1. (a) 在 $\triangle OAY$ 中，$\dfrac{y}{\sin\theta} = \dfrac{OY}{\sin\angle OAY}$。 1M

在 $\triangle OBY$ 中，$\dfrac{1-y}{\sin 3\theta} = \dfrac{OY}{\sin\angle OBY}$。 1M

由於 $\angle OAY = \angle OBY$ （等腰 △ 底角）

$$\frac{y}{\sin\theta} = \frac{1-y}{3\sin\theta - 4\sin^3\theta} \quad \text{1M}$$

$$3y - 4y\sin^2\theta = 1 - y$$

$$4y(1-\sin^2\theta) = 1$$

$$y = \frac{1}{4}\sec^2\theta \quad \text{1}$$

(b) 由於 $0° < 4\theta < 180°$，可得 1M

$$0 < \theta < 45°$$

$$\frac{1}{4}\sec^2 0° < y < \frac{1}{4}\sec^2 45°$$

$$\frac{1}{4} < y < \frac{1}{2} \quad \text{1A}$$

2. (a) $\dfrac{\tan\theta}{1-\cot\theta} + \dfrac{\cot\theta}{1-\tan\theta}$

$$= \frac{\tan^3\theta}{\tan\theta(\tan\theta - 1)} + \frac{1}{\tan\theta(1-\tan\theta)} \quad \text{1M}$$

$$= \frac{\tan^3\theta - 1}{\tan\theta(\tan\theta - 1)}$$

$$= \frac{(\tan\theta - 1)(\tan^2\theta + \tan\theta + 1)}{\tan\theta(\tan\theta - 1)} \quad \text{1M}$$

$$= \frac{\sec^2\theta + \tan\theta}{\tan\theta}$$

$$= 1 + \sec\theta\csc\theta \quad \text{1}$$

(b) $\dfrac{\tan\theta}{1-\cot\theta} + \dfrac{\cot\theta}{1-\tan\theta} = 5$

$$1 + \sec\theta\csc\theta = 5$$

$$\frac{2}{\sin 2\theta} = 4 \quad \text{1M}$$

$$\sin 2\theta = \frac{1}{2}$$

$$2\theta = \frac{5\pi}{6}$$

$$\theta = \frac{5\pi}{12} \quad \text{1A}$$

3. (a) $\dfrac{\sin 2x}{1+\cos 2x} = \dfrac{2\sin x\cos x}{1+(2\cos^2 x-1)}$ 1M

$= \dfrac{\sin x}{\cos x}$

$= \tan x$ 1

(b) $\dfrac{\sin 8y\cos 4y\cos 2y}{(1+\cos 8y)(1+\cos 4y)(1+\cos 2y)} = \tan 4y\cdot\dfrac{\cos 4y\cos 2y}{(1+\cos 4y)(1+\cos 2y)}$ 1M

$= \dfrac{\sin 4y\cos 2y}{(1+\cos 4y)(1+\cos 2y)}$ 1M

$= \tan 2y\cdot\dfrac{\cos 2y}{1+\cos 2y}$

$= \dfrac{\sin 2y}{1+\cos 2y}$

$= \tan y$ 1

4. (a) (i) $\tan 3x = \dfrac{\tan 2x+\tan x}{1-\tan x\tan 2x}$ 1M

$= \dfrac{\frac{2\tan x}{1-\tan^2 x}+\tan x}{1-\frac{2\tan^2 x}{1-\tan^2 x}}$

$= \dfrac{2\tan x+\tan x(1-\tan^2 x)}{1-\tan^2 x-2\tan^2 x}$

$= \dfrac{3\tan x-\tan^3 x}{1-3\tan^2 x}$ 1

(ii) $\tan x\tan(60^\circ-x)\tan(60^\circ+x)$

$= \tan x\left(\dfrac{\tan 60^\circ-\tan x}{1+\tan 60^\circ\tan x}\right)\left(\dfrac{\tan 60^\circ+\tan x}{1-\tan 60^\circ\tan x}\right)$

$= \tan x\times\dfrac{\sqrt{3}-\tan x}{1-\sqrt{3}\tan x}\times\dfrac{\sqrt{3}+\tan x}{1+\sqrt{3}\tan x}$ 1M

$= \dfrac{3\tan x-\tan^3 x}{1-3\tan^2 x}$

$= \tan 3x$ 1

(b) $\tan 5^\circ\tan(60^\circ-5^\circ)\tan(60^\circ+5^\circ) = \tan[3(5^\circ)]$ 1M

$\tan 5^\circ\tan 55^\circ\tan 65^\circ = 15^\circ$

$\left(\dfrac{1}{\tan 85^\circ}\right)\tan 55^\circ\tan 65^\circ = \dfrac{1}{\tan 75^\circ}$

$\tan 55^\circ\tan 65^\circ\tan 75^\circ = \tan 15^\circ$ 1

5. (a) $\sin^2 x \cos^2 x = \dfrac{\sin^2 2x}{4}$ 1M

$= \dfrac{1}{4}\left(\dfrac{1-\cos 4x}{2}\right)$

$= \dfrac{1-\cos 4x}{8}$ 1

(b) (i) $f(x) = (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x$ 1M

$= 1 - 2\left(\dfrac{1-\cos 4x}{8}\right)$ 1M

$= \dfrac{1}{4}\cos 4x + \dfrac{3}{4}$ 1A

(ii) $8\left(\dfrac{1}{4}\cos 4x + \dfrac{3}{4}\right) = 7$ 1M

$\cos 4x = \dfrac{1}{2}$

$4x = \dfrac{\pi}{3}$ 或 $\dfrac{5\pi}{3}$

$x = \dfrac{\pi}{12}$ 或 $\dfrac{5\pi}{12}$ 1A

6. (a) $\sin 3x = \sin(2x + x)$

$= \sin 2x \cos x + \cos 2x \sin x$ 1M

$= 2\sin x \cos^2 x + (1 - 2\sin^2 x)\sin x$

$= 2\sin x(1 - \sin^2 x) + \sin x - 2\sin^3 x$

$= 3\sin x - 4\sin^3 x$ 1

(b) (i) $\dfrac{\sin\left[3\left(x-\frac{\pi}{4}\right)\right]}{\sin\left(x-\frac{\pi}{4}\right)} = \dfrac{\sin 3x \cos\frac{3\pi}{4} - \sin\frac{3\pi}{4}\cos 3x}{\sin x \cos\frac{\pi}{4} - \cos x \sin\frac{\pi}{4}}$ 1M

$= \dfrac{-\frac{1}{\sqrt{2}}(\cos 3x + \sin 3x)}{\frac{1}{\sqrt{2}}(\sin x - \cos x)}$

$= \dfrac{\cos 3x + \sin 3x}{\cos x - \sin x}$ 1

(ii) $\dfrac{\cos 3x + \sin 3x}{\cos x - \sin x} = 2$

$\dfrac{\sin 3\left(x-\frac{\pi}{4}\right)}{\sin\left(x-\frac{\pi}{4}\right)} = 2$ 1M

$\dfrac{3\sin\left(x-\frac{\pi}{4}\right) - 4\sin^3\left(x-\frac{\pi}{4}\right)}{\sin\left(x-\frac{\pi}{4}\right)} = 2$

$3 - 4\sin^2\left(x-\dfrac{\pi}{4}\right) = 2$ 1M

$\sin\left(x-\dfrac{\pi}{4}\right) = \dfrac{1}{2}$ 或 $-\dfrac{1}{2}$（捨去） 1M

$x - \dfrac{\pi}{4} = \dfrac{\pi}{6}$

$x = \dfrac{5\pi}{12}$ 1A

7. (a) $\cot A = 3\cot B$

$\cos A\sin B = 3\cos B\sin A$ 1M

$$\begin{aligned}\sin(A+B) - 2\sin(B-A) &= (\sin A\cos B + \cos A\sin B) - 2(\sin B\cos A - \cos B\sin A)\\ &= 3\sin A\cos B - \cos A\sin B\\ &= 0\end{aligned}$$

因此，$\sin(A+B) = 2\sin(B-A)$。 1

(b) 據 (a)，該方程可被重寫成

$$\sin\left[\left(x+\frac{4\pi}{9}\right)+\left(x+\frac{5\pi}{18}\right)\right] = 2\sin\left[\left(x+\frac{5\pi}{18}\right)-\left(x+\frac{4\pi}{9}\right)\right]$$ 1M

$$\begin{aligned}\sin\left(2x+\frac{13\pi}{18}\right) &= 2\sin\left(-\frac{\pi}{6}\right)\\ &= -1\end{aligned}$$ 1M

$$2x+\frac{13\pi}{18} = \frac{3\pi}{2}$$

$$x = \frac{7\pi}{18}$$ 1A

8. (a) $$\begin{aligned}\cos 3x &= \cos(2x+x)\\ &= \cos 2x\cos x - \sin 2x\sin x\end{aligned}$$ 1M

$$\begin{aligned}&= (2\cos^2 x - 1)\cos x - (2\sin x\cos x)\sin x\\ &= 2\cos^3 x - \cos x - 2\cos x(1-\cos^2 x)\\ &= 4\cos^3 x - 3\cos x\end{aligned}$$ 1

(b) $$\begin{aligned}\sec^3 x - 6\sec^2 x + 8 &= 0\\ (\cos^3 x)(\sec^3 x - 6\sec^2 x + 8) &= 0\\ 1 + 2(4\cos^3 x - 3\cos x) &= 0\\ \cos 3x &= -\frac{1}{2}\end{aligned}$$ 1M

$$3x = \frac{8\pi}{3} \quad 或 \quad \frac{10\pi}{3}$$

$$x = \frac{8\pi}{9} \quad 或 \quad \frac{10\pi}{9}$$ 1A+1A

9. (a) $4(-1)^3 + 2(-1)^2 - 3(-1) - 1 = -4 + 2 + 3 - 1 = 0$
因此，$x + 1$ 為 $4x^3 + 2x^2 - 3x - 1$ 的因式。 1

(b) $\cos 3\theta = \cos(\theta + 2\theta)$
$= \cos\theta\cos 2\theta - \sin\theta\sin 2\theta$ 1M
$= \cos\left(\cos^2\theta - \sin^2\theta\right) - \sin(2\sin\theta\cos\theta)$
$= \cos\theta[\cos^2\theta - (1 - \cos^2\theta)] - 2\cos\theta(1 - \cos^2\theta)$
$= 4\cos^3\theta - 3\cos\theta$ 1A

(c) 代 $x = \cos\dfrac{3\pi}{5}$ 至 $4x^3 + 2x^2 - 3x - 1$，

$4\cos^3\dfrac{3\pi}{5} + 2\cos^2\dfrac{3\pi}{5} - 3\cos\dfrac{3\pi}{5} - 1$
$= \left(4\cos^3\dfrac{3\pi}{5} - 3\cos\dfrac{3\pi}{5}\right) + \left(2\cos^2\dfrac{3\pi}{5} - 1\right)$
$= \cos\dfrac{9\pi}{5} + \cos\dfrac{6\pi}{5}$ 1M
$= \cos\left(3\pi - \dfrac{6\pi}{5}\right) + \cos\dfrac{6\pi}{5}$
$= -\cos\dfrac{6\pi}{5} + \cos\dfrac{6\pi}{5}$
$= 0$ 1M

故此，$x = \cos\dfrac{3\pi}{5}$ 為方程 $4x^3 + 2x^2 - 3x - 1 = 0$ 的根。
據 (a)，

$$4x^3 + 2x^2 - 3x - 1 = (x+1)(4x^2 - 2x - 1)$$
$$= (x+1)\left(x - \frac{1+\sqrt{5}}{4}\right)\left(x - \frac{1-\sqrt{5}}{4}\right)$$

由於 $\dfrac{\pi}{2} < \dfrac{3\pi}{5} < \pi$，$-1 < \cos\dfrac{3\pi}{5} < 0$。
因此，可得 $\cos\dfrac{3\pi}{5} = \dfrac{1-\sqrt{5}}{4}$。 1

10. 設 $t = \tan 22.5°$ 。

$$\tan 45° = \frac{2\tan 22.5°}{1-\tan^2 22.5°} \qquad \text{1M}$$

$$1 = \frac{2t}{1-t^2} \qquad \text{1A}$$

$$0 = t^2 + 2t - 1$$

$$t = \frac{-2+\sqrt{8}}{2} \quad \text{或} \quad \frac{-2-\sqrt{8}}{2}\ \text{（捨去）} \qquad \text{1A}$$

$$= \sqrt{2} - 1 \qquad \text{1A}$$

11. $$\frac{\sin 3\theta}{\sin\theta} + \frac{\cos 3\theta}{\cos\theta} = \frac{\sin 3\theta\cos\theta + \cos 3\theta\sin\theta}{\sin\theta\cos\theta}$$

$$= \frac{\sin 4\theta}{\frac{1}{2}\sin 2\theta} \qquad \text{1A+1A}$$

$$= \frac{2\sin 2\theta\cos 2\theta}{\frac{1}{2}\sin 2\theta} \qquad \text{1A}$$

$$= 4\cos 2\theta \qquad 1$$

12. $$\frac{\tan x - \sin^2 x}{\tan x + \sin^2 x} = \frac{\frac{\sin x}{\cos x} - \sin^2 x}{\frac{\sin x}{\cos x} + \sin^2 x}$$

$$= \frac{\sin x(1-\sin x\cos x)}{\sin x(1+\sin x\cos x)}$$

$$= \frac{1-\sin x\cos x}{1+\sin x\cos x} \qquad \text{1A}$$

$$= \frac{1-\frac{1}{2}\sin 2x}{1+\frac{1}{2}\sin 2x} \qquad \text{1M}$$

$$= \frac{2-\left(1+\frac{1}{2}\sin 2x\right)}{1+\frac{1}{2}\sin 2x}$$

$$= \frac{4}{2+\sin 2x} - 1 \qquad 1$$

$$\text{最小值} = \frac{4}{2+(1)} - 1 \qquad \text{1M}$$

$$= \frac{1}{3} \qquad \text{1A}$$

13. (a) $\sin C = \sin(\pi - (A + B))$ 1A

$= \sin(A + B)$

$= \sin A \cos B + \cos A \sin B$

由於 A 及 B 為銳角，$\cos A = \sqrt{1 - \sin^2 A} = \dfrac{12}{13}$ 及 $\cos B = \dfrac{4}{5}$。 1A

$\sin C = \dfrac{5}{13} \times \dfrac{4}{5} + \dfrac{12}{13} \times \dfrac{3}{5}$ 1A

$= \dfrac{56}{65}$ 1A

(b) 利用正弦定理，

$a : b : c = \sin A : \sin B : \sin C = \dfrac{5}{13} : \dfrac{3}{5} : \dfrac{56}{65} = 25 : 39 : 56$ 1M

所求長度 $= c = \dfrac{56}{25 + 39 + 56} \times 12 = 5.6\,\text{cm}$ 1A

14. (a) $\tan 3\theta = \dfrac{2\tan\theta + \tan\theta}{1 - \tan 2\theta \tan\theta}$ 1A

$= \dfrac{\frac{2\tan\theta}{1-\tan^2\theta} + \tan\theta}{1 - \frac{2\tan^2\theta}{1-\tan^2\theta}}$

$= \dfrac{2\tan\theta + \tan\theta - \tan^3\theta}{1 - \tan^2\theta - 2\tan^2\theta}$

$= \dfrac{\tan^3\theta - 3\tan\theta}{3\tan^2\theta - 1}$ 1

(b) (i) 可得

$$\begin{cases} f(1) = 3 + m - 9 + n = -8 \\ f(2) = 3(8) + 4m - 18 + n = -5 \end{cases}$$ 1M

求解後，可得 $m = -3$ 及 $n = 1$。 1A

(ii) 代 $x = \tan\theta$，

$3\tan^3\theta - 3\tan^2\theta - 9\tan\theta + 1 = 0$ 1M

$3(\tan^3\theta - 3\tan\theta) = 3\tan^2\theta - 1$

$(3\tan^2\theta - 1)(3\tan 3\theta - 1) = 0$ 1M

$\tan 3\theta = \dfrac{1}{3}$ 或 $\tan\theta = \pm\dfrac{1}{\sqrt{3}}$ （捨去） 1A

$3\theta \approx 18.4^\circ$ 或198° 或378°

$\theta \approx 6.14^\circ$ 或66.1° 或126° 1A

由於 $x = \tan\theta$，所求之根為 0.11、2.26 及 -1.37。 1A

15. (a) (i) $\tan x = k \tan y$

$\sin x \cos y = k \cos x \sin y$ 1A

$\sin(x+y) = \sin x \cos y + \cos x \sin y$ 1A

$= k \cos x \sin y + \cos x \sin y$

$= (k+1) \cos x \sin y$ 1

(ii) $(k+1)\sin(x-y) = (k+1)(\sin x \cos y - \cos x \sin y)$ 1A

$= (k+1)(k \cos x \sin y - \cos x \sin y)$

$= (k+1)(k-1)\cos x \sin y$

$= (k-1)\sin(x+y)$ 1M+1

(b) (i) $(k+1)\sin[(\theta+10^\circ)-(\theta-20^\circ)] = (k-1)\sin[(\theta+10^\circ)-(\theta-20^\circ)]$

$(k+1)\sin 30^\circ = (k-1)\sin(2\theta-10^\circ)$ 1A

$\sin(2\theta-10^\circ) = \dfrac{k+1}{2(k-1)}$ 1

(ii) $\sin^2(2\theta-10^\circ) = \dfrac{(k+1)^2}{4(k-1)^2} \leq 1$ 1M

$(k+1)^2 \leq 4(k-1)^2$

$3k^2 - 10k + 3 \geq 0$ 1A

$k \geq 3$ 或 $k \leq \dfrac{1}{3}$ 1A

(c) Put $k = -2$ into (b)(i),

$\sin(2\theta-10^\circ) = \dfrac{1}{6}$ 1M

$2\theta - 10^\circ = \sin^{-1}\dfrac{1}{6}$ 或 $180^\circ - \sin^{-1}\dfrac{1}{6}$

$\theta \approx 9.8^\circ$ 或 90.2° 1A

16. (a) $\cos x = 2\cos^2 \frac{x}{2} - 1 = \frac{2}{1+\tan^2 \frac{x}{2}} - 1 = \frac{1-t^2}{1+t^2}$ 1A

$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = \frac{2\tan\frac{x}{2}}{\sec^2\frac{x}{2}} = \frac{2t}{1+t^2}$ 1A

$$a\cos x + b\sin x = c$$

$$a \times \frac{1-t^2}{1+t^2} + b \times \frac{2t}{1+t^2} = c$$ 1M

$$a(1-t^2) + 2bt = c(1+t^2)$$

$(a+c)t^2 - 2bt + (c-a) = 0$ 1

若方程有解，

$$\Delta = (2b)^2 - 4(a+c)(c-a) \geq 0$$ 1M

$$b^2 - (c^2 - a^2) \geq 0$$

$$a^2 + b^2 \geq c^2$$ 1

(b) (i) 代 $a = 5$、$b = 6$ 及 $c = 7$，可得 $12t^2 - 12t + 2 = 0$。 1M

方程的根為 $\tan\frac{x_1}{2}$ 及 $\tan\frac{x_2}{2}$。

$\tan\frac{x_1}{2} + \tan\frac{x_2}{2} = 1$ 及 $\tan\frac{x_1}{2}\tan\frac{x_2}{2} = \frac{1}{6}$。 1A

$$\tan\frac{x_1+x_2}{2} = \frac{\tan\frac{x_1}{2} + \tan\frac{x_2}{2}}{1 - \tan\frac{x_1}{2}\tan\frac{x_2}{2}}$$ 1A

$$= \frac{1}{1-\frac{1}{6}} = \frac{6}{5}$$ 1A

(ii) $\tan x_1 \tan x_2 = \frac{2\tan\frac{x_1}{2}}{1-\tan^2\frac{x_1}{2}} \times \frac{2\tan\frac{x_2}{2}}{1-\tan^2\frac{x_2}{2}}$ 1A

$$= \frac{4\tan\frac{x_1}{2}\tan\frac{x_2}{2}}{1 - \left(\tan\frac{x_1}{2} + \tan\frac{x_2}{2}\right)^2 + 2\tan\frac{x_1}{2}\tan\frac{x_2}{2} + \left(\tan\frac{x_1}{2}\tan\frac{x_2}{2}\right)^2}$$ 1M

$$= \frac{4 \times \frac{1}{6}}{1 - 1 + 2 \times \frac{1}{6} + \left(\frac{1}{6}\right)^2}$$

$$= \frac{24}{13}$$ 1A

17. (a) $\angle BAE = \dfrac{(5-2)180^\circ}{5} = 108^\circ$

$\angle ABE = \dfrac{180^\circ - 108^\circ}{2} = 36^\circ$ 1A

$\angle CBE = 108^\circ - 36^\circ = 72^\circ$ 1A

在等腰 $\triangle ABE$，$BE = 2AB\cos 36^\circ = 2\cos 36^\circ$ cm。 1M

在梯形 $BCDE$，$BE = CD + 2BH = (1 + 2\cos 72^\circ)$ cm。 1M

$$2\cos 36^\circ = 1 + 2\cos 72^\circ$$

$$\cos 36^\circ - \cos 72^\circ = \frac{1}{2}$$ 1

$$\cos 36^\circ - (2\cos^2 36^\circ - 1) = \frac{1}{2}$$

$$-4\cos^2 36^\circ + 2\cos 36^\circ + 1 = 0$$ 1A

$$\cos 36^\circ = \frac{-2 \pm \sqrt{2^2 + 4(4)}}{2(-4)}$$

$$= \frac{1+\sqrt{5}}{4} \quad 或 \quad \frac{1-\sqrt{5}}{4}\ （捨去）$$ 1A

(b) $\sin 36^\circ = \sqrt{1-\cos^2 36^\circ} = \sqrt{1 - \left(\dfrac{1+\sqrt{5}}{4}\right)^2} = \sqrt{\dfrac{5-\sqrt{5}}{8}}$ 1M

設 M 為 AB 的中點。考慮 $\triangle OAM$，可得 $\angle AOM = 36^\circ$。

半徑 $= \dfrac{1}{2} \div \sin 36^\circ = \dfrac{\sqrt{2}}{\sqrt{5-\sqrt{5}}} = \dfrac{2}{\sqrt{10-2\sqrt{5}}}$ cm 1M+1

(c) $\angle AOP = \dfrac{360^\circ}{10} = 36^\circ$

$\angle PAO = \dfrac{180^\circ - 36^\circ}{2} = 72^\circ$ 1A

$AP = 2AO\cos 72^\circ$ 1M

$$= 2AO\left(\cos 36^\circ - \frac{1}{2}\right)$$

$$= 2 \times \frac{2}{\sqrt{10-2\sqrt{5}}} \times \frac{\sqrt{5}-1}{4}$$ 1A

$$= \frac{5-1}{\sqrt{10-2\sqrt{5}}(\sqrt{5}+1)} = \frac{4}{\sqrt{(10-2\sqrt{5})(6+2\sqrt{5})}}$$

$$= \frac{4}{\sqrt{40+8\sqrt{5}}} = \frac{2}{\sqrt{10+2\sqrt{5}}}\text{ cm}$$ 1A

18. (a) (i) $\sin 108° = \sin 72° \cos 36° + \cos 72° \sin 36°$ 1A

$$= 2\sin 36° \cos^2 36° + (1 - 2\sin^2 36°)\sin 36°$$

$$= 2\sin 36°(1 - \sin^2 36°) + \sin 36° - 2\sin^3 36°$$

$$= 3\sin 36° - 4\sin^3 36°$$ 1A

$$\sin 108° = \sin 72°$$

$$3\sin 36° - 4\sin^3 36° = 2\sin 36° \cos 36°$$ 1A

$$0 = \sin 36°(4\sin^2 36° + 2\cos 36° - 3)$$

$$= -4\cos^2 36° + 2\cos 36° + 1$$ 1A

$$\cos 36° = \frac{-2 \pm \sqrt{2^2 + 4}}{2(-4)}$$

$$= \frac{1+\sqrt{5}}{4} \quad \text{或} \quad \frac{1-\sqrt{5}}{4}\ \text{（捨去）}$$ 1

(ii) $\cos 72° = 2\cos^2 36° - 1 = 2\left(\frac{1+\sqrt{5}}{4}\right)^2 - 1$ 1A

$$= \frac{\sqrt{5}-1}{2}$$ 1A

(b) (i) $\frac{OH}{\sin\phi} = \frac{1}{\sin(180° - 60° - \phi)}$ 1M

$$OH = \frac{\sin\phi}{\sin 120° \cos\phi - \cos 120° \sin\phi}$$

$$= \frac{\tan\phi}{\frac{\sqrt{3}}{2} + \frac{1}{2}\tan\phi} = \frac{2\tan\phi}{\sqrt{3} + \tan\phi}$$ 1A

$$\cos\angle POK = \frac{OK}{1} = \frac{OH\cos 60°}{1}$$ 1M

$$= \frac{\tan\phi}{\sqrt{3} + \tan\phi}$$ 1

(ii) (1) $BN = \sqrt{1^2 - \left(\frac{1}{4}\right)^2} = \frac{\sqrt{15}}{4}$ 1A

$$\tan\phi = \frac{\frac{\sqrt{15}}{4}}{1 + \frac{1}{4}}$$ 1M

$$= \frac{\sqrt{15}}{5}$$ 1A

(2) $\cos\angle POK = \frac{\frac{\sqrt{15}}{5}}{\sqrt{3} + \frac{\sqrt{15}}{5}}$

$$= \frac{\sqrt{5}}{5+\sqrt{5}} \times \frac{5-\sqrt{5}}{5-\sqrt{5}}$$

$$= \frac{\sqrt{5}-1}{4}$$ 1A

據 (a)，$\angle POK = 72°$。 1A

19. (a) $4\cos x\cos 2x\cos 3x - 1 = 2\cos 3x(\cos 3x + \cos x) - 1$ 1M

$= (2\cos 3x\cos x) + (2\cos^2 3x - 1)$ 1M

$= \cos 2x + \cos 4x + \cos 6x$ 1

(b) $\cos 4\theta + \cos 8\theta + \cos 12\theta = -1$

$4\cos 2\theta\cos 4\theta\cos 6\theta - 1 = -1$ 1M

$4\cos 2\theta\cos 4\theta\cos 6\theta = 0$

$\cos 2\theta = 0$ 或 $\cos 4\theta = 0$ 或 $\cos 6\theta = 0$ 1M

因此，$\theta = \frac{\pi}{12}, \frac{\pi}{8}, \frac{\pi}{4}, \frac{3\pi}{8}, \frac{5\pi}{12}$. 1A

20. (a) 當 $n = 1$，

左式 $= \sin\frac{x}{2}\cos x$，右式 $= \sin\frac{x}{2}\cos x$ 1

對 $n = 1$，命題為真。

假設 $\sin\frac{x}{2}\sum_{k=1}^{m}\cos k = \sin\frac{mx}{2}\cos\frac{(m+1)x}{2}$，其中 $m \in \mathbb{Z}^+$。 1M

$\sin\frac{x}{2}\sum_{k=1}^{m+1}\cos kx$

$= \sin\frac{x}{2}\sum_{k=1}^{m}\cos kx + \sin\frac{x}{2}\cos(m+1)x$

$= \sin\frac{mx}{2}\cos\frac{(m+1)x}{2} + \sin\frac{x}{2}\cos(m+1)x$ 1M

$= \frac{1}{2}\left(\sin\frac{(2m+1)x}{2} - \sin\frac{x}{2}\right) + \frac{1}{2}\left(\sin\frac{(2m+3)x}{2} - \sin\frac{(2m+1)x}{2}\right)$ 1M

$= \frac{1}{2}\left(\sin\frac{(2m+3)x}{2} - \sin\frac{x}{2}\right)$

$= \cos\frac{(2m+4)x}{4}\sin\frac{(2m+2)x}{4}$ 1M

$= \sin\frac{(m+1)x}{2}\cos\frac{(m+2)x}{2}$

對 $n = m + 1$，命題為真。

藉數學歸納法，$\forall n \in \mathbb{Z}^+$，命題為真。 1

(b) 代 $x = \frac{\pi}{7}$ 至 $n = 567$ 至 (a)，

$\sin\left(\frac{1}{2}\cdot\frac{\pi}{7}\right)\sum_{k=1}^{567}\cos\frac{k\pi}{7} = \sin\left(\frac{567}{2}\cdot\frac{\pi}{7}\right)\cos\left(\frac{568}{2}\cdot\frac{\pi}{7}\right)$ 1M

$\sin\frac{\pi}{14}\sum_{k=1}^{567}\cos\frac{k\pi}{7} = \sin\frac{\pi}{2}\cos\left(\frac{\pi}{2} + \frac{\pi}{14}\right)$

$= \sin\frac{\pi}{2}\left(-\sin\frac{\pi}{14}\right)$

$\sum_{k=1}^{567}\cos\frac{k\pi}{7} = -1$ 1A

21. (a) 當 $n = 1$，

左式 $= \sin\sin 2\theta$，右式 $= \sin\theta\sin 2\theta =$ 左式

對 $n = 1$，命題為真。 1

假設 $\sin\theta\sum_{k=1}^{p}\sin 2k\theta = \sin p\theta\sin(p+1)\theta$，其中 $p \in \mathbb{Z}^+$。 1M

$$\sin\theta\sum_{k=1}^{p+1}\sin 2k\theta$$

$$= \sin\theta\sum_{k=1}^{p}\sin 2k\theta + \sin\theta\sin[2(p+1)\theta]$$

$$= \sin p\theta\sin(p+1)\theta + \sin\theta\sin(2p+2)\theta \quad \text{1M}$$

$$= \frac{1}{2}[\cos\theta - \cos(2p+1)\theta] + \frac{1}{2}[\cos(2p+1)\theta - \cos(2p+3)\theta] \quad \text{1M}$$

$$= \frac{1}{2}[\cos\theta - \cos(2p+3)\theta]$$

$$= \sin(p+1)\theta\sin(p+2)\theta$$

對 $n = p + 1$，命題為真。

藉數學歸納法，$\forall n \in \mathbb{Z}^+$，命題為真。 1

(b) $$\sum_{k=1}^{111}\sin\frac{k\pi}{11}\cos\frac{k\pi}{11} = \frac{1}{2}\sum_{K=1}^{111}\sin\frac{2k\pi}{11} \quad \text{1M}$$

$$= \frac{1}{2\sin\frac{\pi}{11}}\sin\frac{\pi}{11}\sum_{k=1}^{111}\sin\frac{2k\pi}{11}$$

$$= \frac{1}{2\sin\frac{\pi}{11}}\sin\frac{111\pi}{11}\sin\frac{112\pi}{11} \quad \text{1M}$$

$$= \frac{1}{2\sin\frac{\pi}{11}}\sin\frac{\pi}{11}\sin\frac{2\pi}{11}$$

$$= \frac{1}{2}\sin\frac{2\pi}{11}$$

可得 $a = \frac{1}{2}$ 及 $b = \frac{2}{11}$。 1A

22. (a) $\dfrac{\sin\alpha+\sin\beta}{\cos\alpha+\cos\beta}=\dfrac{2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}}{2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}}$ 1A

$=\tan\dfrac{\alpha+\beta}{2}$ 1

(b) $3\sin\alpha-4\cos\alpha=4\cos\beta-3\sin\beta$

$\dfrac{\sin\alpha+\sin\beta}{\cos\alpha+\cos\beta}=\dfrac{4}{3}$

$\tan\dfrac{\alpha+\beta}{2}=\dfrac{4}{3}$ 1M

$\tan(\alpha+\beta)=\dfrac{2\tan\frac{\alpha+\beta}{2}}{1-\tan^2\frac{\alpha+\beta}{2}}$

$=\dfrac{2\times\frac{4}{3}}{1-\left(\frac{4}{3}\right)^2}$ 1M

$=-\dfrac{24}{7}$ 1A

23. $-\sin(x+y)\sin(x-y)=-\dfrac{1}{2}\left[\cos[(x+y)-(x-y)]-\cos[(x+y)+(x-y)]\right]$ 1A

$=-\dfrac{1}{2}(\cos 2y-\cos 2x)$

$=-\dfrac{1}{2}[(2\cos^2 y-1)-(2\cos^2 x-1)]$ 1M

$=\cos^2 x-\cos^2 y$ 1

24. $\sin 5x+\sin x=\cos 2x$

$2\sin 3x\cos 2x-\cos 2x=0$ 1A

$\cos 2x(2\sin 3x-1)=0$

$\cos 2x=0$ 或 $\sin 3x=\dfrac{1}{2}$

當 $\cos 2x=0$ 時，$2x=90^\circ$。 1M+1A

當 $\sin 3x=\dfrac{1}{2}$ 時，$3x=30^\circ$ 或 150°。

因此，$x=10^\circ$ 或 45° 或 50°。 1A

25. $\sin[(n+m)\theta]\sin[(n-m)\theta] = \frac{1}{2}[\cos 2m\theta - \cos 2n\theta]$ 1A

$= \frac{1}{2}\left[(1-2\sin^2 m\theta)-(1-2\sin^2 n\theta)\right]$

$= \sin^2 n\theta - \sin^2 m\theta$ 1

$\sin^2 3\theta - \sin^2 2\theta - \sin\theta = 0$

$\sin 5\theta \sin\theta - \sin\theta = 0$ 1M

$\sin\theta(\sin 5\theta - 1) = 0$

$\sin\theta = 0$ 或 $\sin 5\theta = 1$ 1A

當 $\sin\theta = 0$ 時，$\theta = 0$ 或 π。

當 $\sin 5\theta = 1$ 時，$5\theta = \frac{\pi}{2}$ 或 $\frac{5\pi}{2}$ 或 $\frac{9\pi}{2}$。

故此，$\theta = 0$ 或 $\frac{\pi}{10}$ 或 $\frac{\pi}{2}$ 或 $\frac{9\pi}{10}$ 或 π。 1A+1A

26. (a) $\sin(A+B)\sin(B-A) = -\dfrac{1}{2}(\cos 2B - \cos 2A)$ 1A

$= -\dfrac{1}{2}[(2\cos^2 B - 1) - (2\cos^2 A - 1)]$

$= \cos^2 A - \cos^2 B$ 1

(b) (i) $\cos^2 A - \cos^2 B + \sin^2 C = \sin(A+B)\sin(B-A) + \sin^2 C$ 1M

$= \sin(\pi - C)\sin(B-A) + \sin^2 C$ 1A

$= \sin C[\sin(B-A) + \sin(A+B)]$

$= 2\sin C \sin B \cos A$ 1

(ii) $\cos^2 A - \cos^2 B - \cos^2 C = -1$

$\cos^2 A - \cos^2 B + \sin^2 C = 0$

$2\cos A \sin C \sin B = 0$ 1M

$\cos A = 0$ 或 $\sin C \sin B = 0$（捨去） 1A

$A = \dfrac{\pi}{2}$

因此，$\triangle ABC$ is a right angled triangle. 1

(c) $\cos^2 x - \sin^2 y = \cos^2 x - \cos^2\left(\dfrac{\pi}{2} - y\right)$

$= \sin\left(x + \dfrac{\pi}{2} - y\right)\sin\left(\dfrac{\pi}{2} - y - x\right)$ 1M

$= \cos(x+y)\cos(x-y)$ 1

$\cos^2 2\theta - \sin^2 3\theta + \cos\theta \sin 5\theta = 0$

$\cos 5\theta \cos\theta + \cos\theta \sin 5\theta = 0$ 1A

$\cos\theta(\cos 5\theta + \sin 5\theta) = 0$

$\cos\theta \cos 5\theta(1 + \tan 5\theta) = 0$ 1M

$\tan 5\theta = -1$ 或 $\cos\theta = 0$ 或 $\cos 5\theta = 0$（捨去）

When $\tan 5\theta = -1$,

$5\theta = \dfrac{3\pi}{4}$ 或 $\dfrac{7\pi}{4}$ 或 $\dfrac{11\pi}{4}$ 或 $\dfrac{15\pi}{4}$ 或 $\dfrac{19\pi}{4}$

$\theta = \dfrac{3\pi}{20}$ 或 $\dfrac{7\pi}{20}$ 或 $\dfrac{11\pi}{20}$ 或 $\dfrac{15\pi}{20}$ 或 $\dfrac{19\pi}{20}$

When $\cos\theta = 0$, $\theta = \dfrac{\pi}{2}$.

因此，$\theta = \dfrac{3\pi}{20}$ 或 $\dfrac{7\pi}{20}$ 或 $\dfrac{11\pi}{20}$ 或 $\dfrac{15\pi}{20}$ 或 $\dfrac{19\pi}{20}$ 或 $\dfrac{\pi}{2}$. 1A+1A

27. (a) $\cos\dfrac{3\pi}{10} = \sin\left(\dfrac{\pi}{2} - \dfrac{3\pi}{10}\right)$ 1M

$= \sin\dfrac{2\pi}{10}$

因此，$\dfrac{\pi}{10}$ 為 $\cos 3\theta = \sin 2\theta$ 的根。 1

(b)
$$\cos 3\theta = \sin 2\theta$$
$$4\cos^3\theta - 3\cos\theta = 2\sin\theta\cos\theta \quad \text{1M}$$
$$\cos\theta(4\cos^2\theta - 2\sin\theta - 3) = 0$$
$$\cos\theta[4(1-\sin^2\theta) - 2\sin\theta - 3] = 0 \quad \text{1M}$$
$$\cos\theta(-4\sin^2\theta - 2\sin\theta + 1) = 0$$
$$\cos\theta = 0 \quad \text{或} \quad -4\sin^2\theta - 2\sin\theta + 1 = 0$$

由於 $\cos\dfrac{\pi}{10} \neq 0$，可得 $\dfrac{\pi}{10}$ 為 $-4\sin^2\theta - 2\sin\theta + 1 = 0$ 的根。

$-4\sin^2\theta - 2\sin\theta + 1 = 0$

$$\sin\theta = \frac{2+\sqrt{2^2-4(-4)(1)}}{2(-4)} \quad \text{或} \quad \frac{2-\sqrt{2^2+4(-4)(1)}}{2(-4)} \quad \text{1A}$$
$$= -\frac{1+\sqrt{5}}{4} \quad \text{或} \quad \frac{\sqrt{5}-1}{4}$$

由於 $\sin\dfrac{\pi}{10} > 0$，可得 $\sin\dfrac{\pi}{10} = \dfrac{\sqrt{5}-1}{4}$。 1A

28. $\sin 3\theta = \sin 2\theta\cos\theta + \cos 2\theta\sin\theta$ 1A

$= 2\sin\theta\cos^2\theta + (1-2\sin^2\theta)\sin\theta$

$= 3\sin\theta - 4\sin^3\theta$ 1A

代 $x = \sin\theta$ 至 $8x^3 - 6x + 1 = 0$， 1M

$$8\sin^3\theta - 6\sin\theta + 1 = 0$$
$$2(4\sin^3\theta - 3\sin\theta) + 1 = 0$$
$$\sin 3\theta = \frac{1}{2} \quad \text{1A}$$
$$3\theta = 30^\circ \text{ 或 } 150^\circ \text{ 或 } 390^\circ \text{ 或 } 510^\circ \text{ 或 } 750^\circ$$
$$\theta = 10^\circ \text{ 或 } 50^\circ \text{ 或 } 130^\circ \text{ 或 } 170^\circ \text{ 或 } 250^\circ$$

方程的根為 $\sin 10^\circ \approx 0.17$、$\sin 50^\circ \approx 0.77$ 及 $\sin 250^\circ \approx -0.94$。 1A+1A

29. (a) $2\sin\alpha\sum_{k=0}^{4}\cos(\theta+2k\alpha)$

$= \sum_{k=0}^{4}[\sin[\theta+(2k+1)\alpha]-\sin[\theta+(2k-1)\alpha]]$ 1M

$= \sum_{k=0}^{4}\sin[\theta+(2k+1)\alpha]-\sum_{k=0}^{4}\sin[\theta+(2k-1)\alpha]$

$= \sin(\theta+9\alpha)-\sin(\theta-\alpha)$ 1

代 $\alpha=\dfrac{\pi}{5}$，

$$2\sin\frac{\pi}{5}\sum_{k=0}^{4}\cos\left(\theta+\frac{2k\pi}{5}\right)=\sin\left(\theta+\frac{9\pi}{5}\right)-\sin\left(\theta-\frac{\pi}{5}\right)$$ 1M

$$=2\sin\pi\cos\left(\theta+\frac{4\pi}{5}\right)$$ 1A

$$=0$$

$$\sum_{k=0}^{4}\cos\left(\theta+\frac{2k\pi}{5}\right)=\frac{0}{2\sin\frac{\pi}{5}}=0$$ 1A

(b) (i) $PD^2=r^2+r^2-2r^2\cos\left(\dfrac{2\times2\pi}{5}-\theta\right)$ 1A

$=2r^2-2r^2\cos\left(\theta+\dfrac{6\pi}{5}\right)$ 1

(ii) $PA^2=2r^2-2r^2\cos\theta$ 1A

$PB^2=2r^2-2r^2\cos\left(\theta+\dfrac{2\pi}{5}\right)$

$PC^2=2r^2-2r^2\cos\left(\theta+\dfrac{4\pi}{5}\right)$

$PE^2=2r^2-2r^2\cos\left(\dfrac{2\pi}{5}-\theta\right)=2r^2-2r^2\cos\left(\theta+\dfrac{8\pi}{5}\right)$ 1A

$PA^2+PB^2+PC^2+PD^2+PE^2=10r^2-2r^2\sum_{k=0}^{4}\cos\left(\theta+\dfrac{2k\pi}{5}\right)$

$=10r^2$ 1

(iii) 由於 $\angle APQ=90^\circ$，可得 $QA^2=QP^2+PA^2$。

同理，$QB^2=QP^2+PB^2$、$QC^2=QP^2+PC^2$ 等。

$QA^2+QB^2+QC^2+QD^2+QE^2=5(2r)^2+(PA^2+PB^2+PC^2+PD^2+PE^2)$ 1M

$=20r^2+10r^2$

$=30r^2$ 1A