then $2k - 5 > 0$. So, $k > \frac{5}{2}$. 1A
Consider the equation $5y^2 + (-4k + 10)y + (k^2 - 8k + 5) = 0$.
If L cuts C at two distinct points,

$$\Delta = (-4k + 10)^2 - 4(5)(k^2 - 8k + 5) > 0$$ 1M
$$-4k^2 + 80k > 0$$
$$0 < k < 20$$ 1A

Combine $k > \frac{5}{2}$ and $0 < k < 20$, we have $\frac{5}{2} < k < 20$.

Thus, it is possible. 1A

3. (a) (i) $\frac{1}{x-1} - \frac{x+2}{x^2-1} = \frac{x+1-(x+2)}{x^2-1}$

$= \frac{-1}{x^2-1}$ 1A

Thus, $A = 0$ and $B = -1$. 1A

(ii) $f(x) = -4x^2 - 16x - 19$

$= -4[x^2 + 2(2)(x) + 2^2] - 3$ 1M

$= -4(x+2)^2 - 3$ 1A

The maximum value of $f(x)$ is -3. 1A

(b) $x^2 + [(k+2)x]^2 + \frac{1}{k-1}x - \frac{1}{k^2-1}(k+2)x + \frac{1}{(k^2-1)^2} = 0$ 1M

$(k^2+4k+5)x^2 + \left(\frac{1}{k-1} - \frac{k+2}{k^2-1}\right)x + \frac{1}{(k^2-1)^2} = 0$

$(k^2+4k+5)x^2 - \frac{1}{k^2-1}x + \frac{1}{(k^2-1)^2} = 0$ 1A

$\Delta = \left(\frac{1}{k^2-1}\right)^2 - 4(k^2+4k+5)\left(\frac{1}{(k^2-1)^2}\right)$ 1M

$= \frac{1}{(k^2-1)^2}(-4k^2 - 16k - 19)$

$= \frac{1}{(k^2-1)^2}[-4(k+2)^2 - 3]$ 1M

$\leq -3\left(\frac{1}{(k^2-1)^2}\right) < 0$

C and L do not intersect.

The claim is agreed. 1A

4. (a) Let the slope of the straight line be m.

The equation of the tangent is $y = mx + 12$.

$x^2 + (mx+12)^2 - 10x + 46(mx+12) - 71 = 0$ 1M

$(1+m^2)x^2 + (70m - 10)x + 625 = 0$

The equation has repeated real roots.

$\Delta = (70m-10)^2 - 4(1+m^2)(625) = 0$ 1M

$2400m^2 - 1400m - 2400 = 0$

$m = \frac{4}{3}$ or $-\frac{3}{4}$

Required equations are $y = \frac{4x}{3} + 12$ and $y = -\frac{3x}{4} + 12$. 1A

(b) Note that $PQ \perp PR$ and K is the mid-point of QR. 1A

P, G, H and K are collinear.

The coordinates of H are $(5,\ -23)$.
Radius of $C = \sqrt{5^2 + 23^2 + 71} = 25$
$PH = \sqrt{5^2 + (12 + 23)^2} = 25\sqrt{2}$
$GK = \frac{1}{3}PK = \frac{1}{3}(25\sqrt{2} + 25)$ 1M
Consider the required ratio of areas.

$$t : 1 = GK : HK$$ 1M

$$= \frac{25\sqrt{2} + 25}{3} : 25$$

$$= \frac{\sqrt{2} + 1}{3} : 1$$

$$t = \frac{\sqrt{2} + 1}{3}$$ 1A

5. (a) (i) $x^2 + (mx)^2 - 400x - 300mx + 40\,000 = 0$ 1M

$(1 + m^2)x^2 - (300m + 400)x + 40\,000 = 0$

$\Delta = (300m + 400)^2 - 4(1 + m^2)(40\,000) > 0$ 1M

$10\,000(-7m^2 + 24m) > 0$

$0 < m < \frac{24}{7}$ 1A

(ii) x-coordinate of $M = \frac{1}{2}\left[\frac{300m + 400}{1 + m^2}\right]$

$= \frac{50(3m + 4)}{1 + m^2}$

y-coordinate of $M = m \cdot \frac{50(3m + 4)}{1 + m^2}$

$= \frac{50m(3m + 4)}{1 + m^2}$ 1

(b) (i) Perpendicular bisector of AB is the line passing through O and the centre of C. 1M

The coordinates of the centre of C are $(200,\ 150)$.

Required equation is

$$y - 0 = \frac{150 - 0}{200 - 0}(x - 0)$$

$$3x - 4y = 0$$ 1A

(ii) Note that L touches C when $m = 0$ or $\frac{24}{7}$.

When $m = 0$, the coordinates of the intersection of L and C are $(200,\ 0)$.

The coordinates of B are $(200,\ 0)$. 1A

When $m = \frac{24}{7}$, we have the coordinates of A are $(56,\ 192)$.

Denote the centre of C by G $(200,\ 150)$.

Suppose G and M are distinct points.

Note that $\angle GMO = 90°$ and $\angle OBG = 90°$.

We have O, B, G, M are concyclic. 1M

Since $\angle OAG = 90°$ and $\angle GMO = 90°$, we have O, A, G, M are concyclic.

Thus, O, A, M, G and B are concyclic.
If G and M coincides, O, A, G, B are also concyclic.
Note that OG is a diameter of the circle.
The coordinates of the centre of the required circle are $(100,\ 75)$. 1M
Required equation is

$$(x-100)^2+(y-75)^2=(0-100)^2+(0-75)^2$$
$$(x-100)^2+(y-75)^2=15\,625$$ 1A

(iii) Denote the centre of the circle AMB by D.

$$\tan\angle BOG=\frac{150}{200}$$
$$\angle BOG\approx 36.9^\circ$$

$\angle ADB=2\angle AOB=2(2\angle BOG)\approx 147^\circ$ 1M

Length of $\Gamma \le 2\pi(125)\times\dfrac{\angle ADB}{360^\circ}$ 1M

$\approx 322<330$

The claim is disagreed. 1A

6. (a) $(x-2)^2+(y-6)^2=r^2$ 1A

(b) (i) Let G be the centre of C'. Then $G\,(-2,\ 6-c)$. 1A
Since $AG\perp PQ$,

$$\frac{6-c-6}{-2-2}\times\left(-\frac{1}{2}\right)=-1$$ 1M
$$c=8$$ 1A

(ii) mid-point of AG lies on PQ, i.e., $(0,\ 2)$ lies on PQ. 1M
The equation of PQ is $y=-\dfrac{x}{2}+2$. 1A

(iii) $(x-2)^2+\left(-\dfrac{x}{2}+2-6\right)^2=r^2$ 1M

$$\frac{5}{4}x^2+20-r^2=0$$

a and d are roots of the equation.
So, $a+d=0$ and $ad=\dfrac{4(20-r^2)}{5}$. 1M

$(a-d)^2=(a+d)^2-4ad$

$=\dfrac{16(r^2-20)}{5}$ 1A

(c) $PQ^2=(a-d)^2+\left[\left(-\dfrac{a}{2}+2\right)-\left(-\dfrac{d}{2}+2\right)\right]^2$

$80=\dfrac{5}{4}(a-d)^2$

$=4(r^2-20)$ 1M

$r^2=40$

$r=2\sqrt{10}$ or $-2\sqrt{10}$ (rejected)

$AB = \sqrt{(2+1)^2 + (6-1)^2} = \sqrt{34} < r$

So, B lies inside C.

The claim is agreed. 1A

7. (a) Slope of $AG = \dfrac{112 - 12}{83 - 158} = \dfrac{-4}{3}$

Required equation is

$$y - 12 = \frac{-4}{3}(x - 158) \quad \text{1M}$$

$$4x + 3y - 668 = 0 \quad \text{1A}$$

(b) $GP = \sqrt{(83-23)^2 + (112-67)^2} = 75$

$AG = \sqrt{(158-83)^2 + (112-12)^2} = 125$

$AP = \sqrt{125^2 - 75^2} = 100$

Denote the intersection of AG and PQ by T.

Note that $\triangle AGP \sim \triangle APT$.

$$\frac{AT}{AP} = \frac{AP}{AG} \quad \text{1M}$$

$$AT = 80$$

$AT : TG = 80 : (125 - 80) = 16 : 9$

Required coordinates $= \left(\dfrac{16(83) + 9(158)}{16+9}, \ \dfrac{16(112) + 9(12)}{16+9} \right)$ 1M

$= (110, \ 76)$ 1A

(c) Denote the incentre and radius of inscribed circle of $\triangle APQ$ by S and r respectively.

Suppose AP touches the inscribed circle at U.

Note that $\triangle AUS \sim \triangle APG$.

$$\frac{AS}{US} = \frac{AG}{PG}$$

$$\frac{80 - r}{r} = \frac{125}{75} \quad \text{1M}$$

$$r = 30 \quad \text{1A}$$

$AS : ST = (80 - 30) : 30 = 5 : 3$

Coordinates of $S = \left(\dfrac{5(110) + 3(158)}{5+3}, \ \dfrac{5(76) + 3(12)}{5+3} \right)$ 1M

$= (128, \ 52)$

Required equation is

$$(x - 128)^2 + (y - 52)^2 = 30^2$$

$$(x - 128)^2 + (y - 52)^2 = 900 \quad \text{1A}$$

(d) Note that A, P, G, Q are concyclic.

Radius of circumcircle of $\triangle APQ = \dfrac{AG}{2} = \dfrac{125}{2}$ 1M

Required ratio $= 30^2 : \left(\dfrac{125}{2} \right)^2$ 1M

$= 144 : 625 \neq 1 : 4$

The claim is disagreed. 1A

8. (a) The coordinates of G are $(47,\ 42)$.

Slope of $TG = \dfrac{42-12}{47-7} = \dfrac{3}{4}$

Required equation is

$$y - 12 = \frac{3}{4}(x-7) \qquad \text{1M}$$

$$3x - 4y + 27 = 0 \qquad \text{1A}$$

(b) Radius of $C = \sqrt{47^2 + 42^2 - 3073} = 30$

Note that the vertical distance between G and T is 30.

One of the tangent through T is horizontal and touch C at $(47,\ 12)$. 1M

Note that $XY \perp TG$. The slope of XY is $-\dfrac{4}{3}$.

Equation of XY is

$$y - 12 = -\frac{4}{3}(x-47)$$

$$4x + 3y - 224 = 0$$

Solve $\begin{cases} 3x - 4y + 27 = 0 \\ 4x + 3y - 224 = 0 \end{cases}$, 1M

we have $x = \dfrac{163}{5}$ and $y = \dfrac{156}{5}$.

The coordinates of K are $\left(\dfrac{163}{5},\ \dfrac{156}{5}\right)$. 1A

Radius of $C = \sqrt{47^2 + 42^2 - 3073} = 30$

$GT = \sqrt{(47-7)^2 + (42-12)^2} = 50$

$TX = \sqrt{50^2 - 30^2} = 40$

Note that $\triangle GTX \sim \triangle XTK$.

$$\frac{TK}{TX} = \frac{TX}{GT}$$

$$\frac{TK}{40} = \frac{40}{50}$$

$$TK = 32$$

We have $TK : KG = 32 : 18 = 16 : 9$.

Let the coordinates of K be $(a,\ b)$.

$$\frac{a-7}{47-a} = \frac{16}{9} \quad \text{and} \quad \frac{b-12}{42-b} = \frac{16}{9}$$

$$a = \frac{163}{5} \qquad b = \frac{156}{5}$$

The coordinates of K are $\left(\dfrac{163}{5},\ \dfrac{156}{5}\right)$.

(c) (i) $XY = 2XK = 2\sqrt{40^2 - 32^2} = 48$

Let r be the radius of the inscribed circle of $\triangle TXY$.

Consider the area of $\triangle TXY$. Note that $TK \perp XY$.

$$\frac{(XY)(TK)}{2} = \frac{(XY)(r)}{2} + \frac{(TX)(r)}{2} + \frac{(TY)(r)}{2}$$ 1M

$$768 = 64r$$

$$r = 12$$

We have $TI : IK = (32 - 12) : 12 = 5 : 3$.

Let $(c,\ d)$ be the coordinates of I.

$$\frac{c-7}{\frac{163}{5} - c} = \frac{5}{3} \quad \text{and} \quad \frac{d-12}{\frac{156}{5} - d} = \frac{5}{3}$$

$$c = 23 \qquad\qquad d = 24$$

The coordinates of I are $(23,\ 24)$. 1A

(ii) We have $\angle TXG + \angle TYG = 90° + 90° = 180°$.

Points T, X, G and Y are concyclic.

J is also the centre of the circumcircle of $TXGY$.

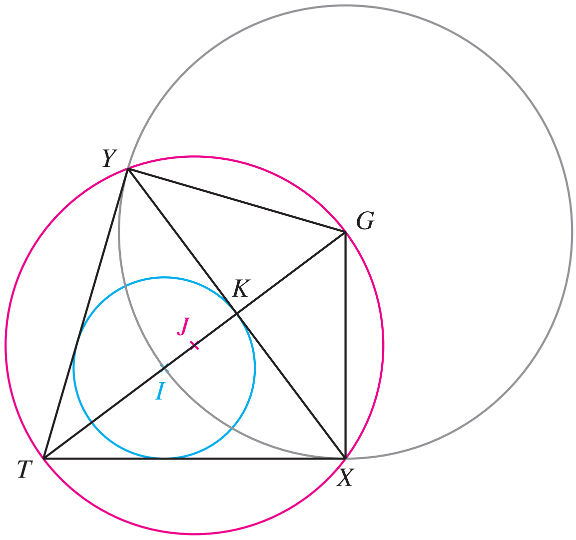
Then J is the mid-point of TG.

The coordinates of J are $(27,\ 27)$. 1A

Note that J lies between I and G.

$$\angle XJG = \angle XIG + \angle IXJ$$
$$> \angle XIG$$

The claim is disagreed. 1A



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