

M2REG-VEC-2425-ASM-SET 3-MATH

Suggested solutions

$$1. \quad \mathbf{p} \cdot \mathbf{c} = 0 \quad 1\text{M}$$

$$k(\mathbf{a} \cdot \mathbf{c}) + 2(\mathbf{b} \cdot \mathbf{c}) = 0$$

$$4k - 2 = 0$$

$$k = \frac{1}{2} \quad 1\text{A}$$

$$2. \quad (a) \quad \mathbf{a} \cdot \mathbf{b} = (5)(2) \cos 120^\circ = -5 \quad 1\text{A}$$

$$(b) \quad (\mathbf{a} + k\mathbf{b}) \cdot (\mathbf{a} + 3\mathbf{b}) = 0 \quad 1\text{M}$$

$$|\mathbf{a}|^2 + (3+k)\mathbf{a} \cdot \mathbf{b} + 3k|\mathbf{b}|^2 = 0$$

$$5^2 + (3+k)(-5) + 3k(4) = 0$$

$$k = -\frac{10}{7} \quad 1\text{A}$$

$$3. \quad (a) \quad \overrightarrow{OP} = \frac{1}{r+1}\overrightarrow{OB} + \frac{r}{r+1}\overrightarrow{OA} \quad 1\text{M}$$

$$= \frac{1}{r+1}(5\mathbf{i} + \mathbf{j}) + \frac{r}{r+1}(2\mathbf{i} + 4\mathbf{j})$$

$$= \frac{5+2r}{r+1}\mathbf{i} + \frac{4r+1}{r+1}\mathbf{j} \quad 1\text{A}$$

$$(b) \quad \overrightarrow{OP} \cdot \overrightarrow{AB} = 0 \quad 1\text{M}$$

$$\left(\frac{5+2r}{r+1}\mathbf{i} + \frac{4r+1}{r+1}\mathbf{j}\right) \cdot (3\mathbf{i} - 3\mathbf{j}) = 0$$

$$\frac{1}{r+1}[(15+6r) - (12r+3)] = 0$$

$$r = 2 \quad 1\text{A}$$

$$4. \quad \overrightarrow{AB} = (2v - 2u - 1)\mathbf{i} + (v + 3u - 2)\mathbf{j} + (-v - u - 2)\mathbf{k} \quad 1\text{A}$$

$$\overrightarrow{AB} \cdot (\mathbf{j} + \mathbf{k}) = 0 \quad 1\text{M}$$

$$v + 3u - 2 + (-v - u - 2) = 0$$

$$u = 2 \quad 1\text{A}$$

$$\overrightarrow{AB} \cdot (-7\mathbf{i} - \mathbf{j} - 2\mathbf{k}) = 0$$

$$-7(2v - 5) - (v + 4) - 2(-v - 4) = 0$$

$$v = 3 \quad 1\text{A}$$

$$\begin{aligned}
 5. \quad (2\sqrt{3})^2 &= |\mathbf{a} - \mathbf{b}|^2 \\
 12 &= (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) & 1M \\
 &= |\mathbf{a}|^2 - 2\mathbf{a} \cdot \mathbf{b} + |\mathbf{b}|^2 \\
 &= 5^2 - 2\mathbf{a} \cdot \mathbf{b} + 3^2
 \end{aligned}$$

$$\mathbf{a} \cdot \mathbf{b} = 11 \quad 1A$$

$$\begin{aligned}
 (2\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} - 4\mathbf{b}) &= 2|\mathbf{a}|^2 - 7\mathbf{a} \cdot \mathbf{b} - 4|\mathbf{b}|^2 \\
 &= 2(5)^2 - 7(11) - 4(3)^2 \\
 &= -63 \neq 0
 \end{aligned}$$

Thus, $2\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - 4\mathbf{b}$ are not orthogonal vectors. 1A

$$\begin{aligned}
 6. \quad & |3\mathbf{a} - 4\mathbf{b}|^2 = 5^2 \quad \text{and} \quad |3\mathbf{a} + 4\mathbf{b}|^2 = 11^2 \\
 & (3\mathbf{a} - 4\mathbf{b}) \cdot (3\mathbf{a} - 4\mathbf{b}) = 25 \quad (3\mathbf{a} + 4\mathbf{b}) \cdot (3\mathbf{a} + 4\mathbf{b}) = 121 & 1M \\
 & 9|\mathbf{a}|^2 - 24\mathbf{a} \cdot \mathbf{b} + 16|\mathbf{b}|^2 = 25 \quad 9|\mathbf{a}|^2 + 24\mathbf{a} \cdot \mathbf{b} + 16|\mathbf{b}|^2 = 121 & 1A \\
 & \text{Therefore,}
 \end{aligned}$$

$$121 - 25 = 48\mathbf{a} \cdot \mathbf{b} \quad 1M$$

$$\mathbf{a} \cdot \mathbf{b} = 2 \quad 1A$$

$$\begin{aligned}
 7. \quad (a) \quad |\mathbf{a} + \mathbf{b}|^2 &= (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) \\
 &= |\mathbf{a}|^2 + 2\mathbf{a} \cdot \mathbf{b} + |\mathbf{b}|^2 & 1M \\
 &= 4 + 2k + 4 \\
 |\mathbf{a} + \mathbf{b}| &= \sqrt{8 + 2k} & 1A
 \end{aligned}$$

$$\begin{aligned}
 (b) \quad \mathbf{a} \cdot (\mathbf{a} + \mathbf{b}) &= |\mathbf{a}||\mathbf{a} + \mathbf{b}| \cos 30^\circ & 1M \\
 |\mathbf{a}|^2 + \mathbf{a} \cdot \mathbf{b} &= 2\sqrt{8 + 2k} \cos 30^\circ \\
 4 + k &= \sqrt{8 + 2k} \sqrt{3} \\
 k^2 + 8k + 16 &= 24 + 6k \\
 k^2 + 2k - 8 &= 0 & 1M \\
 k &= 2 \quad \text{or} \quad -4
 \end{aligned}$$

When $k = -4$, $\mathbf{a} + \mathbf{b}$ is a zero vector and is not possible to have an angle 30° with $\mathbf{a}$. 1A

$$|\mathbf{a} + \mathbf{b}| = \sqrt{8 + 2(2)} = 2\sqrt{3} > 2 = |\mathbf{a}|$$

The claim is agreed. 1A

$$\begin{aligned}
 8. \quad (a) \quad \overrightarrow{PQ} &= \overrightarrow{OQ} - \overrightarrow{OP} = -\mathbf{i} - 7\mathbf{j} & 1A \\
 \overrightarrow{PR} &= \overrightarrow{OR} - \overrightarrow{OP} = 7\mathbf{i} - 4\mathbf{j} & 1A
 \end{aligned}$$

$$\begin{aligned}
 (b) \quad \overrightarrow{PQ} \cdot \overrightarrow{PR} &= |\overrightarrow{PQ}| |\overrightarrow{PR}| \cos \angle QPR \\
 -7 + 28 &= \sqrt{1^2 + 7^2} \sqrt{7^2 + 4^2} \cos \angle QPR & 1M \\
 \angle QPR &\approx 68^\circ & 1A
 \end{aligned}$$

9. (a) $\overrightarrow{OC} = \overrightarrow{OB} - \overrightarrow{OA} = 4\mathbf{i} + 2\mathbf{j}$ 1A
 $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = 7\mathbf{i} - \mathbf{j}$ 1A
 (b) $|\overrightarrow{OB} \cdot \overrightarrow{AC}| = (OB)(AC)|\cos \theta|$ 1M
 $2 = \sqrt{1^2 + 5^2} \sqrt{7^2 + 1^2} |\cos \theta|$ 1A
 $\theta \approx 93^\circ$ or 87° (rejected) 1A
10. $\overrightarrow{AP} = \frac{1}{4}\overrightarrow{AC} + \frac{3}{4}\overrightarrow{AB}$
 $= \frac{19}{4}\mathbf{i} + 5\mathbf{j}$ 1A
 $\overrightarrow{BC} = \overrightarrow{AC} - \overrightarrow{AB} = -9\mathbf{i} + 4\mathbf{j}$ 1A
 $\overrightarrow{PA} \cdot \overrightarrow{BC} = (PA)(BC) \cos \angle APQ$
 $\frac{91}{4} = \sqrt{\left(\frac{19}{4}\right)^2 + 5^2} \sqrt{9^2 + 4^2} \cos \angle APQ$ 1M
 $\angle APQ \approx 70.4^\circ$ 1A
11. (a) $\overrightarrow{OP} = \frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}$ 1A
 (b) (i) $\overrightarrow{OP} \cdot \overrightarrow{OP} = 3^2$
 $\frac{4}{9}|\mathbf{a}|^2 + \frac{1}{9}|\mathbf{b}|^2 + \frac{4}{9}\mathbf{a} \cdot \mathbf{b} = 9$ 1M
 $4 + 4 + \frac{4}{9}\mathbf{a} \cdot \mathbf{b} = 9$
 $\mathbf{a} \cdot \mathbf{b} = \frac{9}{4}$ 1A
 (ii) $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \angle AOB$ 1M
 $\frac{9}{4} = (3)(6) \cos \angle AOB$
 $\angle AOB = \cos^{-1} \frac{1}{8}$ 1A
12. (a) $\overrightarrow{OR} = \frac{1}{4}\overrightarrow{OP} + \frac{3}{4}\overrightarrow{OQ}$ 1M
 $= 10\mathbf{i} - 2\mathbf{j}$ 1A
 $OR = \sqrt{10^2 + 2^2} = \sqrt{104}$ 1A
 Required unit vector $= \frac{1}{\sqrt{104}}(10\mathbf{i} - 2\mathbf{j})$ 1M
 $= \frac{1}{\sqrt{26}}(5\mathbf{i} - \mathbf{j})$ 1A
 (b) $(4\mathbf{i} + \mathbf{j}) \cdot \left(\frac{1}{\sqrt{26}}(5\mathbf{i} - \mathbf{j})\right) = \sqrt{4^2 + 1^2}(1) \cos \angle ROP$ 1M
 $\cos \angle ROP = \frac{19}{\sqrt{26} \times \sqrt{17}}$ 1M
 $\angle ROP \approx 25^\circ$ 1A

$$\begin{aligned}
13. \quad (a) \quad 0 &= \overrightarrow{OP} \cdot \overrightarrow{CM} \\
&= 4\mathbf{a} \cdot [(2-r)\mathbf{a} - s\mathbf{b}] & 1M \\
&= 4 \left[(2-r)(1) - s \left(\frac{1}{3} \right) \right] & 1M \\
3r + s &= 6 & 1
\end{aligned}$$

(b) Let R be the mid-point of OQ . Then $\overrightarrow{OR} = \mathbf{b}$.

$$\begin{aligned}
0 &= \overrightarrow{OR} \cdot \overrightarrow{CR} \\
&= \mathbf{b} \cdot [-r\mathbf{a} + (1-s)\mathbf{b}] & 1M \\
&= -r \left(\frac{1}{3} \right) + (1-s)(1)
\end{aligned}$$

$$r + 3s = 3 \quad 1A$$

$$\text{Solving, we have } r = \frac{15}{8} \text{ and } s = \frac{3}{8}. \quad 1A$$

$$\begin{aligned}
(c) \quad \overrightarrow{CN} &= (2\mathbf{a} + \mathbf{b}) - \left(\frac{15}{8}\mathbf{a} + \frac{3}{8}\mathbf{b} \right) & 1M \\
&= \frac{1}{8}\mathbf{a} + \frac{5}{8}\mathbf{b} & 1A
\end{aligned}$$

Since $OH \perp PQ$ and $CN \perp PQ$, we have $OH \parallel CN$.

$$\overrightarrow{OH} = k \left(\frac{1}{8}\mathbf{a} + \frac{5}{8}\mathbf{b} \right) \quad 1M$$

$$\overrightarrow{QH} = \frac{k}{8}\mathbf{a} + \left(\frac{5k}{8} - 2 \right) \mathbf{b} \quad 1A$$

Since $QH \perp OP$,

$$\begin{aligned}
0 &= \overrightarrow{QH} \cdot \overrightarrow{OP} \\
&= \frac{4k}{8}\mathbf{a} \cdot \mathbf{a} + 4 \left(\frac{5k}{8} - 2 \right) \mathbf{a} \cdot \mathbf{b} & 1M \\
&= \frac{4k}{8} + \frac{4}{3} \left(\frac{5k}{8} - 2 \right) \\
&= \frac{4(8k - 16)}{8}
\end{aligned}$$

$$k = 2 \quad 1A$$

$$\text{Thus, } \overrightarrow{OH} = 2 \left(\frac{1}{8}\mathbf{a} + \frac{5}{8}\mathbf{b} \right) = \frac{1}{4}\mathbf{a} + \frac{5}{4}\mathbf{b}. \quad 1A$$