

M2REG-LIM-2425-ASM-SET 4-MATH

Suggested solutions

$$\begin{aligned}
 1. \quad \frac{d}{dx}(\sqrt{x}) &= \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} & 1M \\
 &= \lim_{h \rightarrow 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} & 1M \\
 &= \lim_{h \rightarrow 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} & 1M \\
 &= \frac{1}{2\sqrt{x}} & 1A
 \end{aligned}$$

$$\begin{aligned}
 2. \quad g(1+h) - g(1) &= \frac{1}{\sqrt{5(1+h)+4}} - \frac{1}{\sqrt{5+4}} \\
 &= \frac{3 - \sqrt{5h+9}}{3\sqrt{5h+9}} \times \frac{3 + \sqrt{5h+9}}{3 + \sqrt{5h+9}} & 1M \\
 &= \frac{-5h}{3\sqrt{5h+9}(3 + \sqrt{5h+9})} & 1 \\
 g'(1) &= \lim_{h \rightarrow 0} \frac{g(1+h) - g(1)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{-5}{3\sqrt{5h+9}(3 + \sqrt{5h+9})} & 1M \\
 &= \frac{-5}{3\sqrt{9}(3 + \sqrt{9})} \\
 &= -\frac{5}{54} & 1A
 \end{aligned}$$

$$\begin{aligned}
 3. \quad \frac{d}{dx}(\cos 3x) &= \lim_{h \rightarrow 0} \frac{\cos[3(x+h)] - \cos 3x}{h} & 1M \\
 &= \lim_{h \rightarrow 0} \frac{-2 \sin\left(3x + \frac{3h}{2}\right) \sin \frac{3h}{2}}{h} & 1M \\
 &= \lim_{h \rightarrow 0} \left[-\frac{\sin \frac{3h}{2}}{\frac{3h}{2}} \cdot 3 \sin\left(3x + \frac{3h}{2}\right) \right] & 1M \\
 &= -3 \sin 3x & 1A
 \end{aligned}$$

$$\begin{aligned}
 4. \quad \frac{d}{dx}(\tan x) &= \lim_{h \rightarrow 0} \frac{\tan(x+h) - \tan x}{h} & 1M \\
 &= \lim_{h \rightarrow 0} \frac{\tan[(x+h) - x][1 + \tan x \tan(x+h)]}{h} & 1M \\
 &= \lim_{h \rightarrow 0} \left[\frac{\sin h}{h} \cdot \frac{1 + \tan x \tan(x+h)}{\cos h} \right] & 1M \\
 &= 1 \cdot (1 + \tan^2 x) \\
 &= \sec^2 x & 1A
 \end{aligned}$$

$$\begin{aligned}
5. \quad \frac{d}{dx}(\sec 4x) &= \lim_{h \rightarrow 0} \frac{\sec[4(x+h)] - \sec 4x}{h} & 1M \\
&= \lim_{h \rightarrow 0} \frac{\cos 4x - \cos(4x+4h)}{h \cos 4x \cos(4x+4h)} \\
&= \lim_{h \rightarrow 0} \frac{2 \sin(4x+2h) \sin 2h}{h \cos 4x \cos(4x+4h)} & 1M \\
&= \lim_{h \rightarrow 0} \left[\frac{\sin 2h}{2h} \cdot \frac{4 \sin(4x+2h)}{\cos 4x \cos(4x+4h)} \right] & 1M \\
&= \frac{4 \sin 4x}{\cos^2 4x} \\
&= 4 \sec 4x \tan 4x & 1A
\end{aligned}$$

$$\begin{aligned}
6. \quad f'(0) &= \lim_{h \rightarrow 0} \frac{e^{2(0+h)} - e^{2(0)}}{h} & 1M \\
&= \lim_{h \rightarrow 0} \frac{e^{2h} - 1}{2h} \cdot 2 & 1M \\
&= 2 & 1A
\end{aligned}$$

$$\begin{aligned}
7. \quad \frac{d}{dx}(e^{x^2}) &= \lim_{h \rightarrow 0} \frac{e^{(x+h)^2} - e^{x^2}}{h} & 1M \\
&= \lim_{h \rightarrow 0} \frac{e^{x^2}(e^{2hx+h^2} - 1)}{h} & 1M \\
&= e^{x^2} \lim_{h \rightarrow 0} \left[\frac{e^{h(2x+h)} - 1}{h(2x+h)} \cdot (2x+h) \right] & 1M \\
&= e^{x^2} \cdot 1 \cdot 2x \\
&= 2xe^{x^2} & 1A
\end{aligned}$$

$$\begin{aligned}
8. \quad \text{Let } u &= x - \frac{\pi}{8}. & 1M \\
\lim_{x \rightarrow \frac{\pi}{8}} \frac{x - \frac{\pi}{8}}{\cos\left(x + \frac{11\pi}{8}\right)} &= \lim_{u \rightarrow 0} \frac{u}{\cos\left(\frac{3\pi}{2} + u\right)} \\
&= \lim_{u \rightarrow 0} \frac{u}{\sin u} & 1M \\
&= 1 & 1A
\end{aligned}$$

$$\begin{aligned}
9. \quad \lim_{x \rightarrow 0} \frac{4x^2}{e^{2x^3} - e^{3x^2}} &= \lim_{x \rightarrow 0} \frac{4x^2}{e^{3x^2}(e^{2x^3-3x^2} - 1)} \\
&= \lim_{x \rightarrow 0} \frac{2x^3 - 3x^2}{e^{2x^3-3x^2} - 1} \cdot \frac{4}{e^{3x^2}(2x-3)} & 1M \\
&= 1 \cdot \frac{4}{0-3} \\
&= -\frac{4}{3} & 1A
\end{aligned}$$

$$\begin{aligned}
10. \quad \lim_{x \rightarrow 0} \frac{e^{3x} + e^{-3x} - 2}{x^2} &= \lim_{x \rightarrow 0} \frac{e^{6x} + 1 - 2e^{3x}}{x^2 e^{3x}} \\
&= \lim_{x \rightarrow 0} \frac{(e^{3x} - 1)^2}{x^2 e^{3x}} && 1\text{M} \\
&= \lim_{x \rightarrow 0} \left[\left(\frac{e^{3x} - 1}{3x} \right)^2 \cdot \frac{9}{e^{3x}} \right] && 1\text{M} \\
&= 1^2 \cdot 9 \\
&= 9 && 1\text{A}
\end{aligned}$$

$$\begin{aligned}
11. \quad (a) \quad \lim_{x \rightarrow 0} \frac{\sqrt{1+x} - 1}{e^{x+1} - e} &= \lim_{x \rightarrow 0} \frac{\sqrt{1+x} - 1}{e(e^x - 1)} \cdot \frac{\sqrt{1+x} + 1}{\sqrt{1+x} + 1} && 1\text{M} \\
&= \lim_{x \rightarrow 0} \frac{x}{e^x - 1} \cdot \frac{1}{e(\sqrt{1+x} + 1)} && 1\text{M} \\
&= 1 \cdot \frac{1}{e(1+1)} \\
&= \frac{1}{2e} && 1\text{A} \\
(b) \quad \lim_{x \rightarrow 0} \frac{e^{5x} - \sin 3x - 1}{e^{-2x} - \cos 4x} &= \lim_{x \rightarrow 0} \frac{e^{5x} - \sin 3x - 1}{e^{-2x} - (1 - 2 \sin^2 2x)} && 1\text{M} \\
&= \lim_{x \rightarrow 0} \frac{\frac{e^{5x}-1}{x} - \frac{\sin 3x}{x}}{\frac{e^{-2x}-1}{x} + \frac{2 \sin^2 2x}{x}} && 1\text{M} \\
&= \lim_{x \rightarrow 0} \frac{\frac{e^{5x}-1}{5x} \cdot 5 - \frac{\sin 3x}{3x} \cdot 3}{\frac{e^{-2x}-1}{-2x} \cdot (-2) + \frac{\sin 2x}{2x} \cdot 4 \sin 2x} && 1\text{M} \\
&= \frac{1 \cdot 5 - 1 \cdot 3}{1 \cdot (-2) + 1 \cdot 0} \\
&= -1 && 1\text{A}
\end{aligned}$$

$$\begin{aligned}
12. \quad (a) \quad \lim_{x \rightarrow 0} \frac{e^{2x} + e^{-2x} - 2}{\cos^2 2x - \cos 2x} &= \lim_{x \rightarrow 0} \frac{(e^x - e^{-x})^2}{\cos 2x(\cos 2x - 1)} & 1M \\
&= \lim_{x \rightarrow 0} \frac{e^{-2x}(e^{2x} - 1)^2}{\cos 2x(-2 \sin^2 x)} & 1M \\
&= \lim_{x \rightarrow 0} \left(\frac{e^{2x} - 1}{2x} \right)^2 \left(\frac{x}{\sin x} \right)^2 \cdot \frac{2e^{-2x}}{-\cos 2x} & 1M \\
&= 1^2 \cdot 1^2 \cdot (-2) \\
&= -2 & 1A
\end{aligned}$$

$$\begin{aligned}
(b) \quad \lim_{x \rightarrow 0} \frac{\sqrt{\cos 2x} - 1}{2^{2x} - 2^{x+1} + 1} & \\
&= \lim_{x \rightarrow 0} \frac{\sqrt{\cos 2x} - 1}{(2^x - 1)^2} \cdot \frac{\sqrt{\cos 2x} + 1}{\sqrt{\cos 2x} + 1} & 1M \\
&= \lim_{x \rightarrow 0} \frac{\cos 2x - 1}{(2^x - 1)^2(\sqrt{\cos 2x} + 1)} \\
&= \lim_{x \rightarrow 0} \frac{-2 \sin^2 x}{(e^{x \ln 2} - 1)^2(\sqrt{\cos 2x} + 1)} & 1M \\
&= \lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \right)^2 \left(\frac{x \ln 2}{e^{x \ln 2} - 1} \right)^2 \cdot \frac{-2}{(\ln 2)^2(\sqrt{\cos 2x} + 1)} & 1M \\
&= 1^2 \cdot 1^2 \cdot \frac{-2}{(\ln 2)^2(1 + 1)} \\
&= -\frac{1}{(\ln 2)^2} & 1A
\end{aligned}$$

$$\begin{aligned}
13. \quad \lim_{x \rightarrow 0} \frac{(e^x - e^{-x})^2}{1 - \cos 2x} &= \lim_{x \rightarrow 0} \frac{(e^{2x} - 1)^2}{e^{2x}[1 - (1 - 2 \sin^2 x)]} & 1M \\
&= \lim_{x \rightarrow 0} \left[\left(\frac{e^{2x} - 1}{2x} \right)^2 \cdot \left(\frac{x}{\sin x} \right)^2 \cdot \frac{2}{e^{2x}} \right] & 1M \\
&= 1^2 \cdot 1^2 \cdot 2 \\
&= 2 & 1A
\end{aligned}$$