M2REG-INVM-2425-ASM-SET 3-MATH
Suggested solutions

1. (a) $\det Q = \dfrac{5}{11} + \dfrac{6}{11} = 1$ 1A

$$Q^{-1} = \frac{1}{1}\begin{pmatrix} 1 & 2 \\ -\frac{3}{11} & \frac{5}{11} \end{pmatrix}^T$$

$$= \begin{pmatrix} 1 & -\frac{3}{11} \\ 2 & \frac{5}{11} \end{pmatrix}$$ 1A

$$QPQ^{-1} = \begin{pmatrix} \frac{5}{11} & \frac{3}{11} \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 5 & 3 \\ 10 & 6 \end{pmatrix}\begin{pmatrix} 1 & -\frac{3}{11} \\ 2 & \frac{5}{11} \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 3 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & -\frac{3}{11} \\ 2 & \frac{5}{11} \end{pmatrix}$$

$$= \begin{pmatrix} 11 & 0 \\ 0 & 0 \end{pmatrix}$$ 1A

(b) $QPQ^{-1} = \begin{pmatrix} 11 & 0 \\ 0 & 0 \end{pmatrix}$

$$P = Q^{-1}\begin{pmatrix} 11 & 0 \\ 0 & 0 \end{pmatrix}Q$$ 1M

$$P^n = Q^{-1}\begin{pmatrix} 11 & 0 \\ 0 & 0 \end{pmatrix}^n Q$$ 1M

$$= Q^{-1}\begin{pmatrix} 11^n & 0 \\ 0 & 0 \end{pmatrix}Q$$

$$= \begin{pmatrix} 1 & -\frac{3}{11} \\ 2 & \frac{5}{11} \end{pmatrix}\begin{pmatrix} 11^n & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \frac{5}{11} & \frac{3}{11} \\ -2 & 1 \end{pmatrix}$$ 1A

$$= \begin{pmatrix} 5 \cdot 11^{n-1} & 3 \cdot 11^{n-1} \\ 10 \cdot 11^{n-1} & 6 \cdot 11^{n-1} \end{pmatrix}$$ 1A

2. (a) (i) $P^2 = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 0 & -2 \\ 0 & 1 & 0 \\ -4 & 0 & 3 \end{pmatrix}$ 1A

$P^3 = \begin{pmatrix} 3 & 0 & -2 \\ 0 & 1 & 0 \\ -4 & 0 & 3 \end{pmatrix}\begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & -1 \end{pmatrix} = \begin{pmatrix} -7 & 0 & 5 \\ 0 & 1 & 0 \\ 10 & 0 & -7 \end{pmatrix}$

$P^3 + P^2 - 3P + I = \begin{pmatrix} -7 & 0 & 5 \\ 0 & 1 & 0 \\ 10 & 0 & -7 \end{pmatrix} + \begin{pmatrix} 3 & 0 & -2 \\ 0 & 1 & 0 \\ -4 & 0 & 3 \end{pmatrix} - 3\begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$= \mathbf{0}$ 1

(ii) $P^3 + P^2 - 3P + I = \mathbf{0}$

$I = 3P - P^2 - P^3$

$= P(3I - P - P^2)$ 1M

$P^{-1} = 3I - P - P^2$ 1M

$= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$ 1A

(b) (i) $P^{-1}AP = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 2 & 0 & 3 \end{pmatrix}\begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & -1 \end{pmatrix}$

$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = M$ 1

(ii) $\det\left(P^{-1}AP\right) = \det M$

$\det P^{-1} \cdot \det A \cdot \det P = \det M$

$\dfrac{1}{\det P} \cdot \det A \cdot \det P = \det M$

$\det A = \det M$

$= 2(1)(1) \neq 0$ 1M

Thus, A and M are non-singular. 1

(iii) $M^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ 1A

$(M^{-1})^{50} = \begin{pmatrix} 2^{-50} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ 1A

$$
\begin{aligned}
M^{-1} &= (P^{-1}AP)^{-1} \\
&= P^{-1}A^{-1}P \\
A^{-1} &= PM^{-1}P^{-1} && \text{1M} \\
(A^{-1})^{50} &= (PM^{-1}P^{-1})^{50} \\
&= P(M^{-1})^{50}P^{-1} && \text{1M} \\
&= \begin{pmatrix} 2-2^{-50} & 0 & 1-2^{-50} \\ 0 & 1 & 0 \\ 2^{-49}-2 & 0 & 2^{-49}-1 \end{pmatrix} && \text{1A}
\end{aligned}
$$

3. (a) (i) For $i = 1, 2$,

$$A\begin{pmatrix}x_i\\y_i\end{pmatrix} = \lambda_i\begin{pmatrix}x_i\\y_i\end{pmatrix}$$

$$(A - \lambda_i I)\begin{pmatrix}x_i\\y_i\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \qquad \text{1M}$$

If $\det(A - \lambda_i I) \neq 0$, then $(A - \lambda_i I)^{-1}$ exists and

$\begin{pmatrix}x_i\\y_i\end{pmatrix} = (A - \lambda_i I)^{-1}\begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$, which is a contradiction.

Thus, $\det(A - \lambda_i I) = 0$.

λ_1 and λ_2 satisfy $\det(A - \lambda I) = 0$. 1

(ii) $\begin{vmatrix}5-\lambda & 0\\21 & -2-\lambda\end{vmatrix} = 0$

$(5-\lambda)(-2-\lambda) = 0$ 1M

$$\lambda = -2 \quad \text{or} \quad 5$$

Thus, $\lambda_2 = -2$ and $\lambda_2 = 5$. 1A

(b) (i) For $i = 1, 2$.

$$A\begin{pmatrix}x_i\\y_i\end{pmatrix} = \lambda_i\begin{pmatrix}x_i\\y_i\end{pmatrix}$$

$$\begin{pmatrix}5x_i\\21x_i - 2y_i\end{pmatrix} = \begin{pmatrix}\lambda_i x_i\\\lambda_i y_i\end{pmatrix}$$

$21x_i - 2y_i = \lambda_i y_i$

$21x_i = (\lambda_i + 2)y_i$ 1M

If $y_i = 0$, then $x_i = 0$, which is a contradiction.

Thus, y_1 and y_2 are non-zero real numbers. 1

(ii) When $\lambda_1 = -2$,

$21x_1 = (-2+2)y_1$

$\frac{x_1}{y_1} = 0$ 1A

When $\lambda_2 = 5$,

$21x_2 = (5+2)y_2$

$\frac{x_2}{y_2} = \frac{1}{3}$ 1A

Thus, $P = \begin{pmatrix}0 & \frac{1}{3}\\1 & 1\end{pmatrix}$. 1A

(iii) $\det P = \begin{vmatrix}0 & \frac{1}{3}\\1 & 1\end{vmatrix} = -\frac{1}{3} \neq 0$

Thus, P is invertible. 1

$$P^{-1} = \frac{1}{\left(-\frac{1}{3}\right)} \begin{pmatrix} 1 & -1 \\ -\frac{1}{3} & 0 \end{pmatrix}^T$$

$$= \begin{pmatrix} -3 & 1 \\ 3 & 0 \end{pmatrix}$$ 1A

(c) (i) $$AP = \begin{pmatrix} 5 & 0 \\ 21 & -2 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{3} \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \frac{5}{3} \\ -2 & 5 \end{pmatrix}$$

$$P \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{3} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -2 & 0 \\ 0 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \frac{5}{3} \\ -2 & 5 \end{pmatrix}$$

$$= AP$$ 1

(ii) $$AP = P \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$$P^{-1}AP = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$$(P^{-1})A(P^{-1})^{-1} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & -\lambda_2 \end{pmatrix}$$

Thus, QAQ^{-1} is a diagonal matrix if $Q = P^{-1}$. 1

4. (a) (i) $\det A = -k$

$$A^{-1} = \frac{1}{-k}\begin{pmatrix} 0 & -1 \\ -k & 1 \end{pmatrix}^T \qquad \text{1M}$$

$$= \frac{1}{k}\begin{pmatrix} 0 & k \\ 1 & -1 \end{pmatrix} \qquad \text{1A}$$

(ii) $$A^{-1}MA = \frac{1}{k}\begin{pmatrix} 0 & k \\ 1 & -1 \end{pmatrix}\begin{pmatrix} k & 2-k \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & k \\ 1 & 0 \end{pmatrix}$$

$$= \frac{1}{k}\begin{pmatrix} 0 & k \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 2 & k^2 \\ 2 & 0 \end{pmatrix} \qquad \text{1M+1A}$$

$$= \frac{1}{k}\begin{pmatrix} 2k & 0 \\ 0 & k^2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 \\ 0 & k \end{pmatrix} \qquad \text{1A}$$

(iii) $$(A^{-1}MA)^n = \begin{pmatrix} 2 & 0 \\ 0 & k \end{pmatrix}^n$$

$$A^{-1}M^nA = \begin{pmatrix} 2^n & 0 \\ 0 & k^n \end{pmatrix} \qquad \text{1M}$$

$$M^n = A\begin{pmatrix} 2^n & 0 \\ 0 & k^n \end{pmatrix}A^{-1} \qquad \text{1M}$$

$$= \frac{1}{k}\begin{pmatrix} 1 & k \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2^n & 0 \\ 0 & k^n \end{pmatrix}\begin{pmatrix} 0 & k \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{k}\begin{pmatrix} 2^n & k^{n+1} \\ 2^n & 0 \end{pmatrix}\begin{pmatrix} 0 & k \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{k}\begin{pmatrix} k^{n+1} & k2^n - k^{n+1} \\ 0 & k2^n \end{pmatrix}$$

$$= \begin{pmatrix} k^n & 2^n - k^n \\ 0 & 2^n \end{pmatrix} \qquad \text{1A}$$

(b) Put $k = -1$ into M.

$$\begin{pmatrix} x_{2n+2} \\ x_{2n+1} \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x_{2n} \\ x_{2n-1} \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 3 \\ 0 & 2 \end{pmatrix}^2 \begin{pmatrix} x_{2n-2} \\ x_{2n-3} \end{pmatrix} \qquad \text{1M}$$

$$= \ldots$$

$$= \begin{pmatrix} -1 & 3 \\ 0 & 2 \end{pmatrix}^n \begin{pmatrix} x_2 \\ x_1 \end{pmatrix} \qquad \text{1A}$$

$$= \begin{pmatrix} (-1)^{n+1} & 2^n - (-1)^n \\ 0 & 2^n \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2^n + 2(-1)^{n+1} \\ 2^n \end{pmatrix} \qquad \text{1A}$$

Thus, $x_n = \begin{cases} 2^{\frac{n}{2}-1} + 2(-1)^{\frac{n}{2}} & \text{when } n = 2, 4, 6, \ldots \\ 2^{\frac{n-1}{2}} & \text{when } n = 1, 3, 5, \ldots \end{cases}$ 1A

5. (a) (i) $X^2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{pmatrix}$

$X^2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 3 \\ 3 & 2 & 4 \\ 4 & 3 & 6 \end{pmatrix}$

$$X^3 + mX^2 + nX = I$$

$\begin{pmatrix} 2 & 1 & 3 \\ 3 & 2 & 4 \\ 4 & 3 & 6 \end{pmatrix} + m\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{pmatrix} + n\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ 1M

We have $2 + m = 1$ and $3 + m + n = 0$. 1M

Solving, we have $m = -1$ and $n = -2$. 1A

(ii) $X^3 - X^2 - 2X = I$

$X(X^2 - X - 2I) = I = (X^2 - X - 2I)X$ 1M

$X^{-1} = X^2 - X - 2I$

$= \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ 1A

(b) (i) $Z = XY^3X^{-1}$

$= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} -2 & 0 & -4 \\ 3 & 1 & 3 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ 1A

(ii) $\det(Z) = (2)(-2)(1)$

$= -4$

$\det(Z) = \det\left(XY^3X^{-1}\right)$

$= \det X(\det Y^3)[\det\left(X^{-1}\right)]$

$= \det\left(XX^{-1}\right)(\det Y)^3$

$= (\det Y)^3$ 1M

$\det Y = \sqrt[3]{-4} \neq 0$

Thus, Y is non-singular. 1

(iii) $Z = XY^3X^{-1}$

$$Z^{-1} = (XY^3X^{-1})^{-1}$$
$$= X(Y^3)^{-1}X^{-1}$$
$$= X(Y^{-1})^3X^{-1} \quad \text{1M}$$
$$(Z^{-1})^{20} = X(Y^{-1})^{60}X^{-1}$$
$$(Y^{-1})^{60} = X^{-1}(Z^{-1})^{20}X^{-1} \quad \text{1M}$$
$$= X^{-1}\begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}^{20} X^{-1} \quad \text{1A}$$
$$= \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 2^{-20} & 0 & 0 \\ 0 & 2^{-20} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 2^{-20} & 0 & 0 \\ 1-2^{-20} & 1 & 1-2^{-20} \\ 0 & 0 & 2^{-20} \end{pmatrix} \quad \text{1A}$$

6. (a) $\begin{vmatrix} 1-\lambda & 3 \\ 2 & 2-\lambda \end{vmatrix} = 0$ 1A

$(1-\lambda)(2-\lambda)-6=0$ 1A

$\lambda^2 - 3\lambda - 4 = 0$

$\lambda = -1 \quad \text{or} \quad 4$ 1A

(b) $\lambda_1 = -1.$ $\begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_1 + 3y_1 \\ 2x_1 + 2y_1 \end{pmatrix} = \begin{pmatrix} -x_1 \\ -y_1 \end{pmatrix}.$ 1M

Thus, $x_1 + 3y_1 = -x_1$ and $2x_1 + 2y_1 = -y_1$.

A possible solution is $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}.$ 1A

$\lambda_2 = 4.$ $\begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_2 + 3y_2 \\ 2x_2 + 2y_2 \end{pmatrix} = \begin{pmatrix} 4x_2 \\ 4y_2 \end{pmatrix}$

Thus, $x_2 + 3y_2 = 4x_2$ and $2x_2 + 2y_2 = 4y_2$.

A possible solution is $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$ 1A

Let $P = \begin{pmatrix} -3 & 1 \\ 2 & 1 \end{pmatrix}$. Then $\det P = -3 - 2 = -5 \neq 0$.

$P^{-1} = \frac{1}{-5}\begin{pmatrix} 1 & -1 \\ -2 & -3 \end{pmatrix}$ 1A

$$\begin{aligned} P^{-1}\begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}P &= -\frac{1}{5}\begin{pmatrix} 1 & -1 \\ -2 & -3 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} -3 & 1 \\ 2 & 1 \end{pmatrix} \\ &= -\frac{1}{5}\begin{pmatrix} 1 & -1 \\ -2 & -3 \end{pmatrix}\begin{pmatrix} 3 & 4 \\ -2 & 4 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix} \end{aligned}$$ 1A

(c) $\left[P^{-1}\begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}P\right]^{1996} = \begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix}^{1996}$

$P^{-1}\begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}^{1996}P = \begin{pmatrix} 1 & 0 \\ 0 & 4^{1996} \end{pmatrix}$ 1M

$\begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}^{1996} = P\begin{pmatrix} 1 & 0 \\ 0 & 4^{1996} \end{pmatrix}P^{-1}$ 1M

$$\begin{aligned} &= -\frac{1}{5}\begin{pmatrix} -3 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 4^{1996} \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -2 & -3 \end{pmatrix} \\ &= \frac{1}{5}\begin{pmatrix} -3 & 4^{1996} \\ 2 & 4^{1996} \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 2 & 3 \end{pmatrix} \\ &= \frac{1}{5}\begin{pmatrix} 3 + 2 \cdot 4^{1996} & -3 + 3 \cdot 4^{1996} \\ -2 + 2 \cdot 4^{1996} & 2 + 3 \cdot 4^{1996} \end{pmatrix} \end{aligned}$$ 1A

7. (a) (i) Assume on the contrary, P is a singular matrix, i.e., $|P| = a + b = 0$. 1A

Then $ab = (-b)b = -b^2 \le 0$, which is a contradiction. 1M

Therefore, $|P|$ is a non-singular matrix. 1

(ii) $$P^{-1} = \frac{1}{a+b}\begin{pmatrix} 1 & -b \\ 1 & a \end{pmatrix}^T$$

$$= \frac{1}{a+b}\begin{pmatrix} 1 & 1 \\ -b & a \end{pmatrix}$$ 1A

$P^{-1}AP$

$$= \frac{1}{a+b}\begin{pmatrix} 1 & 1 \\ -b & a \end{pmatrix}\begin{pmatrix} 4-b & a \\ b & 4-a \end{pmatrix}\begin{pmatrix} a & -1 \\ b & 1 \end{pmatrix}$$

$$= \frac{1}{a+b}\begin{pmatrix} 4 & 4 \\ ab-(4-b)b & (4-a)a-ab \end{pmatrix}\begin{pmatrix} a & -1 \\ b & 1 \end{pmatrix}$$ 1M

$$= \frac{1}{a+b}\begin{pmatrix} 4a+4b & 0 \\ a^2b-(4-b)ab+(4-a)ab-ab^2 & -ab+(4-b)b+(4-a)a-ab \end{pmatrix}$$

$$= \frac{1}{a+b}\begin{pmatrix} 4(a+b) & 0 \\ 0 & -a^2-2ab-b^2+4a+4b \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 0 \\ 0 & 4-(a+b) \end{pmatrix}$$ 1A

(iii) By (a)(ii),

$$(P^{-1}AP)^n = \begin{pmatrix} 4 & 0 \\ 0 & 4-(a+b) \end{pmatrix}^n$$

$$P^{-1}A^nP = \begin{pmatrix} 4^n & 0 \\ 0 & (4-a-b)^n \end{pmatrix}$$ 1M+1M

$$A^n = P\begin{pmatrix} 4^n & 0 \\ 0 & (4-a-b)^n \end{pmatrix}P^{-1}$$

Thus, $d_1 = 4^n$ and $d_2 = (4-a-b)^n$. 1A

(b) Since $4 \cdot 1 = 4 > 0$, put $a = 4$ and $b = 1$ into matrix A, we have 1A

$$B^k = P\begin{pmatrix} 4^k & 0 \\ 0 & (4-4-1)^k \end{pmatrix}P^{-1}$$

$$= P\begin{pmatrix} 4^k & 0 \\ 0 & (-1)^k \end{pmatrix}P^{-1}$$ 1M

Therefore,

$$B + B^3 + B^5 + \ldots + B^{2n-1} = \sum_{k=1}^{n} P \begin{pmatrix} 4^{2k-1} & 0 \\ 0 & (-1)^{2k-1} \end{pmatrix} P^{-1} \qquad \text{1M}$$

$$= P \begin{pmatrix} \sum_{k=1}^{n} 4^{2k-1} & 0 \\ 0 & \sum_{k=1}^{n} (-1)^{2k-1} \end{pmatrix} P^{-1} \qquad \text{1A}$$

$$= P \begin{pmatrix} \frac{4(1-16^n)}{1-16} & 0 \\ 0 & -n \end{pmatrix} P^{-1} \qquad \text{1M}$$

$$= \begin{pmatrix} 4 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{4}{15}(16^n - 1) & 0 \\ 0 & -n \end{pmatrix} \cdot \frac{1}{4+1} \begin{pmatrix} 1 & 1 \\ -1 & 4 \end{pmatrix}$$

$$= \frac{1}{5} \begin{pmatrix} \frac{16}{15}(16^n - 1) & n \\ \frac{4}{15}(16^n - 1) & -n \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 4 \end{pmatrix}$$

$$= \frac{1}{5} \begin{pmatrix} \frac{16}{15}(16^n - 1) - n & \frac{16}{15}(16^n - 1) + 4n \\ \frac{4}{15}(16^n - 1) + n & \frac{4}{15}(16^n - 1) - 4n \end{pmatrix} \qquad \text{1A}$$

$$= \frac{1}{75} \begin{pmatrix} 16(16^n - 1) - 15n & 16(16^n - 1) + 60n \\ 4(16^n - 1) + 15n & 4(16^n - 1) - 60n \end{pmatrix}$$

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