| Solution | Marks |
|---|---|

**REG-2324-MOCK-SET 6-MATH-EP(M2)**
**Suggested solutions**

1. $\dfrac{d}{dx}[\ln(x+6)] = \lim_{h\to 0}\dfrac{\ln(x+h+6)-\ln(x+6)}{h}$ — 1M

$= \lim_{h\to 0}\ln\left(1+\dfrac{h}{x+6}\right)^{\frac{1}{h}}$

$= \lim_{h\to 0}\ln\left[\left(1+\dfrac{h}{x+6}\right)^{\frac{x+6}{h}}\right]^{\frac{1}{x+6}}$ — 1M

$= \ln e^{\frac{1}{x+6}}$

$= \dfrac{1}{x+6}$ — 1A

2. (a) $(1-x)(1+x)^n = (1-x)\left(1+nx+\dfrac{n(n-1)}{2}x^2+\ldots\right)$

$= 1+(n-1)x+\left(\dfrac{n(n-1)}{2}-n\right)x^2+\ldots$ — 1M

Thus, $a=1$, $b=n-1$ and $c=\dfrac{n^2-3n}{2}$. — 1A

(b) $(n-1)-1 = \dfrac{n^2-3n}{2}-(n-1)$ — 1M

$0 = \dfrac{n^2}{2}-\dfrac{7n}{2}+3$

$n=6$ or $1$ (rejected) — 1A

3. $\dfrac{dy}{dx} = e^{-x}(\cos x+\sin x)+(\sin x-\cos x)(-e^{-x})$ — 1M

$= 2e^{-x}\cos x$ — 1A

$\dfrac{d^2y}{dx^2} = 2e^{-x}(-\sin x)-2e^{-x}\cos x$

$= -2e^{-x}(\sin x+\cos x)$ — 1A

$-2e^{-x}(\sin x+\cos x)+2(2e^{-x}\cos x)+ke^{-x}(\sin x-\cos x)=0$

$2\cos x-2\sin x+k(\sin x-\cos x)=0$

$(k-2)(\sin x-\cos x)=0$

$k=2$ — 1A

| Solution | Marks |
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| 4. (a) Let $x = \sin\theta$. Then $\mathrm{d}x = \cos\theta \, \mathrm{d}\theta$. | |
| $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \int \frac{\cos\theta}{\sqrt{1-\sin^2\theta}} \mathrm{d}\theta$ | 1M |
| $\displaystyle = \int \mathrm{d}\theta$ | |
| $= \sin^{-1} x + \text{constant}$ | 1A |
| (b) $\displaystyle\int_{\frac{1}{2}}^{\frac{1}{\sqrt{2}}} \sqrt{\frac{1-x}{1+x}} \mathrm{d}x = \int_{\frac{1}{2}}^{\frac{1}{\sqrt{2}}} \sqrt{\frac{(1-x)^2}{(1+x)(1-x)}} \mathrm{d}x$ | 1M |
| $\displaystyle = \int_{\frac{1}{2}}^{\frac{1}{\sqrt{2}}} \left( \frac{1}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}} \right) \mathrm{d}x$ | |
| $\displaystyle = \int_{\frac{1}{2}}^{\frac{1}{\sqrt{2}}} \frac{\mathrm{d}x}{\sqrt{1-x^2}} + \frac{1}{2} \int_{\frac{3}{4}}^{\frac{1}{2}} \frac{\mathrm{d}u}{\sqrt{u}}$ where $u = 1 - x^2$ | 1M |
| $\displaystyle = \left[ \sin^{-1} x \right]_{\frac{1}{2}}^{\frac{1}{\sqrt{2}}} + \frac{1}{2} \left[ 2u^{\frac{1}{2}} \right]_{\frac{3}{4}}^{\frac{1}{2}}$ | 1M |
| $\displaystyle = \frac{\pi}{12} + \frac{\sqrt{2}-\sqrt{3}}{2}$ | 1A |
| 5. (a) $\begin{vmatrix} 1 & 0 & a \\ 2 & -1 & -1 \\ -1 & a & 2 \end{vmatrix} = -2 + 2a^2 - a + a$ | |
| $= 2(a+1)(a-1)$ | |
| $(E)$ has a unique solution if and only if $(a+1)(a-1) \neq 0$. | 1M |
| Therefore, $a \neq 1$ and $a \neq -1$. | 1A |
| Required range is $a < -1$ or $-1 < a < 1$ or $a > 1$. | |
| (b) When $a = -1$, the augmented matrix of $(E)$ becomes | |
| $\left(\begin{array}{ccc\|c} 1 & 0 & -1 & b \\ 2 & -1 & -1 & 0 \\ -1 & -1 & 2 & 1 \end{array}\right) \sim \left(\begin{array}{ccc\|c} 1 & 0 & -1 & b \\ 0 & -1 & 1 & -2b \\ 0 & -1 & 1 & b+1 \end{array}\right) \sim \left(\begin{array}{ccc\|c} 1 & 0 & -1 & b \\ 0 & -1 & 1 & -2b \\ 0 & 0 & 0 & 3b+1 \end{array}\right)$ | 1M |
| If $(E)$ is consistent, then $3b + 1 = 0$, i.e., $b = -\frac{1}{3}$. | 1A |
| The augmented matrix of $(E)$ is therefore $\left(\begin{array}{ccc\|c} 1 & 0 & -1 & -\frac{1}{3} \\ 0 & -1 & 1 & \frac{2}{3} \\ 0 & 0 & 0 & 0 \end{array}\right)$. | |
| Let $z = t$, where $t \in \mathbb{R}$. Then $y = \dfrac{3t-2}{3}$ and $x = \dfrac{3t-1}{3}$. | |
| The solution set is $\left\{ \left( \dfrac{3t-1}{3}, \dfrac{3t-2}{3}, t \right) : t \in \mathbb{R} \right\}$. | 1A |

| Solution | Marks |
|---|---|
| 6. (a) $\overrightarrow{OA}\cdot\overrightarrow{OB}=(OA)(OB)\cos\theta$ | |
| $\cos\theta=\dfrac{(3a)(2a)}{(3a)(2a\sqrt{2})}$ | 1M |
| $\theta=\dfrac{\pi}{4}$ | 1A |
| (b) (i) $\overrightarrow{OG}\cdot\overrightarrow{AB}=0$ | |
| $[2a\mathbf{i}+b\mathbf{j}]\cdot[-a\mathbf{i}+2a\mathbf{j}]=0$ | 1M |
| $2a(-a+b)=0$ | |
| $b=a$ or $a=0$ (rejected) | 1A |
| (ii) $\overrightarrow{BG}\cdot\overrightarrow{OA}=[(b-2a)\mathbf{j}]\cdot[3a\mathbf{i}]=0$ | 1M |
| So, $BG\perp OA$ and therefore $G$ is the orthocentre of $\triangle OAB$. | 1 |
| 7. (a) Let $u=x^2$, then $\mathrm{d}u=2x\,\mathrm{d}x$. | |
| $\int 2x^3\cos\left(x^2\right)\mathrm{d}x=\int u\cos u\,\mathrm{d}u$ | 1M |
| $=u\sin u-\int\sin u\,\mathrm{d}u$ | 1M |
| $=x^2\sin\left(x^2\right)+\cos\left(x^2\right)+\text{constant}$ | 1A |
| (b) (i) $y=\int 2x^3\cos\left(x^2\right)\mathrm{d}x=x^2\sin\left(x^2\right)+\cos\left(x^2\right)+C$ | |
| Substitute $\left(\sqrt{\pi},\ -1\right)$, we have $C=0$ | 1M |
| The equation of $\Gamma$ is $y=x^2\sin\left(x^2\right)+\cos\left(x^2\right)$ | 1A |
| (ii) Slope of tangent $=2\pi^{\frac{3}{2}}\cos\pi=-2\pi^{\frac{3}{2}}$ | 1M |
| The equation is | |
| $y+1=-2\pi^{\frac{3}{2}}(x-\sqrt{\pi})$ | |
| $y=-2\pi^{\frac{3}{2}}x+2\pi^2-1$ | 1A |

| Solution | Marks |
|---|---|
| 8. (a) $A^2 + I = \begin{pmatrix} 2 & 5 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 5 \\ -1 & -2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ | |
| $= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ | 1M |
| $= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ | |
| $= \mathbf{0}$ | 1 |
| (b) $A^4 = (A^2)^2$ | |
| $= (-I)^2$ | 1M |
| $= I$ | 1 |
| (c) $A^{4n+3} = (A^4)^n A^3$ | |
| $= I^n A^3$ | 1M |
| $= A^2 A$ | |
| $= (-I)A$ | 1m |
| $= \begin{pmatrix} -2 & -5 \\ 1 & 2 \end{pmatrix}$ | 1A |

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9. (a) When $n = 1$,

L.H.S. $= \cos x$ and R.H.S. $= \dfrac{\sin 2x}{\sin x} = \dfrac{2\sin x\cos x}{2\sin x} = \cos x =$ L.H.S.

It is true for $n = 1$. **1**

Assume $\displaystyle\sum_{m=1}^{k}\cos(2m-1)x = \frac{\sin 2kx}{2\sin x}$, where $k \in \mathbb{Z}^+$. **1**

$$\begin{aligned}\sum_{m=1}^{k+1}\cos(2m-1)x &= \sum_{m=1}^{k}\cos(2m-1) + \cos(2k+1)x\\ &= \frac{\sin 2kx}{2\sin x} + \cos(2k+1)x \quad \textbf{1}\\ &= \frac{1}{2\sin x}[\sin 2kx + 2\cos(2k+1)x\sin x]\\ &= \frac{1}{2\sin x}[\sin 2kx + \sin(2k+2)x + \sin(-2kx)]\\ &= \frac{\sin(2k+2)x}{2\sin x} \quad \textbf{1}\end{aligned}$$

It is true for $n = k + 1$.

By M.I., it is true $\forall n \in \mathbb{Z}^+$. **1**

(b) $\cos^2\dfrac{\pi}{r} + \cos^2\dfrac{3\pi}{r} + \cos^2\dfrac{5\pi}{r} + \ldots + \cos^2\dfrac{(2r-1)\pi}{r}$

$$\begin{aligned}&= \sum_{m=1}^{r}\cos^2\frac{(2m-1)\pi}{r}\\ &= \frac{1}{2}\sum_{m=1}^{r}\left(1 + \cos\left[(2m-1)\frac{2\pi}{r}\right]\right) \quad \textbf{1M}\\ &= \frac{r}{2} + \frac{1}{2}\sum_{m=1}^{r}\cos\left[(2m-1)\frac{2\pi}{r}\right]\\ &= \frac{r}{2} + \frac{\sin\left[2r\times\frac{2\pi}{r}\right]}{4\sin\frac{2\pi}{r}} \quad \textbf{1M}\\ &= \frac{r}{2} \quad \textbf{1}\end{aligned}$$

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|---|---|
| 10. (a) $\overrightarrow{OQ} = \frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}$ | 1M |
| Let $PC : CR = \beta : 1 - \beta$. | |
| $\overrightarrow{OC} = \beta\overrightarrow{OR} + (1-\beta)\overrightarrow{OP}$ | |
| $= \frac{3\beta}{8}\mathbf{b} + \frac{1-\beta}{2}\mathbf{a}$ | 1M |
| Since $OC // OQ$, | |
| $\frac{1-\beta}{2} \div \frac{2}{3} = \frac{3\beta}{8} \div \frac{1}{3}$ | 1M |
| $\beta = \frac{2}{5}$ | |
| Thus, $PC : CR = \frac{2}{5} : \frac{3}{5} = 2 : 3$. | 1A |
| (b) (i) $\overrightarrow{OC} = \frac{3 \times \frac{2}{5}}{8}(3\mathbf{i} - \mathbf{j} - 2\mathbf{k}) + \frac{1 - \frac{2}{5}}{2}(\mathbf{i} - 3\mathbf{k}) = \frac{3}{4}\mathbf{i} - \frac{3}{20}\mathbf{j} - \frac{6}{5}\mathbf{k}$ | 1A |
| (ii) $\overrightarrow{OA} \times \overrightarrow{OB} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -3 \\ 3 & -1 & -2 \end{vmatrix}$ | 1M |
| $= -3\mathbf{i} - 7\mathbf{j} - \mathbf{k}$ | 1A |
| (iii) Area of $\triangle OAB = \frac{1}{2}\left\|\overrightarrow{OA} \times \overrightarrow{OB}\right\|$ | |
| $= \frac{1}{2}\sqrt{3^2 + 7^2 + 1^2}$ | |
| $= \frac{\sqrt{59}}{2}$ | |
| By considering the volume of tetrahedron $OABD$, | |
| $\frac{1}{3}\left(\frac{\sqrt{59}}{2}\right)(CD) = \frac{59}{2}$ | 1M |
| $CD = 3\sqrt{59}$ | 1A |
| Since $\overrightarrow{CD} // \overrightarrow{OA} \times \overrightarrow{OB}$, | |
| $\overrightarrow{CD} = \pm\frac{3\sqrt{59}}{\left\|\overrightarrow{OA} \times \overrightarrow{OB}\right\|}\overrightarrow{OA} \times \overrightarrow{OB}$ | 1M |
| $= \pm 3(\overrightarrow{OA} \times \overrightarrow{OB})$ | |
| $= \pm 3(-9\mathbf{i} - 21\mathbf{j} - 3\mathbf{k})$ | |
| $\overrightarrow{OD} = \overrightarrow{OC} + \overrightarrow{CD}$ | |
| $= -\frac{33}{4}\mathbf{i} - \frac{423}{20}\mathbf{j} - \frac{21}{5}\mathbf{k}$ or $\frac{39}{4}\mathbf{i} + \frac{417}{20}\mathbf{j} + \frac{9}{5}\mathbf{k}$ | |
| The possible coordinates of $D$ are $\left(-\frac{33}{4}, -\frac{423}{20}, -\frac{21}{5}\right)$ and $\left(\frac{39}{4}, \frac{417}{20}, \frac{9}{5}\right)$. | 1A+1A |

| Solution | Marks |
|---|---|
| 11. (a) (i) $\lvert A\rvert = m + 1 \neq 0$. | |
| $A^{-1} = \dfrac{1}{m+1}\begin{pmatrix} 1 & -1 \\ 1 & m \end{pmatrix}$ | 1M+1A |
| (ii) $C = \dfrac{1}{m+1}\begin{pmatrix} 1 & -1 \\ 1 & m \end{pmatrix}\begin{pmatrix} 1-k & k \\ 1 & 0 \end{pmatrix}\begin{pmatrix} m & 1 \\ -1 & 1 \end{pmatrix}$ | |
| $= \dfrac{1}{m+1}\begin{pmatrix} -k & k \\ 1-k+m & k \end{pmatrix}\begin{pmatrix} m & 1 \\ -1 & 1 \end{pmatrix}$ | 1M |
| $= \dfrac{1}{m+1}\begin{pmatrix} -km-k & 0 \\ m-km+m^2-k & 1+m \end{pmatrix}$ | |
| $= \begin{pmatrix} -k & 0 \\ m-k & 1 \end{pmatrix}$ | 1 |
| (b) (i) When $m = k$, $C^n = \begin{pmatrix} -k & 0 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} (-k)^n & 0 \\ 0 & 1 \end{pmatrix}$ | 1A |
| Since $pC + qI = \begin{pmatrix} -pk+q & 0 \\ 0 & p+q \end{pmatrix}$, we have $\begin{cases} (-k)^n = -pk+q \\ 1 = p+q \end{cases}$ | 1M |
| Solving, we have $p = \dfrac{1-(-k)^n}{1+k}$ and $q = \dfrac{k+(-k)^n}{1+k}$ | 1A |
| (ii) $C^n = pC + qI$ | |
| $(A^{-1}BA)^n = p(A^{-1}BA) + qI$ | |
| $A^{-1}B^nA = pA^{-1}BA + qI$ | 1M |
| $B^n = p(AA^{-1}BAA^{-1}) + qAA^{-1}$ | |
| $= pB + qI$ | 1 |
| (c) Put $k = 3$ such that $B = \begin{pmatrix} -2 & 3 \\ 1 & 0 \end{pmatrix}$. | |
| $B^n = pB + qI$ | |
| $= \dfrac{1-(-3)^n}{1+3}\begin{pmatrix} -2 & 3 \\ 1 & 0 \end{pmatrix} + \dfrac{3+(-3)^n}{1+3}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ | |
| $= \dfrac{1}{4}\begin{pmatrix} 1+3(-3)^n & 3+(-3)^{n+1} \\ 1-(-3)^n & 3+(-3)^n \end{pmatrix}$ | |
| $\begin{pmatrix} x_{2016} \\ x_{2015} \end{pmatrix} = B\begin{pmatrix} x_{2015} \\ x_{2014} \end{pmatrix}$ | |
| $= B^2\begin{pmatrix} x_{2014} \\ x_{2013} \end{pmatrix}$ | 1M |
| $= B^{2014}\begin{pmatrix} x_2 \\ x_1 \end{pmatrix}$ | 1A |
| $= \dfrac{1}{4}\begin{pmatrix} 1+3(-3)^{2014} & 3+(-3)^{2015} \\ 1-(-3)^{2014} & 3+(-3)^{2014} \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ | |
| Thus, $x_{2016} = \dfrac{1}{4}[2+6(-3)^{2014}+3+(-3)^{2015}] = \dfrac{5-(-3)^{2015}}{4}$ | 1A |

| Solution | Marks |
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| 12. (a) $y$-intercept $= \frac{12}{-3} - \frac{12}{3} - 1 = -9$ | 1A |
| $0 = \frac{12}{x-3} - \frac{12}{x+3} - 1$ | |
| $(x-3)(x+3) = 12[(x+3)-(x-3)]$ | |
| $x^2 - 81 = 0$ | |
| $x = \pm 9$ | |
| $x$-intercepts are 9 and $-9$. | 1A |
| (b) (i) $f'(x) = -\frac{12}{(x-3)^2} + \frac{12}{(x+3)^2}$ | 1A |
| $f''(x) = \frac{24}{(x-3)^3} - \frac{24}{(x+3)^3}$ | 1A |
| (ii) When $f'(x) = 0$, | |
| $\frac{12}{(x-3)^2} = \frac{12}{(x+3)^2}$ | 1M |
| $x - 3 = -(x+3)$ or $x - 3 = x + 3$ (rejected) | |
| $x = 0$ | |
| <table><tr><th>$x$</th><th>$x < -3$</th><th>$-3 < x < 0$</th><th>$0 < x < 3$</th><th>$x > 3$</th></tr><tr><td>$f'(x)$</td><td>+</td><td>+</td><td>−</td><td>−</td></tr></table> | 1M |
| The maximum point is $(0, -9)$ and there is no minimum point. | 1A |
| When $f''(x) = 0$, | |
| $\frac{24}{(x-3)^3} = \frac{24}{(x+3)^3}$ | |
| $x + 3 = x - 3$ | |
| There is no solution. | |
| Thus, there is no point of inflexion. | 1A |
| (c) $x = -3$ and $x = 3$ are vertical asymptote. | 1A |
| Since $y = -1 + \frac{12}{x-3} - \frac{12}{x+3}$, $y = -1$ is a horizontal asymptote. | 1A |
| (d) $A = \int_9^k \left(\frac{12}{x-3} - \frac{12}{x+3} - 1 - (-1)\right) dx$ | 1M |
| $= \Big[12 \ln \lvert x-3 \rvert - 12 \ln \lvert x+3 \rvert\Big]_9^k$ | 1M |
| $= 12 \ln \frac{k-3}{6} - 12 \ln \frac{k+3}{12}$ | |
| $= 12 \ln \frac{k-3}{k+3} + 12 \ln 2$ | |
| Since $k > 9$, $0 < \frac{k-3}{k+3} < 1$ and $\ln \frac{k-3}{k+3} < 0$. | |
| We have $A < 12 \ln 2$. | 1 |

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|---|---|
| 13. (a) (i) Put $y = 0$ into $3x^2 - 2y - 4 = 0$, we have $x = \pm\sqrt{\frac{4}{3}}$. | 1A |
| The points $\left(\pm\sqrt{\frac{4}{3}},\ 0\right)$ lies on the curve $3x^2 + 2y + k = 0$, | |
| $3 \times \frac{4}{3} + 2(0) + k = 0$ | |
| $k = -4$ | 1A |
| (ii) The equation of $BC$ can be rewritten as $y = -\frac{3}{2}x^2 + 2$. | |
| The coordinates of vertex are $(0,\ 2)$. | 1A |
| (b) (i) When $0 \le h \le 2$, | |
| Volume $= \pi \int_0^h \left[\frac{2y+4}{3} - \frac{4-2y}{3}\right] dx$ | 1M+1A |
| $= \pi \int_0^h \frac{4y}{3}\, dy$ | |
| $= \pi \left[\frac{2y^2}{3}\right]_0^h$ | 1M |
| $= \frac{2\pi h^2}{3}$ cubic units | 1A |
| (ii) When $2 < h \le 6$, | |
| Volume $= \frac{2\pi(2)^2}{3} + \pi \int_2^h \frac{2y+4}{3}\, dy$ | 1M |
| $= \frac{8\pi}{3} + \pi\left[\frac{4y}{3} + \frac{y^2}{3}\right]_2^h$ | |
| $= \pi\left(\frac{h^2}{3} + \frac{4h}{3} - \frac{4}{3}\right)$ cubic units | 1A |
| (c) Let $V$ cubic units be the volume of water in the cup at time $t$ seconds. | |
| When $V = \frac{28\pi}{3}$, we have $V > \frac{8\pi}{3}$ and so $h > 2$. | 1M |
| $\frac{28\pi}{3} = \pi\left(\frac{h^2}{3} + \frac{4h}{3} - \frac{4}{3}\right)$ | |
| $h^2 + 4h - 32 = 0$ | 1M |
| $h = 4$ or $-8$ (rejected) | |
| For $2 < h \le 6$, $\frac{dV}{dt} = \pi\left(\frac{2h}{3} + \frac{4}{3}\right)\frac{dh}{dt}$ | 1M |
| When $\frac{dV}{dt} = \frac{8\pi}{3}$ and $h = 4$, | |
| $\frac{8\pi}{3} = \pi\left(\frac{2 \times 4}{3} + \frac{4}{3}\right)\frac{dh}{dt}$ | |
| $\frac{dh}{dt} = \frac{2}{3}$ | |
| Required rate is $\frac{2}{3}$ units per second. | 1A |



**END OF PAPER**