REG-COT-2425-ASM-SET 3-MATH
Suggested solutions

Conventional Questions

1. (a) $a = \dfrac{(PQ)r}{2} + \dfrac{(QR)r}{2} + \dfrac{(PR)r}{2}$ 1M

$= \dfrac{r}{2}(PQ + QR + PR)$

$= \dfrac{pr}{2}$ 1

(b) $AB = \sqrt{(9-2)^2 + (18+6)^2} = 25$, $BC = \sqrt{32^2 + 24^2} = 40$ and $AC = 39$.

Let the radius of the inscribed circle of $\triangle ABC$ be r.

$\dfrac{(41-2)(18+6)}{2} = \dfrac{(25+40+39)r}{2}$ 1M

$r = 9$ 1A

Required y-coordinate $= -6 + 9 = 3$ 1A

2. (a) $PB = PD$ and $\angle PBD = \dfrac{180° - x}{2} = 90° - \dfrac{x}{2}$ 1M

$\angle BAD = \angle PBD = 90° - \dfrac{x}{2}$ 1A

$\angle ABC = 90°$

$\angle AQB = 180° - 90° - \left(90° - \dfrac{x}{2}\right) = \dfrac{x}{2}$ 1A

(b) (i) Since $PB = PD = PR$, we have $\angle BRD = \angle PDR$. 1M

$\angle BRD + \angle PDR = x$

$\angle BRD = \dfrac{x}{2}$ 1A

(ii) Note that $\angle BRD = \angle BQD = \dfrac{x}{2}$.

B, D, Q, R are concyclic. 1M

Since $PB = PD = PR$, P is the centre of the circle BDR,

and hence is the centre of the circle $BDQR$.

Thus, P is the centre of the circumcircle of $\triangle BDQ$.

The claim is agreed. 1A

3. (a) $\angle BPA = 90°$ *(given)*

$\angle GMA = 90°$ *(property of circumcentre)*

$= \angle BPA$

$BC // GM$ *(corr. ∠s equal)*

$\angle BDC = \angle MDG$ *(vert. opp. ∠s)*

$\angle CBD = \angle GMD$ *(alt. ∠s, BC//GM)*

$\triangle BCD \sim \triangle MGD$ *(AA)*

Marking Scheme		
Case 1	Any correct proof with correct reasons.	3
Case 2	Any correct proof without reasons.	2
Case 3	Incomplete proof with any one correct step with reason.	1

(b) (i) $GM = \sqrt{r^2 - \left(\frac{a}{2}\right)^2} = \frac{1}{2}\sqrt{4r^2 - a^2}$ 1A

The coordinates of G are $\left(\frac{a}{2}, \frac{\sqrt{4r^2 - a^2}}{2}\right)$.

$$BG = r$$

$$\left(\frac{a}{2} - b\right)^2 + \left(\frac{\sqrt{4r^2 - a^2}}{2} - h\right)^2 = r^2$$ 1M

$$\left(\frac{\sqrt{4r^2 - a^2}}{2} - h\right)^2 = r^2 - \frac{(a - 2b)^2}{4}$$

Since $h > GM > 0$, $\frac{\sqrt{4r^2 - a^2}}{2} - h < 0$.

$$\frac{\sqrt{4r^2 - a^2}}{2} - h = -\sqrt{r^2 - \frac{(a - 2b)^2}{4}}$$

$$h = \frac{\sqrt{4r^2 - a^2}}{2} + \frac{\sqrt{4r^2 - (a - 2b)^2}}{\sqrt{4}}$$

$$= \frac{\sqrt{4r^2 - a^2 + 4ab - 4b^2} + \sqrt{4r^2 - a^2}}{2}$$ 1

(ii) $BD : DM = 2 : 1$ 1A

$BC : GM = BD : DM = 2 : 1$ and $BC = 2GM = \sqrt{4r^2 - a^2}$ 1M

$CP = h - \sqrt{4r^2 - a^2} = \frac{\sqrt{4r^2 - a^2 + 4ab - 4b^2} - \sqrt{4r^2 - a^2}}{2}$

(Slope of AB) × (slope of OC)

$= \frac{h - 0}{b - a} \times \frac{CP}{b}$ 1M

$= \frac{1}{4b(b - a)}\left[(\sqrt{4r^2 - a^2 + 4ab - 4b^2})^2 - (\sqrt{4r^2 - a^2})^2\right]$

$= \frac{4b(a - b)}{4b(b - a)}$

$= -1$

Since $BC \perp OA$ and $OC \perp AB$, C is the orthocentre of $\triangle OAB$. 1

(iii) $h = \frac{\sqrt{4r^2 - a^2 + 4ab - 4b^2} + \sqrt{4r^2 - a^2}}{2} = 49$

$CP = h - \sqrt{4r^2 - a^2} = 31$

x-coordinate of $D = \frac{2\left(\frac{80}{2}\right) + 31}{1 + 2} = 37$ 1M

$$\text{Required area} = \frac{(40-31)(49)}{2} - \frac{(49-31)(37-31)}{2}$$
$$= \frac{333}{2}$$ 1A

4. (a) $\angle PGT = \angle QGT$ *(tangent properties)*

$PR = QR$ *(equal ∠s, equal chords)*

$\angle RQP = \angle RPQ$ *(base ∠s, isos. △s)*

$\angle RQT = \angle RPQ$ *(∠ in alt. segment)*

$= \angle RQP$

Thus, QR is the angle bisector of $\angle PQT$.

Marking Scheme		
Case 1	Any correct proof with correct reasons.	3
Case 2	Any correct proof without reasons.	2
Case 3	Incomplete proof with any one correct step with reason.	1

(b) (i) Slope $= \frac{7+68}{12+88} = \frac{3}{4}$

Required equation is

$$y - 7 = \frac{3}{4}(x-12)$$
$$3x - 4y - 8 = 0$$ 1A

(ii) Note that GT is the angle bisector of $\angle PTQ$.

The in-centre of $\triangle PQT$ is R. 1M

$GR = GQ = 7 + 68 = 75$

$GT = \sqrt{(12+88)^2 + (6+68)^2} = 125$

We have $GR : RT = 75 : (125-75) = 3 : 2$.

Let the coordinates of R be (a, b).

$$\frac{12-a}{a+88} = \frac{3}{2} \quad \text{and} \quad \frac{7-b}{b+68} = \frac{3}{2}$$ 1M
$$a = -48 \qquad\qquad b = -38$$

The coordinates of the in-centre of $\triangle PQT$ are $(-48, -38)$. 1A

(iii) Suppose the inscribed circle of $\triangle PQT$ touches QT at U.

Radius of inscribed circle of $\triangle PQT$

$= RU$

$= -38 + 68$ 1M

$= 30$

Note that $GPTQ$ is a cyclic quadrilateral.

The circumcircle of $\triangle PQT$ passes through G.

Radius of circumcircle of $\triangle PQT$

$= \dfrac{GT}{2}$ 1M

$= \dfrac{125}{2}$

Required ratio

$= 2(30)\pi : 2\left(\dfrac{125}{2}\right)\pi$

$= 12 : 25$

$\neq 1 : 2$

The claim is disagreed. 1A

5. (a) Equation of C is $(x+6)^2 + (y-4)^2 = r^2$. 1A

$$\left(\frac{14+ky}{4}+6\right)^2 + (y-4)^2 = r^2$$

$$\left(\frac{k^2}{16}+1\right)y^2 + \left(\frac{19k}{4}-8\right)y + \frac{425}{4} - r^2 = 0$$

$$(k^2+16)y^2 + 4(19k-32)y + 1700 - 16r^2 = 0$$

L is a tangent to C.

$$[4(19k-32)]^2 - 4(k^2+16)(1700-16r^2) = 0$$ 1M

$$r^2 = \frac{1700}{16} - \frac{(19k-32)^2}{4(k^2+16)}$$

$$= \frac{(4k+38)^2}{k^2+16}$$ 1A

(b) (i) The coordinates of E are $(-4,\ -10)$. 1A

L passes through the point D.

$$4(8) - k(6) - 14 = 0$$

$$k = 3$$

Let the coordinates of F be $(a,\ b)$.

$$DF = EF$$

$$\sqrt{(a-8)^2 + (b-6)^2} = \sqrt{(a+4)^2 + (b+10)^2}$$

$$-24a - 32b - 16 = 0$$

$$a = -\frac{4b}{3} - \frac{2}{3}$$

G is the circumcentre of $\triangle DEF$.

$$GF = GD$$

$$\sqrt{(a+6)^2 + (b-4)^2} = \sqrt{(8+6)^2 + (6-4)^2} \qquad \text{1M}$$

$$\left(-\frac{4b}{3} - \frac{2}{3} + 6\right)^2 + (b-4)^2 = 200$$

$$\frac{25b^2}{9} - \frac{200b}{9} - \frac{1400}{9} = 0$$

$$b = 4 \pm 6\sqrt{2}$$

The coordinates of F are $\left(-6+8\sqrt{2},\ 4-6\sqrt{2}\right)$ or $\left(-6-8\sqrt{2},\ 4+6\sqrt{2}\right)$. 1A

(ii) L is a tangent to the inscribed circle of $\triangle DFH$.

By (a), if G is the in-centre, then $r^2 = \dfrac{[4(3)+38)^2]}{3^2+16} = 100$. 1M

The radius of the inscribed circle of $\triangle DFH$ is 10.

Let K be the point of contact of the inscribed circle and L.

$$\sin \angle GDK = \frac{r}{DG} \qquad \text{1M}$$

$$\angle GDK = 45^\circ$$

Note that E lies on L. We have $\angle FDH = \angle FDE$.

The coordinates of the mid-point of EF are $\left(-5+4\sqrt{2},\ -3-3\sqrt{2}\right)$ or $\left(-5-4\sqrt{2},\ -3+3\sqrt{2}\right)$.

Neither of the coordinates match with G.

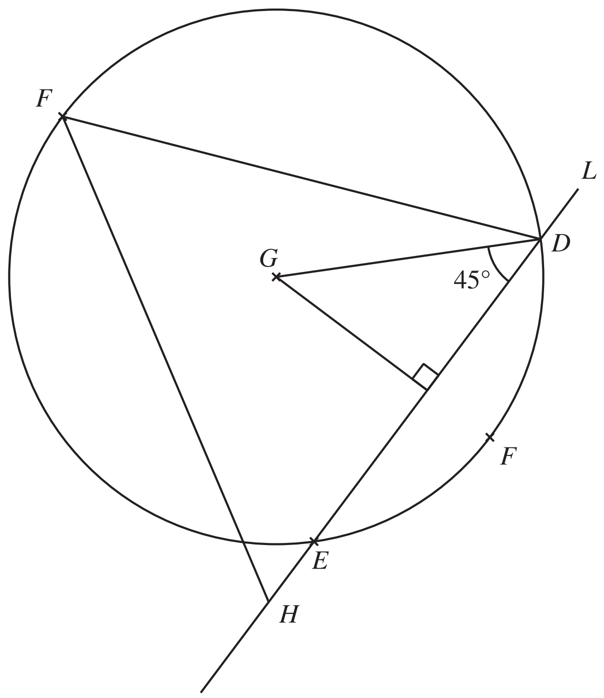
EF is not a diameter of the circle DEF.

Hence, $\angle FDE \neq 90^\circ$ and $\angle FDH = \angle FDE \neq 90^\circ$. 1M

We have $\angle FDH \neq 2\angle GDK$ and DG is not an angle bisector of $\angle FDH$.

G is not the in-centre of $\triangle DFH$.

The claim is disagreed. 1A



6. (a) $f(-2) = k(-2)^2 + (2k-2)(-2) - 4$

$= 4k - 4k + 4 - 4$

$= 0$

The graph of $y = f(x)$ passes through A. 1

(b) (i) $f(x) = kx^2 + (2k-2)x - 4$

$= k\left[x - 2(x)\left(\frac{1-k}{k}\right) + \left(\frac{1-k}{k}\right)^2\right] - \frac{(k+1)^2}{k}$ 1M

$= k\left[x - \frac{1-k}{k}\right]^2 - \frac{(k+1)^2}{k}$

The coordinates of Q are $\left(\frac{1-k}{k},\ -\frac{(k+1)^2}{k}\right)$. 1A

(ii) The coordinates of C are $(0,\ -4)$. 1A

Let the coordinates of R be $\left(\frac{1-k}{k},\ r\right)$, where r is a constant. 1M

$$RA = RC$$

$$\sqrt{\left(\frac{1-k}{k}+2\right)^2 + r^2} = \sqrt{\left(\frac{1-k}{k}\right)^2 + (r+4)^2}$$ 1M

$$\left(\frac{1-k}{k}\right)^2 + 4\left(\frac{1-k}{k}\right) + 4 + r^2 = \left(\frac{1-k}{k}\right)^2 + r^2 + 8r + 16$$

$$4\left(\frac{1-k}{k}\right) - 12 = 8r$$

$$r = \frac{1-4k}{2k}$$ 1A

The coordinates of R are $\left(\frac{1-k}{k},\ \frac{1-4k}{2k}\right)$.

(iii) We have $\angle RCQ = 90°$.

$$\frac{\frac{1-4k}{2k}+4}{\frac{1-k}{k}-0} \times \frac{-\frac{(k+1)^2}{k}+4}{\frac{1-k}{k}-0} = -1$$ 1M+1M

$$\frac{1+4k}{2k} \times \frac{-k^2+2k-1}{k} = -\left(\frac{1-k}{k}\right)^2$$

$$\frac{1+4k}{2k} \times \frac{(k-1)^2}{k} = \frac{(1-k)^2}{k^2}$$

$$1 + 4k = 2$$

$$k = \frac{1}{4}$$ 1A

The coordinates of R are $(3,\ 0)$.

Required area $= AR^2\pi$

$= (3+2)^2\pi$

$= 25\pi$ 1A

END OF PAPER