Solution	Marks
REG-2324-MOCK-SET 5-MATH-EP(M2)	
Suggested solutions	
1. $f(1+h) = \dfrac{1}{e^{4(1+h)-1}}$	
$= \dfrac{1}{e^{4h+3}}$	1A
$f'(1) = \lim_{h\to 0} \dfrac{f(1+h)-f(1)}{h}$	1M
$= \lim_{h\to 0} \dfrac{\frac{1}{e^{4h+3}} - \frac{1}{e^3}}{h}$	
$= \lim_{h\to 0} \dfrac{1-e^{4h}}{he^{4h+3}}$	
$= \lim_{h\to 0} \dfrac{e^{4h}-1}{4h} \cdot \dfrac{-4}{e^{4h+3}}$	1M
$= 1 \cdot \dfrac{-4}{e^3}$	
$= -\dfrac{4}{e^3}$	1A
2. (a) $(1+4x+4x^2)^n = (1+2x)^{2n}$	1M
$= 1 + C_1^{2n}(2x) + C_2^{2n}(2x)^2 + \ldots$	
$= 1 + 4nx + 4n(2n-1)x^2 + \ldots$	
$(1-x)^9 = 1 - 9x + 36x^2 + \ldots$	
$4n(2n-1) + 4n(-9) + 36 = 4$	1M
$8n^2 - 40n + 32 = 0$	
$n = 4$ or 1 (rejected)	1A
(b) $(1+4x+4x^2)^4(1-x)^9$	
$= (1+2x)^8(1-x)^9$	
$= (1+16x+112x^2+448x^3+\ldots)(1-9x+36x^2-84x^3+\ldots)$	
Required coefficient $= (448)(1) + (112)(-9) + (16)(36) + (1)(-84)$	1M
$= -68$	1A

Solution	Marks
3. (a) $\overrightarrow{OP} = \dfrac{3}{7}\mathbf{a} + \dfrac{4}{7}\mathbf{b}$	1A
(b) (i) $\lvert\mathbf{a} \times \mathbf{b}\rvert = \lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert \sin \angle AOB$	1M
$294 = (35)(14) \sin \angle AOB$	
$\sin \angle AOB = \dfrac{3}{5}$	
Since $\angle AOB$ is an acute angle, $\cos \angle AOB = \dfrac{\sqrt{5^2 - 3^2}}{5} = \dfrac{4}{5}$.	
$\mathbf{a} \cdot \mathbf{b} = \lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert \cos \angle AOB$	1M
$= (35)(14)\left(\dfrac{4}{5}\right)$	
$= 392$	1A
(ii) $\overrightarrow{OP} \cdot \overrightarrow{OP} = \left(\dfrac{3}{7}\mathbf{a} + \dfrac{4}{7}\mathbf{b}\right) \cdot \left(\dfrac{3}{7}\mathbf{a} + \dfrac{4}{7}\mathbf{b}\right)$	
$= \dfrac{9}{49}\lvert\mathbf{a}\rvert^2 + \dfrac{24}{49}\mathbf{a} \cdot \mathbf{b} + \dfrac{16}{49}\lvert\mathbf{b}\rvert^2$	
$= \dfrac{9}{49}(35)^2 + \dfrac{24}{49}(392) + \dfrac{16}{49}(14)^2$	1M
$= 481$	
$\left\lvert\overrightarrow{OP}\right\rvert = \sqrt{481}$	1A
4. (a) Let $u = 1 + x^3$. Then $\mathrm{d}u = 3x^2\,\mathrm{d}x$.	1M
$\displaystyle\int \frac{x^5}{\sqrt{1+x^3}}\,\mathrm{d}x = \frac{1}{3}\int \frac{u-1}{\sqrt{u}}\,\mathrm{d}u$	1M
$\displaystyle = \frac{1}{3}\int (u^{\frac{1}{2}} - u^{-\frac{1}{2}})\,\mathrm{d}u$	
$= \dfrac{2}{9}u^{\frac{3}{2}} - \dfrac{2}{3}u^{\frac{1}{2}} + \text{constant}$	
$= \dfrac{2}{9}(1+x^3)^{\frac{3}{2}} - \dfrac{2}{3}(1+x^3)^{\frac{1}{2}} + \text{constant}$	1A
(b) $\displaystyle y = \int \frac{9x^5}{2\sqrt{1+x^3}}\,\mathrm{d}x$	1M
$= \dfrac{9}{2}\left[\dfrac{2}{9}(1+x^3)^{\frac{3}{2}} - \dfrac{2}{3}(1+x^3)^{\frac{1}{2}}\right] + C$	1M
$= (1+x^3)^{\frac{3}{2}} - 3\sqrt{1+x^3} + C$	
$3 = 1 - 3 + C$	1M
$C = 5$	
Required equation is $y = (1+x^3)^{\frac{3}{2}} - 3\sqrt{1+x^3} + 5$.	1A

Solution	Marks

5. (a) When $n = 1$,

L.H.S. $= 1(3-1) = 2$ and R.H.S. $= 1^2(1+1) = 2 =$ L.H.S.

It is true for $n = 1$. **1**

Assume $\displaystyle\sum_{k=1}^{p} k(3k-1) = p^2(p+1)$, where $p \in \mathbb{Z}^+$. **1M**

$$\begin{aligned}\sum_{k=1}^{p+1} k(3k-1) &= \sum_{k=1}^{p} k(3k-1) + (p+1)(3p+2)\\ &= p^2(p+1) + (p+1)(3p+2) \quad \textbf{1M}\\ &= (p+1)(p^2+3p+2)\\ &= (p+1)^2(p+2)\end{aligned}$$

It is true for $n = p+1$.

By M.I., it is true $\forall n \in \mathbb{Z}^+$. **1**

(b) $$\begin{aligned}\sum_{k=1}^{n} k(3k+1) &= \sum_{k=1}^{n} k(3k-1) + \sum_{k=1}^{n} 2k \quad \textbf{1M}\\ &= n^2(n+1) + 2\cdot\frac{n(n+1)}{2} \quad \textbf{1M}\\ &= n(n+1)^2 \quad \textbf{1A}\end{aligned}$$

6. (a) (i) $\Delta = \begin{vmatrix} 3 & -1 & 2\\ 1 & 2 & 3\\ -1 & 5 & h\end{vmatrix}$

$= 7h - 28$

Since (E) has a unique solution, $\Delta \neq 0$. **1M**

Thus, we have $h \neq 4$. **1A**

(ii) $$z = \frac{\begin{vmatrix} 3 & -1 & -1\\ 1 & 2 & 2\\ -1 & 5 & k\end{vmatrix}}{7h-28} \quad \textbf{1M}$$

$$= \frac{7k-35}{7h-28}$$

$$= \frac{k-5}{h-4} \quad \textbf{1A}$$

(b) When $h = 4$, the augmented matrix of (E) is

$$\left(\begin{array}{ccc|c} 3 & -1 & 2 & -1\\ 1 & 2 & 3 & 2\\ -1 & 5 & 4 & k\end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2\\ 3 & -1 & 2 & -1\\ -1 & 5 & 4 & k\end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2\\ 0 & -7 & -7 & -7\\ 0 & 7 & 7 & k+2\end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2\\ 0 & 1 & 1 & 1\\ 0 & 1 & 1 & k-5\end{array}\right) \quad \textbf{1M}$$

We have $k = 5$.

The solution set is $\{(-t,\ 1-t,\ t) : t \in \mathbb{R}\}$. **1A**

Solution	Marks
7. (a) Let r cm be the radius of the water surface in the container.	
$\frac{r}{h} = \tan 30°$	
$r = \frac{h}{\sqrt{3}}$	1M
$V = \frac{1}{3}\pi\left(\frac{h}{\sqrt{3}}\right)^2 h$	
$= \frac{\pi}{9}h^3$	1
$A = \pi r\sqrt{r^2 + h^2}$	
$= \pi \cdot \frac{h}{\sqrt{3}}\left(\frac{2h}{\sqrt{3}}\right)$	
$= \frac{2\pi}{3}h^2$	1
(b) When $V = \frac{64\pi}{9}$,	
$\frac{\pi}{9}h^3 = \frac{64\pi}{9}$	
$h = 4$	1A
When $h = 4$,	
$V = \frac{\pi}{9}h^3$	
$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\pi}{9}(3h^2)\frac{\mathrm{d}h}{\mathrm{d}t}$	1M
$8\pi = \frac{\pi}{9}(48)\frac{\mathrm{d}h}{\mathrm{d}t}$	
$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{3}{2}$	
Also,	
$\frac{\mathrm{d}A}{\mathrm{d}t} = \left(\frac{4\pi}{3}h\right)\frac{\mathrm{d}h}{\mathrm{d}t}$	1M
$= \frac{16\pi}{3}\left(\frac{3}{2}\right)$	
$= 8\pi$	
Required rate is 8π cm^2/s.	1A

Solution	Marks
8. (a) $\tan 3x = \dfrac{\tan 2x + \tan x}{1 - \tan 2x \tan x}$	1M
$= \dfrac{\frac{2\tan x}{1-\tan^2 x} + \tan x}{1 - \left(\frac{2\tan x}{1-\tan^2 x}\right)\tan x}$	1M
$= \dfrac{2\tan x + \tan x(1 - \tan^2 x)}{1 - \tan^2 x - 2\tan^2 x}$	
$= \dfrac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$	1
(b) $\tan 2x = \cot 3x$	
$\dfrac{2\tan x}{1 - \tan^2 x} = \dfrac{1 - 3\tan^2 x}{3\tan x - \tan^3 x}$	1M
$2\tan x(3\tan x - \tan^3 x) = (1 - 3\tan^2 x)(1 - \tan^2 x)$	1M
$6\tan^2 x - 2\tan^4 x = 1 - 4\tan^2 x + 3\tan^4 x$	
$5\tan^4 x - 10\tan^2 x + 1 = 0$	1
(c) $\tan 2x = \cot 3x$	
$5\tan^4 x - 10\tan^2 x + 1 = 0$	
$\tan^2 x = \dfrac{5 \pm 2\sqrt{5}}{5}$	
$\tan x = \sqrt{\dfrac{5 \pm 2\sqrt{5}}{5}}$	1A
$x \approx 0.314$ or 0.942	1A

Solution	Marks

9. (a) (i) $f'(x) = \dfrac{162(x+1)^3 - 162(x-1)(3)(x+1)^2}{(x+1)^6}$

$= \dfrac{324(2-x)}{(x+1)^4}$ 1A

$f''(x) = \dfrac{-324(x+1)^4 - 324(2-x)(4)(x+1)^3}{(x+1)^8}$

$= \dfrac{972(x-3)}{(x+1)^5}$ 1A

(ii) When $f'(x) = 0$, $x = 2$.

x	$x < -1$	$-1 < x < 2$	$x > 2$
$f'(x)$	$+$	$+$	$-$

1M

The maximum point is $(2,\ 6)$. 1A

No minimum point.

When $f''(x) = 0$, $x = 3$.

x	$x < -1$	$-1 < x < 3$	$x > 3$
$f''(x)$	$+$	$-$	$+$

1M

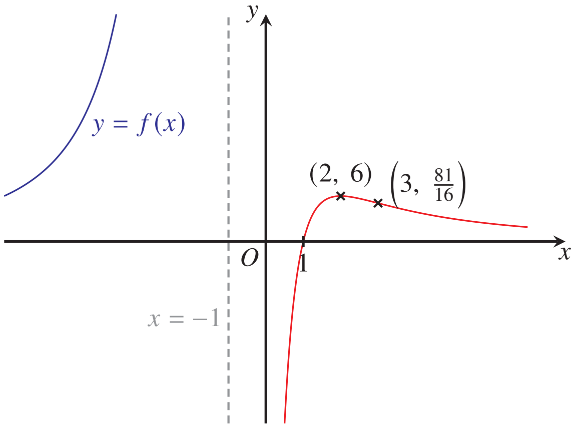
The point of inflexion is $\left(3,\ \dfrac{81}{16}\right)$. 1A

(b) Vertical asymptote is $x = -1$. 1A

Horizontal asymptote is $y = 0$. 1A

(c) (Shape, extrema and point of inflexion) 1A

(All correct) 1A



(d) Let $u = x + 1$. Then $\mathrm{d}u = \mathrm{d}x$.

Volume $= \pi \displaystyle\int_0^1 \left[\frac{162(x-1)}{(x+1)^3}\right]^2 \mathrm{d}x$ 1M

$= 26\,244\pi \displaystyle\int_1^2 \frac{(u-2)^2}{u^6}\,\mathrm{d}u$

$= 26\,244\pi \displaystyle\int_1^2 (u^{-4} - 4u^{-5} + 4u^{-6})\,\mathrm{d}u$ 1M

$= 26\,244\left[\dfrac{u^{-3}}{-3} + u^{-4} - \dfrac{4u^{-5}}{5}\right]_1^2$

$= \dfrac{67\,797\pi}{20}$ 1A

Solution	Marks

10. (a) (i) $\cos^4 x \sin^2 x = \cos^2 x \left(\frac{2\sin x \cos x}{2}\right)^2$

$= \left(\frac{1+\cos 2x}{2}\right)\left(\frac{\sin^2 2x}{4}\right)$ 1M

$= \frac{\sin^2 2x + \cos 2x \sin^2 2x}{8}$ 1

(ii) Let $x = \frac{\pi}{2} - u$. Then $\mathrm{d}x = -\mathrm{d}u$.

$\int_0^{\frac{\pi}{2}} \sin^4 x \cos^2 x \,\mathrm{d}x = -\int_{\frac{\pi}{2}}^{0} \sin^4\left(\frac{\pi}{2}-u\right)\cos^2\left(\frac{\pi}{2}-u\right)\mathrm{d}u$ 1M

$= \int_0^{\frac{\pi}{2}} \cos^4 u \sin^2 u \,\mathrm{d}u$

$= \int_0^{\frac{\pi}{2}} \frac{\sin^2 2u + \cos 2u \sin^2 2u}{8}\,\mathrm{d}u$

$= \frac{1}{16}\int_0^{\frac{\pi}{2}} (1-\cos 4u)\,\mathrm{d}u + \frac{1}{16}\int_0^{\frac{\pi}{2}} \sin^2 2u \,\mathrm{d}(\sin 2u)$ 1M+1M

$= \frac{1}{16}\left[u - \frac{\sin 4u}{4}\right]_0^{\frac{\pi}{2}} + \frac{1}{16}\left[\frac{\sin^3 2u}{3}\right]_0^{\frac{\pi}{2}}$

$= \frac{\pi}{32}$ 1A

(b) (i) Let $x = k - u$. Then $\mathrm{d}x = -\mathrm{d}u$.

$\int_0^{k} f(x)\,\mathrm{d}x = \int_0^{\frac{k}{2}} f(x)\,\mathrm{d}x + \int_{\frac{k}{2}}^{k} f(x)\,\mathrm{d}x$

$= \int_0^{\frac{k}{2}} f(x)\,\mathrm{d}x - \int_{\frac{k}{2}}^{0} f(k-u)\,\mathrm{d}u$ 1M

$= \int_0^{\frac{k}{2}} f(x)\,\mathrm{d}x + \int_0^{\frac{k}{2}} f(x)\,\mathrm{d}x$

$= 2\int_0^{\frac{k}{2}} f(x)\,\mathrm{d}x$ 1

(ii) Let $x = k - w$. Then $\mathrm{d}x = -\mathrm{d}w$.

$\int_0^{k} x f(x)\,\mathrm{d}x = -\int_k^{0} (k-w) f(k-w)\,\mathrm{d}w$ 1M

$= k\int_0^{k} f(w)\,\mathrm{d}w - \int_0^{k} w f(w)\,\mathrm{d}w$

$= k\int_0^{k} f(x)\,\mathrm{d}x - \int_0^{k} x f(x)\,\mathrm{d}x$

$2\int_0^{k} x f(x)\,\mathrm{d}x = k\int_0^{k} f(x)\,\mathrm{d}x$

$\int_0^{k} x f(x)\,\mathrm{d}x = \frac{k}{2}\int_0^{k} f(x)\,\mathrm{d}x$

$= k\int_0^{\frac{k}{2}} f(x)\,\mathrm{d}x$ 1

Solution	Marks
(iii) Let $f(x) = \cos^4 x \sin^2 x$.	
Then $f(\pi - x) = \cos^4(\pi - x)\sin^2(\pi - x) = \cos^4 x \sin^2 x = f(x)$.	1M
Volume $= \pi \int_0^{\pi} x \sin^4 x \cos^2 x \, dx$	1M
$= \pi^2 \int_0^{\frac{\pi}{2}} \sin^4 x \cos^2 x \, dx$	
$= \dfrac{\pi^3}{32}$	1A

Solution	Marks
11. (a) (i) $\begin{pmatrix} a & -3 \\ -3 & a+8 \end{pmatrix}\begin{pmatrix} 1 \\ -3 \end{pmatrix} = \lambda\begin{pmatrix} 1 \\ -3 \end{pmatrix}$	
$\begin{pmatrix} a+9 \\ -3a-27 \end{pmatrix} = \begin{pmatrix} \lambda \\ -3\lambda \end{pmatrix}$	
We have $\lambda = a+9$.	1A
(ii) $\begin{pmatrix} a & -3 \\ -3 & a+8 \end{pmatrix}\begin{pmatrix} b \\ 1 \end{pmatrix} = \mu\begin{pmatrix} b \\ 1 \end{pmatrix}$	
$\begin{pmatrix} ab-3 \\ -3b+a+8 \end{pmatrix} = \begin{pmatrix} \mu b \\ \mu \end{pmatrix}$	
We have $ab-3 = \mu b$ and $-3b+a+8 = \mu$.	
$ab-3 = (-3b+a+8)b$	1M
$3b^2 - 8b - 3 = 0$	
$b = 3$ or $-\frac{1}{3}$ (rejected)	
$\mu = -3(3)+a+8 = a-1$	1A
(iii) (1) $M^T M = \begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ -3 & 1 \end{pmatrix}$	
$= \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix}$	1A
(2) When $n = 1$,	
R.H.S. $= \frac{1}{10}MDM^T$	
$= \frac{1}{10}\begin{pmatrix} 1 & 3 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} a+9 & 0 \\ 0 & a-1 \end{pmatrix}\begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix}$	
$= \frac{1}{10}\begin{pmatrix} a+9 & 3a-3 \\ -3a-27 & a-1 \end{pmatrix}\begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix}$	
$= \begin{pmatrix} a & -3 \\ -3 & a+8 \end{pmatrix}$	
It is true for $n = 1$.	1
Assume $A^k = \frac{1}{10}MD^kM^T$, where $k \in \mathbb{Z}^+$.	1
$A^{k+1} = A^kA$	
$= \frac{1}{10}MD^kM^T\left(\frac{1}{10}MDM^T\right)$	1M
$= \frac{1}{100}MD^k(M^TM)DM^T$	
$= \frac{1}{100}MD^k(10I)DM^T$	
$= \frac{1}{10}MD^{k+1}M^T$	
It is true for $n = k+1$.	
By M.I., it is true $\forall n \in \mathbb{Z}^+$.	1

Solution	Marks
(iv) Put $a = 2$, then $\lambda = 11$, $\mu = 1$ and $A = \begin{pmatrix} 2 & -3 \\ -3 & 10 \end{pmatrix}$.	
$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 2 & -3 \\ -3 & 10 \end{pmatrix}^{2019} \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{0}$$	
$$\begin{pmatrix} x & y \end{pmatrix} \left(\frac{1}{10} MDM^T \right) \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{0}$$	
$$\frac{1}{10} \begin{pmatrix} x - 3y & 3x + y \end{pmatrix} \begin{pmatrix} 11^{2019} & 0 \\ 0 & 1^{2019} \end{pmatrix} \begin{pmatrix} x - 3y \\ 3x + y \end{pmatrix} = \mathbf{0}$$	1M
$$\begin{pmatrix} 11^{2019}(x - 3y) & 3x + y \end{pmatrix} \begin{pmatrix} x - 3y \\ 3x + y \end{pmatrix} = \mathbf{0}$$	
$$\begin{pmatrix} 11^{2019}(x - 3y)^2 + (3x + y)^2 \end{pmatrix} = \mathbf{0}$$	
We have $11^{2019}(x - 3y)^2 + (3x + y)^2 = 0$.	
Thus, $x - 3y = 0$ and $3x + y = 0$.	1M
Solving, we have $x = y = 0$.	1

Solution	Marks
12. (a) (i) $\overrightarrow{AB} = 2\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$	
$\overrightarrow{AC} = -\mathbf{i} + \mathbf{k}$	
$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & -2 \\ -1 & 0 & 1 \end{vmatrix}$	
$= 2\mathbf{i} + 2\mathbf{k}$	1A
(ii) Let $\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC}$.	
$\overrightarrow{AD} = \mathbf{i} - \mathbf{j} + 5\mathbf{k}$	
$\overrightarrow{ED} = \dfrac{\overrightarrow{AD} \cdot \mathbf{n}}{\mathbf{n} \cdot \mathbf{n}}(\mathbf{n})$	1M
$= \dfrac{2 + 10}{2^2 + 2^2}(\mathbf{n})$	
$= 3\mathbf{i} + 3\mathbf{k}$	1A
(iii) Required volume $= \dfrac{1}{3}\left(\dfrac{1}{2}\right)\left\lvert\overrightarrow{AB} \times \overrightarrow{AC}\right\rvert(ED)$	1M
$= \dfrac{1}{6}\sqrt{2^2 + 0^2 + 2^2}\sqrt{3^2 + 0^2 + 3^2}$	
$= 2$	1A
(b) (i) $\overrightarrow{BF}$ is the projection of $\overrightarrow{BD}$ on $\overrightarrow{BA}$.	
$\overrightarrow{BD} = -\mathbf{i} - 3\mathbf{j} + 7\mathbf{k}$	
$\overrightarrow{BA} = -2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$	
$\overrightarrow{BF} = \dfrac{\overrightarrow{BD} \cdot \overrightarrow{BA}}{\overrightarrow{BA} \cdot \overrightarrow{BA}}(\overrightarrow{BA})$	1M
$= -\dfrac{11}{3}\mathbf{i} - \dfrac{11}{3}\mathbf{j} + \dfrac{11}{3}\mathbf{k}$	
$\overrightarrow{DF} = \overrightarrow{BF} + \overrightarrow{DB}$	1M
$= -\dfrac{8}{3}\mathbf{i} - \dfrac{2}{3}\mathbf{j} - \dfrac{10}{3}\mathbf{k}$	1A
(ii) $\overrightarrow{BF} \cdot \overrightarrow{EF} = \overrightarrow{BF} \cdot (\overrightarrow{ED} + \overrightarrow{DF})$	1M
$= \left(-\dfrac{11}{3}\mathbf{i} - \dfrac{11}{3}\mathbf{j} + \dfrac{11}{3}\mathbf{k}\right) \cdot \left(\dfrac{1}{3}\mathbf{i} - \dfrac{2}{3}\mathbf{j} - \dfrac{1}{3}\mathbf{k}\right)$	
$= 0$	
Thus, $\overrightarrow{BF}$ is perpendicular to $\overrightarrow{EF}$.	1A
(c) Required angle is $\angle DFE$.	1M
$\sin \angle DFE = \dfrac{ED}{DF}$	
$= \dfrac{\sqrt{3^2 + 0^2 + 3^2}}{\sqrt{\left(\frac{8}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{10}{3}\right)^2}}$	1M
$= \dfrac{3\sqrt{21}}{14}$	
$\angle DFE = \sin^{-1}\dfrac{3\sqrt{21}}{14}$	1A

END OF PAPER