M2REG-TRIG-2425-ASM-SET 3-MATH
Suggested solutions

1. (a)
$$\begin{aligned}\tan B &= \tan\left(\frac{\pi}{4} - A\right) \\ &= \frac{\tan\frac{\pi}{4} - \tan A}{1 + \tan\frac{\pi}{4}\tan A} \\ &= \frac{1 - \frac{1}{1+2m}}{1 + 1 \cdot \frac{1}{1+2m}} && \text{1M} \\ &= \frac{(1+2m) - 1}{(1+2m) + 1} \\ &= \frac{m}{1+m} && \text{1}\end{aligned}$$

(b)
$$\begin{aligned}\tan(A - B) &= \frac{\tan A - \tan B}{1 + \tan A \tan B} \\ &= \frac{\frac{1}{1+2m} - \frac{m}{1+m}}{1 + \left(\frac{1}{1+2m}\right)\left(\frac{m}{1+m}\right)} && \text{1M} \\ &= \frac{(1+m) - m(1+2m)}{(1+2m)(1+m)} \div \frac{(1+2m)(1+m) + m}{(1+2m)(1+m)} \\ &= \frac{1 - 2m^2}{1 + 4m + 2m^2} && \text{1A}\end{aligned}$$

2. (a)
$$\begin{aligned}\cot(\angle A + \angle B) &= \frac{1}{\tan(\angle A + \angle B)} \\ &= \frac{1 - \tan\angle A \tan\angle B}{\tan\angle A + \tan\angle B} && \text{1M} \\ &= \frac{1 - (\sqrt{a} + \sqrt{a-1})(\sqrt{a} - \sqrt{a-1})}{(\sqrt{a} + \sqrt{a-1}) + (\sqrt{a} - \sqrt{a-1})} \\ &= \frac{1 - [a - (a-1)]}{2\sqrt{a}} \\ &= 0 && \text{1A}\end{aligned}$$

(b)
$$\begin{aligned}\cot(\angle A + \angle B) &= 0 \\ \angle A + \angle B &= 90^\circ && \text{1A} \\ \angle C &= 180^\circ - (\angle A + \angle B) \\ &= 90^\circ\end{aligned}$$

Thus, $\triangle ABC$ is a right-angled triangle. 1A

3. (a)

$$k\cos(\alpha-\beta)=\cos(\alpha+\beta)$$

$$k(\cos\alpha\cos\beta+\sin\alpha\sin\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta \qquad \text{1M}$$

$$(k+1)\sin\alpha\sin\beta=(1-k)\cos\alpha\cos\beta$$

$$(k+1)\tan\alpha\tan\beta=1-k$$

$$\tan\alpha\tan\beta=\frac{1-k}{1+k} \qquad 1$$

(b)

$$\frac{1-k}{1+k}=\tan(2\beta)\tan\beta$$

$$=\frac{2\tan\beta}{1-\tan^2\beta}\cdot\tan\beta$$

$$=\frac{2\tan^2\beta}{1-\tan^2\beta} \qquad \text{1A}$$

$$(1-k)(1-\tan^2\beta)=2(1+k)\tan^2\beta$$

$$(3+k)\tan^2\beta=1-k$$

$$\tan^2\beta=\frac{1-k}{3+k}$$

$$\tan\beta=\pm\sqrt{\frac{1-k}{3+k}} \qquad \text{1A}$$

Note that when $\pi<\alpha<2\pi$, we have $\frac{\pi}{2}<\beta<\pi$ and $\tan\beta<0$. 1A

Thus, $\tan\beta=-\sqrt{\frac{1-k}{3+k}}$. 1A

4. Since O is the orthocentre of $\triangle ABC$, we have $CD\perp AB$ and $BE\perp AC$.

$$\tan\alpha=\frac{CE}{BE}$$

$$=\frac{4\sqrt{6}}{6\sqrt{6}}$$

$$=\frac{2}{3} \qquad \text{1A}$$

$$\tan\beta=\frac{BD}{CD}$$

$$=\frac{3}{15}$$

$$=\frac{1}{5} \qquad \text{1A}$$

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

$$=\frac{\frac{2}{3}+\frac{1}{5}}{1-\frac{2}{3}\times\frac{1}{5}}$$

$$=1 \qquad \text{1A}$$

Since $0<\alpha+\beta<\pi$, we have $\alpha+\beta=\frac{\pi}{4}$. 1

5. $\tan(x+y)-\tan x-\tan y-\tan x\tan y\tan(x+y)$

$= \tan(x+y)(1-\tan x\tan y)-(\tan x+\tan y)$

$= \dfrac{\tan x+\tan y}{1-\tan x\tan y}(1-\tan x\tan y)-(\tan x+\tan y)$ 1M+1M

$= (\tan x+\tan y)-(\tan x+\tan y)$

$= 0$

Thus, $\tan(x+y)-\tan x-\tan y=\tan x\tan y\tan(x+y)$. 1

6. $\cos(x+y+z)=\cos(x+y)\cos z-\sin(x+y)\sin z$ 1M

$= (\cos x\cos y-\sin x\sin y)\cos z-(\sin x\cos y+\cos x\sin y)\sin z$ 1M

$= \cos x\cos y\cos z-\sin x\sin y\cos z-\sin x\cos y\sin z-\cos x\sin y\sin z$

$= \cos x\cos y\cos z(1-\tan x\tan y-\tan x\tan z-\tan y\tan z)$ 1

7. $\dfrac{\cos(6\alpha+\beta)}{\sin 3\alpha}+2\sin(3\alpha+\beta)$

$= \dfrac{\cos[(3\alpha+\beta)+3\alpha]+2\sin(3\alpha+\beta)\sin 3\alpha}{\sin 3\alpha}$

$= \dfrac{\cos(3\alpha+\beta)\cos 3\alpha-\sin(3\alpha+\beta)\sin 3\alpha+2\sin(3\alpha+\beta)\sin 3\alpha}{\sin 3\alpha}$ 1M

$= \dfrac{\cos(3\alpha+\beta)\cos 3\alpha+\sin(3\alpha+\beta)\sin 3\alpha}{\sin 3\alpha}$

$= \dfrac{\cos[(3\alpha+\beta)-3\alpha]}{\sin 3\alpha}$ 1M

$= \dfrac{\cos\beta}{\sin 3\alpha}$ 1

8. $\dfrac{\cos A-\cos B\cos(A-B)}{\sin A-\cos B\sin(A-B)}$

$= \dfrac{\cos A-\cos B(\cos A\cos B+\sin A\sin B)}{\sin A-\cos B(\sin A\cos B-\cos A\sin B)}$ 1M

$= \dfrac{\cos A-\cos A\cos^2 B-\sin A\sin B\cos B}{\sin A-\sin A\cos^2 B+\cos A\sin B\cos B}$

$= \dfrac{\cos A(1-\cos^2 B)-\sin A\sin B\cos B}{\sin A(1-\cos^2 B)+\cos A\sin B\cos B}$

$= \dfrac{\cos A\sin^2 B-\sin A\sin B\cos B}{\sin A\sin^2 B+\cos A\sin B\cos B}$

$= \dfrac{\sin B(\cos A\sin B-\sin A\cos B)}{\sin B(\sin A\sin B+\cos A\cos B)}$

$= \dfrac{\sin(B-A)}{\cos(B-A)}$ 1M

$= \tan(B-A)$ 1

9. (a) $\dfrac{\cos(\alpha+\beta)}{\cos(\alpha-\beta)} = \dfrac{\cos\alpha\cos\beta - \sin\alpha\sin\beta}{\cos\alpha\cos\beta + \sin\alpha\sin\beta}$ 1M

$$= \frac{\frac{\cos\alpha\cos\beta}{\sin\alpha\cos\beta} - \frac{\sin\alpha\sin\beta}{\sin\alpha\cos\beta}}{\frac{\cos\alpha\cos\beta}{\sin\alpha\cos\beta} + \frac{\sin\alpha\sin\beta}{\sin\alpha\cos\beta}}$$

$$= \frac{\cot\alpha - \tan\beta}{\cot\alpha + \tan\beta}$$ 1

(b) We have $\cos(\alpha+\beta)(\cot\alpha+\tan\beta) = \cos(\alpha-\beta)(\cot\alpha-\tan\beta)$.

$$\frac{1}{\cos 1^\circ}\sum_{i=1}^{90}\cos(2i-1)^\circ[\cot i^\circ + \tan(i-1)^\circ]$$

$$= \frac{1}{\cos 1^\circ}\sum_{i=1}^{90}\cos[i+(i-1)]^\circ[\cot i^\circ + \tan(i-1)^\circ]$$

$$= \frac{1}{\cos 1^\circ}\sum_{i=1}^{90}\cos[i-(i-1)]^\circ[\cot i^\circ - \tan(i-1)^\circ]$$ 1M

$$= (\cot 1^\circ + \cot 2^\circ + \cot 3^\circ + \ldots + \cot 90^\circ) - (\tan 0^\circ + \tan 1^\circ + \tan 2^\circ + \ldots + \tan 89^\circ)$$

$$= (\tan 89^\circ + \tan 88^\circ + \tan 87^\circ + \ldots + \tan 0^\circ) - (\tan 0^\circ + \tan 1^\circ + \tan 2^\circ + \ldots + \tan 89^\circ)$$ 1M

$$= 0$$ 1A

10. $\tan\dfrac{A}{2}\tan\dfrac{B}{2} + \tan\dfrac{B}{2}\tan\dfrac{C}{2} + \tan\dfrac{C}{2}\tan\dfrac{A}{2}$

$$= \tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{180^\circ - A - B}{2} + \tan\frac{180^\circ - A - B}{2}\tan\frac{A}{2}$$ 1M

$$= \tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\cot\left(\frac{A}{2}+\frac{B}{2}\right) + \cot\left(\frac{A}{2}+\frac{B}{2}\right)\tan\frac{A}{2}$$

$$= \tan\frac{A}{2}\tan\frac{B}{2} + \left(\tan\frac{B}{2} + \tan\frac{A}{2}\right)\cdot\frac{1-\tan\frac{A}{2}\tan\frac{B}{2}}{\tan\frac{A}{2}+\tan\frac{B}{2}}$$ 1M

$$= \tan\frac{A}{2}\tan\frac{B}{2} + 1 - \tan\frac{A}{2}\tan\frac{B}{2}$$

$$= 1$$ 1

11. $\dfrac{\tan 10x - \tan 8x}{1 + \tan 10x\tan 8x} = \dfrac{\sqrt{3}}{3}$

$$\tan(10x - 8x) = \frac{\sqrt{3}}{3}$$ 1M

$$\tan 2x = \frac{\sqrt{3}}{3}$$

$$2x = \frac{\pi}{6} \text{ or } \frac{7\pi}{6} \text{ or } \frac{13\pi}{6} \text{ or } \frac{19\pi}{6}$$ 1M

$$x = \frac{\pi}{12} \text{ or } \frac{7\pi}{12} \text{ or } \frac{13\pi}{12} \text{ or } \frac{19\pi}{12}$$ 1A

12. (a) $\tan(\alpha+\beta)(\tan\alpha+\tan\beta)+2$

$$= \frac{(\tan\alpha+\tan\beta)^2}{1-\tan\alpha\tan\beta}+2 \qquad \text{1M}$$

$$= \frac{\tan^2\alpha+2\tan\alpha\tan\beta+\tan^2\beta+2(1-\tan\alpha\tan\beta)}{1-\tan\alpha\tan\beta}$$

$$= \frac{\tan^2\alpha+\tan^2\beta+2}{1-\tan\alpha\tan\beta} \qquad \text{1M}$$

$$= \frac{(\tan^2\alpha+1)+(\tan^2\beta+1)}{1-\tan\alpha\tan\beta}$$

$$= \frac{\sec^2\alpha+\sec^2\beta}{1-\tan\alpha\tan\beta} \qquad 1$$

(b)

$$\tan\left(\alpha+\frac{\pi}{4}\right)\left(\tan\alpha+\tan\frac{\pi}{4}\right) = -8$$

$$\tan\left(\alpha+\frac{\pi}{4}\right)\left(\tan\alpha+\tan\frac{\pi}{4}\right)+2 = -6$$

$$\frac{\sec^2\alpha+\sec^2\frac{\pi}{4}}{1-\tan\alpha\tan\frac{\pi}{4}} = -6 \qquad \text{1M}$$

$$\frac{1+\tan^2\alpha+2}{1-\tan\alpha} = -6$$

$$\tan^2\alpha+3 = -6+6\tan\alpha$$

$$\tan^2\alpha-6\tan\alpha+9 = 0 \qquad \text{1A}$$

$$\tan\alpha = 3$$

$$\alpha \approx 1.25 \quad \text{or} \quad 4.39 \qquad \text{1A}$$

13. (a) $\tan A$ and $\tan B$ are roots of the equation $x^2 + 6x - 3 = 0$.

We have $\tan A + \tan B = -6$ and $\tan A \tan B = -3$. 1A

$$\begin{aligned}(\tan A - \tan B)^2 &= \tan^2 A - 2\tan A \tan B + \tan^2 B \\ &= (\tan A + \tan B)^2 - 4\tan A \tan B \quad \text{1M} \\ &= (-6)^2 - 4(-3) \\ &= 48 \quad \text{1A}\end{aligned}$$

Since A is obtuse angle and B is acute angle, $\tan A < 0$ and $\tan B > 0$.

$\tan A - \tan B = -\sqrt{48} = -4\sqrt{3}$ 1A

(b) $$\begin{aligned}\tan(A - B) &= \frac{\tan A - \tan B}{1 + \tan A \tan B} \quad \text{1M} \\ &= \frac{-4\sqrt{3}}{1 + (-3)} \\ &= 2\sqrt{3} \quad \text{1A}\end{aligned}$$

(c) Since $\tan(A - B) > 0$, angle $A - B$ is an acute angle.

$$\begin{aligned}\sec(A - B) &= \frac{\sqrt{(2\sqrt{3})^2 + 1}}{1} \\ &= \sqrt{13} \quad \text{1A}\end{aligned}$$

$$\begin{aligned}\csc(A - B) &= \frac{\sqrt{13}}{2\sqrt{3}} \\ &= \frac{\sqrt{39}}{6} \quad \text{1A}\end{aligned}$$

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