REG-COT-2425-ASM-SET 4-MATH
Suggested solutions

Conventional Questions

1. $20^2 = 14^2 + 30^2 - 2(14)(30)\cos\angle BAC$ 1M

$\angle BAC \approx 34.0°$
O is the orthocentre of $\triangle ABC$.
We have $AE \perp OB$. 1M
$CE = AE - AC$
$= 30\cos\angle BAC - 14$ 1M
$\approx 10.9\,\text{cm}$ 1A

2. $\angle ABC = 180° - 50° - 58° = 72°$
Consider $\triangle OBC$.
$\angle OBC = \dfrac{72°}{2} = 36°$ 1M
$\angle OCB = \dfrac{58°}{2} = 29°$
$\angle BOC = 180° - 36° - 29° = 115°$
$\dfrac{OB}{\sin 29°} = \dfrac{7}{\sin 115°}$ 1M
$OB \approx 3.74\,\text{cm}$

Let r cm be the radius of the inscribed circle.
$\sin 36° = \dfrac{r}{OB}$ 1M
$r \approx 2.20$ 1A

Required radius is 2.20 cm.

3. (a) (i) Since G is centroid of $\triangle OAB$, P and Q are mid-points of AB and OA respectively.
$PQ // OB$ and $QP = \dfrac{1}{2}OB$ *(mid-point theorem)*
In $\triangle PGQ$ and $\triangle OGB$,

$\angle PGQ = \angle OGB$ *(vert. opp. ∠s)*
$\angle PQG = \angle OBG$ *(alt. ∠s, $QP // OB$)*
$\angle QPG = \angle BOG$ *(alt. ∠s, $QP // OB$)*
$\triangle PGQ \sim \triangle OGB$ *(AAA)*

Marking Scheme		
Case 1	Any correct proof with correct reasons.	3
Case 2	Any correct proof without reasons.	2
Case 3	Incomplete proof with any one correct step with reason.	1

(ii) $\triangle PGQ \sim \triangle OGB$ *(proved)*

$$\frac{GQ}{BG} = \frac{QP}{OB} \quad \textit{(corr. sides., } \sim \triangle s\textit{)}$$

$$= \frac{\frac{1}{2}OB}{OB} \quad \textit{(proved)}$$

$$= \frac{1}{2}$$

Thus, $BG : GQ = 2 : 1$.

Marking Scheme		
Case 1	Any correct proof with correct reasons.	2
Case 2	Any correct proof without reasons.	1

(b) (i) Let $(m,\ n)$ and $(b,\ 0)$ be the coordinates of Q and B respectively.

Since $BG : GQ = 2 : 1$,

$$\frac{n-1}{1-0} = \frac{1}{2}$$

$$n = \frac{3}{2}$$

Since Q lies on OA, when $n = \frac{3}{2}$, $m = 2 \times \frac{3}{2} = 3$.

The coordinates of Q are $\left(3,\ \frac{3}{2}\right)$. 1M

$$\frac{b-5}{5-3} = \frac{2}{1}$$

$$b = 9$$

The coordinates of B are $(9,\ 0)$. 1A

(ii) Let $S\ (p,\ q)$ be the circumcentre of $\triangle OAB$.

Since S lies on the perpendicular bisector of OB, $p = \frac{9}{2}$. 1M

Since $OA \perp SQ$,

$$\frac{1}{2} \times \frac{q - \frac{3}{2}}{\frac{9}{2} - 3} = -1$$ 1M+1M

$$q = -\frac{3}{2}$$

The coordinates of the circumcentre are $\left(\frac{9}{2},\ -\frac{3}{2}\right)$. 1A

4. (a) (i) $\angle BOD = 90°$ *(property of orthocentre)*

$\angle ACB = 90°$ *($\angle$ in semi-circle)*

$= \angle BOD$

Thus, B, C, O and D are concyclic. *(converse of $\angle$s in the same segment)*

Marking Scheme		
Case 1	Any correct proof with correct reasons.	2
Case 2	Any correct proof without reasons.	1

(ii) Since incentre of $\triangle BCD$ lies on AB, $\angle DBA = \angle ABC$. 1

Since B, O, C and D are concyclic, $\angle DBA = \angle DCO$. 1

So, $\angle DCO = \angle ABC$ and OC is a tangent to the circle ABC.

The claim is agreed. 1

(b) $\angle DBO = \angle CBO$ *(proved)*

$\angle CDO = \angle CBO$ *($\angle$s in the same segment)*

$= \angle DBO$

$\angle DOB = \angle AOD$ *(common $\angle$)*

$\triangle ADO \sim \triangle DBO$ *(AA)*

$$\frac{OD}{AO} = \frac{OB}{OD}$$ 1M

$$OD^2 = 2 \cdot 4$$

$$OD = 2\sqrt{2}$$

Coordinate of D are $\left(0,\ 2\sqrt{2}\right)$. 1A

Since $\angle DOB = 90°$, DB is a diameter of the circle.

Coordinates of centre are $\left(2,\ \sqrt{2}\right)$. Required equation is

$$(x-2)^2 + (y-\sqrt{2})^2 = (0-2)^2 + (0-\sqrt{2})^2$$ 1M

$$(x-2)^2 + (y-\sqrt{2})^2 = 6$$ 1A

5. (a) $f(x) = \frac{1}{2}(4x^2 + 16kx + 16k^2 + 4k - 2)$

$= 2x^2 + 8kx + 8k^2 + 2k - 1$ 1A

(b) $f(x) = 2x^2 + 8kx + 8k^2 + 2k - 1$

$= 2[x^2 + 2(x)(2k) + (2k)^2] + 2k - 1$ 1M

$= 2(x + 2k)^2 + 2k - 1$

The coordinates of the vertex are $(-2k,\ 2k - 1)$. 1A

(c) The coordinates of P are $(-2k,\ 2k - 1)$.

The coordinates of Q are $(2k,\ 1 - 2k)$. 1A

The coordinates of the mid-point of PQ are $(0,\ 0)$. 1M

Thus, O is the mid-point of PQ.

OR is a median of $\triangle PQR$, and passes through G.

The claim is agreed. 1A

6. (a) $$\frac{\text{area of } \triangle BAD}{\text{area of } \triangle CAD} = \frac{BD}{DC}$$

$$\frac{\frac{1}{2}(AB)(AD)\sin\angle BAD}{\frac{1}{2}(AC)(AD)\sin\angle CAD} = \frac{BD}{DC}$$ 1M

$$\frac{AB}{AC} = \frac{BD}{DC}$$ 1

(b) (i) $AC = \sqrt{(24k)^2 + (18k)^2} = 30k$

$$\frac{BD}{DC} = \frac{AB}{AC}$$

$$= \frac{15}{30k}$$ 1M

$$= \frac{1}{2k}$$

Let $(a,\ b)$ be the coordinates of D.

$$\frac{a-0}{24k-a} = \frac{1}{2k} \quad \text{and} \quad \frac{15-b}{b-18k} = \frac{1}{2k}$$ 1M

$$a = \frac{24k}{2k+1} \qquad b = \frac{48k}{2k+1}$$

Slope of $AD = \frac{48k}{2k+1} \div \frac{24k}{2k+1} = 2$ 1

(ii) (1) The x-coordinate of I is 3.

Since I lies on the line $y = 2x$, the coordinates of I are $(3,\ 6)$. 1M

Equation of BI is

$$y - 15 = \frac{15-6}{0-3}(x-0)$$ 1M

$$y = -3x + 15$$

Equation of AC is

$$y - 0 = \frac{18k-0}{24k-0}(x-0)$$

$$y = \frac{3x}{4}$$

Solving, we have $x = 4$ and $y = 3$.

Required coordinates are $(4,\ 3)$. 1A

(2) Denote the intersection of AC and BI by E.

$$\frac{AE}{EC} = \frac{BA}{BC}$$

$$\frac{\sqrt{4^2+3^2}}{\sqrt{(24k-4)^2+(18k-3)^2}} = \frac{15}{\sqrt{(24k)^2+(18k-15)^2}}$$ 1M

$$(24k)^2 + (18k-15)^2 = 9[(24k-4)^2 + (18k-3)^2]$$

$$0 = 7200k^2 - 2160k$$

$$k = 0.3 \quad \text{or} \quad 0 \text{ (rejected)}$$

The coordinates of C are $(7.2,\ 5.4)$.

$AC^2 + BC^2 = (7.2^2 + 5.4^2) + (7.2^2 + (15-5.4)^2) = 225$

$AB^2 = 15^2 = 225 = AC^2 + BC^2$ 1M

Thus, $\angle ACB = 90°$.

AB is a diameter of the circumcircle of $\triangle ABC$.

The claim is agreed. 1A

7. (a) (i) $r^2 = \left(\frac{k-4}{-2}\right)^2 + \left(\frac{2+k}{-2}\right)^2 - (-3k-6)$ 1M

$= \frac{k^2}{2} + 2k + 11$ 1A

(ii) $r^2 = \frac{k^2}{2} + 2k + 11$

$= \frac{1}{2}(k^2 + 4k + 4) + 9$ 1M

$= \frac{1}{2}(k+2)^2 + 9$

$r = \sqrt{\frac{1}{2}(k+2)^2 + 9}$

When $k = -2$, the value of r is the least.

Required equation is $x^2 + y^2 - 6x = 0$. 1A

(b) Equation of L is $y = mx + h$. 1A

$$x^2 + (mx+h)^2 - 6x = 0$$ 1M

$$(1+m^2)x^2 + (2mh-6)x + h^2 = 0$$

The equation has only one real root.

$$\Delta = (2mh-6)^2 - 4(1+m^2)h^2 = 0$$ 1M

$$-24mh - 4h^2 + 36 = 0$$

$$m = \frac{9-h^2}{6h}$$ 1

(c) Coordinates of A are $(3,\ 0)$.

Let M be the mid-point of OA, then $M\left(\frac{3}{2},\ 0\right)$.

Coordinates of centroid $= \left(\frac{1(0) + 2\left(\frac{3}{2}\right)}{1+2},\ \frac{1(h) + 2(0)}{1+2}\right)$ 1M

$= \left(1,\ \frac{h}{3}\right)$

Centroid lies on the circle.

$$1^2 + \left(\frac{h}{3}\right)^2 - 6(1) = 0$$ 1M

$$h^2 = 45$$

$$h = \sqrt{45} \quad \text{or} \quad -\sqrt{45} \text{ (rejected)}$$

Slope of $PQ = m$

$$\frac{\sqrt{45} - 0}{0 - q} = \frac{9 - (\sqrt{45})^2}{6(\sqrt{45})}$$ 1M

$$q = \frac{15}{2}$$

$$\text{Area of } \triangle OPQ = \frac{(\sqrt{45})\left(\frac{15}{2}\right)}{2}$$ 1M

$$\approx 25.2$$

$$> 25$$

The claim is disagreed. 1A

END OF PAPER