

REG-AOT-2425-ASM-SET 5-MATH

Suggested solutions

Conventional Questions

1. (a) Area of $\triangle ACD = \frac{1}{2}(20)(24) \sin 60^\circ$ 1M
 $= 120\sqrt{3} \text{ cm}^2$ 1A
- (b) Let M be the foot of perpendicular from A to CD .
 Required angle is $\angle AMB$. 1A
 In $\triangle ACD$,
 $CD^2 = 20^2 + 24^2 - 2(20)(24) \cos 60^\circ$ 1M
 $CD \approx 22.3 \text{ cm}$
 $20^2 = CD^2 + 24^2 - 2(CD)(24) \cos \angle ADC$
 $\angle ADC \approx 51.1^\circ$
 In $\triangle ADM$, $AM = AD \sin \angle ADC \approx 18.7 \text{ cm}$ 1M
 In $\triangle ABM$,
 $\sin \angle AMB = \frac{18}{AM}$
 $\angle AMB \approx 74.7^\circ$ 1A
 Required angle is 74.7° .
- (c) AB is the shortest distance from A to the plane BCD .
 Let h be the shortest distance from B to the plane ACD .
 By considering volume of the tetrahedron,

$$\frac{1}{3}(\text{area of } \triangle ACD)(h) = \frac{1}{3}(\text{area of } \triangle BCD)(AB)$$

$$\frac{1}{3} \times \frac{1}{2}(CD)(AM)(h) = \frac{1}{3} \times \frac{1}{2}(CD)(BM)(AB)$$

$$h = \frac{BM}{AM}(AB)$$
 1M
 Since $AM > BM$, $h < AB$.
 The claim is agreed. 1A
2. (a) $\frac{18}{\sin \angle ABD} = \frac{22}{\sin 80^\circ}$ 1M
 $\angle ABD \approx 53.7^\circ$ or 126° (rejected) 1A
- (b) (i) $25^2 = 22^2 + 18^2 - 2(22)(18) \cos \angle ADC$ 1M
 $\angle ADC \approx 76.6^\circ$ 1A
- (ii) Let E be a point on BD such that $AE \perp BD$.
 Let F be a point on CD such that $EF \perp BD$.
 Required angle is $\angle AEF$. 1M
 $AE = 18 \sin 80^\circ \approx 17.7 \text{ cm}$ 1M

$$DE = 18 \cos 80^\circ \approx 3.13 \text{ cm}$$

$$EF = DE \tan \angle CDB = DE \tan \angle ABD \approx 4.25 \text{ cm}$$

$$DF = \sqrt{DE^2 + EF^2} \approx 5.28 \text{ cm}$$

Consider $\triangle ACD$.

$$AF^2 = 18^2 + DF^2 - 2(18)(DF) \cos \angle ADC$$

$$AF \approx 17.5 \text{ cm}$$

Consider $\triangle AEF$.

$$AF^2 = AE^2 + EF^2 - 2(AE)(EF) \cos \angle AEF$$

$$\angle AEF \approx 80.7^\circ$$

$$> 80^\circ$$

The claim is agreed.

1A

3. (a) Let $OP = h$ m.

$$\text{Then } AO = \frac{h}{\tan 16.5^\circ}, BO = \frac{h}{\tan 20^\circ} \text{ and } CO = \frac{h}{\tan 18^\circ}.$$

In $\triangle ABO$,

$$AO^2 = BO^2 + 340^2 - 2(BO)(340) \cos \angle ABO$$

1M

$$\cos \angle ABO = \frac{340^2 + BO^2 - AO^2}{2(BO)(340)}$$

In $\triangle BOC$,

$$CO^2 = BO^2 + 430^2 - 2(BO)(430) \cos \angle OBC$$

$$\cos \angle OBC = \frac{BO^2 + 430^2 - CO^2}{2(BO)(430)}$$

Since $\angle ABO = 180^\circ - \angle OBC$,

$$\frac{340^2 + BO^2 - AO^2}{2(BO)(340)} = -\frac{BO^2 + 430^2 - CO^2}{2(BO)(430)}$$

1M

$$43 \left[(340^2) + \left(\frac{h}{\tan 20^\circ} \right)^2 - \left(\frac{h}{\tan 16.5^\circ} \right)^2 \right] = -34 \left[\left(\frac{h}{\tan 20^\circ} \right)^2 + (430)^2 - \left(\frac{h}{\tan 18^\circ} \right)^2 \right]$$

$$h^2 \left[\frac{43 + 34}{\tan^2 20^\circ} - \frac{43}{\tan^2 16.5^\circ} - \frac{34}{\tan^2 18^\circ} \right] = -34(430)^2 - 43(340)^2$$

1M

$$h \approx 221$$

1A

(b) Let H be a point on AC such that $PH \perp AC$ and $OH \perp AC$. Required angle is $\angle PHO$.

1A

Using the value of h , $AO \approx 745$ m, $BO \approx 607$ m and $CO \approx 680$ m.

$$AO^2 = CO^2 + (340 + 430)^2 - 2(CO)(340 + 430) \cos \angle OCA$$

1M

$$\angle OCA \approx 61.5^\circ$$

$$OH = CO \sin \angle OCA \approx 597 \text{ m}$$

1A

$$\angle PHO = \tan^{-1} \frac{h}{OH} \approx 20.3^\circ$$

1A

4. (a) $MS = 2 \sin 60^\circ = \sqrt{3} \text{ cm}$
 $PS = \sqrt{2^2 + 2^2} = 2\sqrt{2} \text{ cm}$
 $\sin \angle MPS = \frac{\sqrt{3}}{2\sqrt{2}}$ 1M
 $\angle MPS \approx 37.8^\circ$ 1A
- (b) Let A be a point on PS such that $MA \perp PS$.
Let B be a point on ST such that $AB \perp PS$.
Required angle is $\angle MAB$. 1A
 $MA = MP \sin \angle MPS \approx 1.37 \text{ cm}$ 1M
 $PA = MP \cos \angle MPS \approx 1.77 \text{ cm}$
 $AS = PS - PA \approx 1.06 \text{ cm}$
 $AB = AS \tan 45^\circ \approx 1.06 \text{ cm}$
 $BS = \frac{AS}{\cos 45^\circ} = 1.5 \text{ cm}$
 $BM^2 = BS^2 + MS^2 - 2(BS)(MS) \cos 30^\circ$ 1M
 $BM \approx 0.866 \text{ cm}$
 $BM^2 = AB^2 + MA^2 - 2(AB)(MA) \cos \angle MAB$
 $\angle MAB \approx 39.2^\circ$ 1A
Required angle is 39.2° .
- (c) Let E be a point on WS such that $DE \perp WS$.
Let F be a point on RQ such that $DF \perp RQ$.
Area of $\triangle DSW = \frac{(WS)(DE)}{2} = \frac{WS\sqrt{DF^2 + EF^2}}{2}$ 1M
Since WS and EF are constant, area of $\triangle DSW$ is minimum when DF is minimum, i.e.,
 $DF = 0$.
When D is at P , $DF \neq 0$. The area is therefore not minimum when D is at P . 1A
5. (a) $AC = \sqrt{78^2 + 104^2}$
 $= 130 \text{ cm}$
 $AB = \sqrt{130^2 - 120^2}$ 1M
 $= 50 \text{ cm}$ 1A
 $\angle BAD = \tan^{-1} \frac{120}{50} + \tan^{-1} \frac{104}{78}$
 $\approx 121^\circ$
 $BD^2 = 50^2 + 78^2 - 2(50)(78) \cos \angle BAD$ 1M
 $BD = 112 \text{ cm}$ 1A
- (b) (i) Note that $GA = GB = GC = GD = \frac{AC}{2} = 65 \text{ cm}$.
 $VG = \sqrt{169^2 - 65^2}$
 $= 156 \text{ cm}$ 1A
- (ii) Let H be the projection of G on the plane VBD .
Let $h \text{ cm}$ be the required distance.
Consider the area of $\triangle BDG$.

$$\text{Let } s = \frac{65 + 65 + 112}{2} = 121 \text{ cm.}$$

$$\text{Area} = \sqrt{121(121 - 65)^2(121 - 112)} = 1848 \text{ cm}^2 \quad 1\text{M}$$

Consider the area of $\triangle VBD$.

$$\text{Let } s = \frac{169 + 169 + 112}{2} = 225 \text{ cm.}$$

$$\text{Area} = \sqrt{225(225 - 169)^2(225 - 112)} = 840\sqrt{113} \text{ cm}$$

Consider the volume of the tetrahedron $VBGD$.

$$\frac{1}{3}(1848)(156) = \frac{1}{3}(840\sqrt{113})(h) \quad 1\text{M}$$

$$h \approx 32.3 \quad 1\text{A}$$

Required distance is 32.3 cm.

6. (a) $\angle BAC = 180^\circ - 56^\circ - 82^\circ = 42^\circ$

$$\frac{BC}{\sin 42^\circ} = \frac{20}{\sin 56^\circ} \quad 1\text{M}$$

$$BC \approx 16.1 \text{ cm} \quad 1\text{A}$$

(b) (i) $10^2 = BC^2 + 20^2 - 2(BC)(20) \cos \angle ABC \quad 1\text{M}$

$$\angle ABC \approx 29.8^\circ \quad 1\text{A}$$

(ii) Let E be a point on BD such that $CE \perp BD$.

Let F be a point on AB such that $EF \perp BD$.

Required angle is $\angle CEF$. 1M

$$BF = BC \approx 16.1 \text{ cm}$$

$$CE = FE = BC \sin \frac{82^\circ}{2} \approx 10.6 \text{ cm}$$

$$CF^2 = BC^2 + BF^2 - 2(BC)(BF) \cos \angle ABC \quad 1\text{M}$$

$$CF \approx 8.29 \text{ cm}$$

$$CF^2 = EF^2 + CE^2 - 2(EF)(CE) \cos \angle CEF \quad 1\text{M}$$

$$\angle CEF \approx 46.1^\circ > 45^\circ$$

The claim is agreed. 1A

7. (a) $AD = \frac{\sqrt{30^2 + 40^2}}{2} = 25 \text{ cm}$

$$\angle BAC = \cos^{-1} \frac{30}{50} = \cos^{-1} \frac{3}{5}$$

$$BD^2 = 25^2 + 30^2 - 2(25)(30) \cos \angle BAC \quad 1\text{M}$$

$$BD = 25 \text{ cm} \quad 1\text{A}$$

$$\angle ABD = \angle BAD = \cos^{-1} \frac{3}{5}$$

$$AF = 30 \sin \angle ABD = 24 \text{ cm} \quad 1\text{A}$$

$$\cos \angle BAF = \frac{24}{30} = \frac{4}{5}$$

$$FE = \frac{30}{\cos \angle BAF} - 24 = 13.5 \text{ cm}$$

1A

- (b) (i) Required angle is $\angle AFE$.

$$\cos \angle AFE = \frac{13.5}{24}$$

1M

$$\angle AFE \approx 55.8^\circ$$

1A

- (ii) Since $AF \perp BD$ and $FE \perp BD$,
the claim is agreed.

1A

$$\begin{aligned} \text{(iii) Area of } \triangle BCD &= \frac{\text{area of } \triangle ABC}{2} \\ &= \frac{(30)(40)}{2 \times 2} \\ &= 300 \text{ cm}^2 \end{aligned}$$

$$\text{Required volume} = \frac{1}{3}(300)(24 \sin \angle AFE)$$

1M

$$\approx 1984 \text{ cm}^3$$

1A

- (iv) Area of $\triangle ABD$ is equal to the area of $\triangle BCD$, i.e. 300 cm^2 .

By considering the volume of the tetrahedron,

$$\frac{1}{3}(300)(AB \sin \alpha) = \frac{1}{3}(300)(BC \sin \beta)$$

1M

$$\frac{\sin \alpha}{\sin \beta} = \frac{40}{30} > 1$$

So, $\sin \alpha > \sin \beta$ and therefore $\alpha > \beta$.

1A