

M2REG-TRIG-2425-ASM-SET 6-MATH

建議題解

1. θ 在第二象限或第三象限。

當 θ 在第二象限，

$$\begin{aligned}\csc \theta &= \frac{23}{\sqrt{23^2 - 5^2}} \\ &= \frac{23}{6\sqrt{14}}\end{aligned}\quad 1M$$

$$\cot \theta = -\frac{5}{6\sqrt{14}}$$

$$\begin{aligned}\csc \theta - \cot \theta &= \frac{23}{6\sqrt{14}} - \left(-\frac{5}{6\sqrt{14}}\right) \\ &= \frac{\sqrt{14}}{3}\end{aligned}\quad 1A$$

當 θ 在第三象限，

$$\begin{aligned}\csc \theta &= -\frac{23}{\sqrt{23^2 - 5^2}} \\ &= -\frac{23}{6\sqrt{14}}\end{aligned}$$

$$\cot \theta = \frac{5}{6\sqrt{14}}$$

$$\begin{aligned}\csc \theta - \cot \theta &= -\frac{23}{6\sqrt{14}} - \frac{5}{6\sqrt{14}} \\ &= -\frac{\sqrt{14}}{3}\end{aligned}\quad 1A$$

因此， $\csc \theta - \cot \theta = -\frac{\sqrt{14}}{3}$ 或 $\frac{\sqrt{14}}{3}$ 。

$$2. \frac{(1 - \cot \theta)(1 + \cot \theta)}{2}$$

$$= \frac{(1 - \cot^2 \theta) - 2}{2}\quad 1M$$

$$= \frac{-(\cot^2 \theta + 1)}{2}$$

$$= \frac{-\csc^2 \theta}{2}\quad 1M$$

$$= -\frac{1}{2 \sin^2 \theta}$$

$$= -\frac{1}{2a^2}\quad 1A$$

3. (a) $(\csc x + \cot x)(\csc x - \cot x) = \csc^2 x - \cot^2 x$

$$\frac{3}{2}(\csc x - \cot x) = 1 \quad 1M$$

$$\csc x - \cot x = \frac{2}{3} \quad 1$$

(b) $(\csc x + \cot x) + (\csc x - \cot x) = \frac{3}{2} + \frac{2}{3}$ 1M

$$\csc x = \frac{13}{12} \quad 1A$$

$$\cot x = \frac{3}{2} - \frac{13}{12} = \frac{5}{12} \quad 1A$$

4. $\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$ 1M

$$\frac{5}{7} = \frac{\tan A + \tan B}{1 - \left(-\frac{1}{6}\right)}$$

$$\tan A + \tan B = \frac{5}{6} \quad 1M$$

考慮兩根之和及兩根之積，

$\tan A$ 及 $\tan B$ 為方程 $6x^2 - 5x - 1 = 0$ 的根。 1M+1

5. (a) $\tan\left(2 \cdot \frac{\pi}{8}\right) = \frac{2 \tan \frac{\pi}{8}}{1 - \tan^2 \frac{\pi}{8}}$ 1M

$$1 = \frac{2 \tan \frac{\pi}{8}}{1 - \tan^2 \frac{\pi}{8}}$$

$$1 - \tan^2 \frac{\pi}{8} = 2 \tan \frac{\pi}{8}$$

$$\tan^2 \frac{\pi}{8} + 2 \tan \frac{\pi}{8} - 1 = 0$$

因此， $\tan \frac{\pi}{8}$ 為方程 $x^2 + 2x - 1 = 0$ 。 1

(b) $x^2 + 2x - 1 = 0$

$$x = \frac{-2 \pm \sqrt{2^2 - 4(1)(-1)}}{2(1)} \quad 1M$$

$$= -1 \pm \sqrt{2}$$

留意 $\tan \frac{\pi}{8} > 0 > -1 - \sqrt{2}$ 。 1M

因此， $\tan \frac{\pi}{8} = \sqrt{2} - 1$ 。 1A

6.	$\sec 4A = 4$	
	$\cos 4A = \frac{1}{4}$	
	$2 \cos^2 2A - 1 = \frac{1}{4}$	1M
	$\cos^2 2A = \frac{5}{8}$	
	$\sin^2 2A = 1 - \frac{5}{8}$	1M
	$= \frac{3}{8}$	
	$\cos^4 A + \sin^4 A = (\cos^2 A + \sin^2 A)^2 - 2 \sin^2 A \cos^2 A$	1M
	$= 1^2 - 2 \left(\frac{\sin 2A}{2} \right)^2$	
	$= 1 - \frac{1}{2} \sin^2 2A$	
	$= 1 - \frac{1}{2} \left(\frac{3}{8} \right)$	
	$= \frac{13}{16}$	1A

7. (a) 當 $n = 1$,

$$\text{左式} = \cos x$$

$$\text{右式} = \frac{\sin 2x}{2 \sin x} = \frac{2 \sin x \cos x}{2 \sin x} = \cos x = \text{左式}$$

對 $n = 1$, 命題為真。

1

假設 $\cos x \cos 2x \cos 4x \dots \cos(2^{k-1}x) = \frac{\sin(2^k x)}{2^k \sin x}$, 其中 $\sin x \neq 0$ 及 $k \in \mathbb{Z}^+$ 。

1M

$$\cos x \cos 2x \cos 4x \dots \cos(2^{k-1}x) \cos(2^k x)$$

$$= \frac{\sin(2^k x)}{2^k \sin x} \cos(2^k x)$$

1M

$$= \frac{1}{2^k \sin x} \left(\frac{2 \sin(2^k x) \cos(2^k x)}{2} \right)$$

$$= \frac{\sin(2^{k+1} x)}{2^{k+1} \sin x}$$

1

對 $n = k + 1$, 命題為真。

藉數學歸納法, $\forall n \in \mathbb{Z}^+$, 命題為真。

1

$$(b) \cos \frac{\pi}{3} \cos \frac{2\pi}{3} \cos \frac{4\pi}{3} \dots \cos \frac{256\pi}{3}$$

$$= \cos \frac{\pi}{3} \cos \frac{2\pi}{3} \cos \frac{4\pi}{3} \dots \cos \frac{2^{9-1}\pi}{3}$$

$$= \frac{\sin \frac{2^9 \pi}{3}}{2^9 \sin \frac{\pi}{3}}$$

1M

$$= \frac{\sin \left(170\pi + \frac{2\pi}{3} \right)}{512 \sin \frac{\pi}{3}}$$

$$= \frac{\sin \frac{2\pi}{3}}{512 \sin \frac{\pi}{3}}$$

$$= \frac{1}{512}$$

1A

$$8. \sin 3x + 2 \sin 5x + \sin 7x = (\sin 3x + \sin 7x) + 2 \sin 5x$$

$$= 2 \sin 5x \cos 2x + 2 \sin 5x$$

1M

$$= 2 \sin 5x (\cos 2x + 1)$$

$$= 2 \sin 5x [(2 \cos^2 x - 1) + 1]$$

1M

$$= 4 \cos^2 x \sin 5x$$

1

$$9. \cos 4\theta \cos 2\theta - \sin 8\theta \sin 6\theta$$

$$= \frac{1}{2}(\cos 6\theta + \cos 2\theta) - \frac{1}{2}(\cos 2\theta - \cos 14\theta)$$

1M

$$= \frac{1}{2}(\cos 6\theta + \cos 14\theta)$$

$$= \frac{1}{2} \cdot 2 \cos \frac{6\theta + 14\theta}{2} \cos \frac{14\theta - 6\theta}{2}$$

1M

$$= \cos 10\theta \cos 4\theta$$

1

10. (a) $\cos 3\theta \cos^3 \theta + \sin 3\theta \sin^3 \theta$

$$= \cos 3\theta \cos \theta \cos^2 \theta + \sin 3\theta \sin \theta \sin^2 \theta$$

$$= \frac{1}{2}(\cos 4\theta + \cos 2\theta) \cdot \frac{1}{2}(1 + \cos 2\theta) + \frac{1}{2}(\cos 2\theta - \cos 4\theta) \cdot \frac{1}{2}(1 - \cos 2\theta) \quad 1M$$

$$= \frac{1}{4}[(\cos 4\theta + \cos 2\theta + \cos 4\theta \cos 2\theta + \cos^2 2\theta) - (\cos 4\theta - \cos 2\theta - \cos 4\theta \cos 2\theta + \cos^2 2\theta)]$$

$$= \frac{\cos 2\theta}{2}(1 + \cos 4\theta)$$

$$= \frac{\cos 2\theta}{2}[1 + (2\cos^2 2\theta - 1)] \quad 1M$$

$$= \cos^3 2\theta \quad 1$$

(b) $\cos^3 \frac{2\pi}{12} = \cos \frac{3\pi}{12} \cos^3 \frac{\pi}{12} + \sin \frac{3\pi}{12} \sin^3 \frac{\pi}{12} \quad 1M$

$$\left(\frac{\sqrt{3}}{2}\right)^3 = \frac{1}{\sqrt{2}} \cos^3 \frac{\pi}{12} + \frac{1}{\sqrt{2}} \sin^3 \frac{\pi}{12}$$

$$\frac{3\sqrt{3}}{8} = \frac{1}{\sqrt{2}} \left(\cos^3 \frac{\pi}{12} + \sin^3 \frac{\pi}{12} \right)$$

$$\frac{3\sqrt{6}}{8} = \left(\cos \frac{\pi}{12} + \sin \frac{\pi}{12} \right) \left(\cos^2 \frac{\pi}{12} - \cos \frac{\pi}{12} \sin \frac{\pi}{12} + \sin^2 \frac{\pi}{12} \right) \quad 1M$$

$$\frac{3\sqrt{6}}{8} = \left(\cos \frac{\pi}{12} + \sin \frac{\pi}{12} \right) \left(1 - \frac{1}{2} \sin \frac{\pi}{6} \right)$$

$$\frac{3\sqrt{6}}{8} = \left(\cos \frac{\pi}{12} + \sin \frac{\pi}{12} \right) \left[1 - \frac{1}{2} \left(\frac{1}{2} \right) \right]$$

$$\cos \frac{\pi}{12} + \sin \frac{\pi}{12} = \frac{\sqrt{6}}{2} \quad 1A$$

$$11. \quad (a) \quad \sin \frac{n\pi}{24} \sin \frac{\pi}{24} = -\frac{1}{2} \left[\cos \left(\frac{n\pi}{24} + \frac{\pi}{24} \right) - \cos \left(\frac{n\pi}{24} - \frac{\pi}{24} \right) \right] \quad 1M$$

$$= -\frac{1}{2} \left[\cos \frac{(n+1)\pi}{24} - \cos \frac{(n-1)\pi}{24} \right] \quad 1$$

$$(b) \quad \sum_{k=1}^5 \sin \frac{2k\pi}{24} \sin \frac{\pi}{24} = \sum_{k=1}^5 \left[-\frac{1}{2} \left[\cos \frac{(2k+1)\pi}{24} - \cos \frac{(2k-1)\pi}{24} \right] \right] \quad 1M$$

$$= -\frac{1}{2} \left[\sum_{k=1}^5 \cos \frac{(2k+1)\pi}{24} - \sum_{k=1}^5 \cos \frac{(2k-1)\pi}{24} \right]$$

$$= -\frac{1}{2} \left(\cos \frac{11\pi}{24} - \cos \frac{\pi}{24} \right)$$

$$= -\frac{1}{2} \left[\cos \left(\frac{\pi}{2} - \frac{\pi}{24} \right) - \cos \frac{\pi}{24} \right]$$

$$= -\frac{1}{2} \left(\sin \frac{\pi}{24} - \cos \frac{\pi}{24} \right) \quad 1A$$

$$\sum_{k=1}^5 \sin \frac{k\pi}{12} = \frac{\sum_{k=1}^5 \sin \frac{2k\pi}{24} \sin \frac{\pi}{24}}{\sin \frac{\pi}{24}} \quad 1M$$

$$= \frac{-\frac{1}{2} (\sin \frac{\pi}{24} - \cos \frac{\pi}{24})}{\sin \frac{\pi}{24}}$$

$$= -\frac{1}{2} + \frac{1}{2} \cot \frac{\pi}{24}$$

$$= -\frac{1}{2} + \frac{1}{2} (2 + \sqrt{2} + \sqrt{3} + \sqrt{6})$$

$$= \frac{1 + \sqrt{2} + \sqrt{3} + \sqrt{6}}{2} \quad 1A$$