

REG-2324-MOCK-SET 4-MATH-EP(M2)**Suggested solutions**

$$\begin{aligned}
 1. \quad f'(1) &= \lim_{h \rightarrow 0} \frac{(1+h+1) \ln(1+h) - 0}{h} \\
 &= \lim_{h \rightarrow 0} \frac{h+2}{h} \ln(1+h) \\
 &= \lim_{h \rightarrow 0} (2+h) \ln(1+h)^{\frac{1}{h}} \\
 &= 2 \ln e \\
 &= 2
 \end{aligned}$$

1M

1M+1A

1A

$$\begin{aligned}
 2. \quad (1+x)^m(1-2x)^n &= \left[1 + mx + \frac{m(m-1)}{2}x^2 + \dots \right] \left[1 - 2nx + \frac{n(n-1)}{2}4x^2 + \dots \right] \\
 &= 1 + (m-2n)x + \left[\frac{m(m-1)}{2} - 2mn + \frac{n(n-1)}{2} \right] x^2 + \dots
 \end{aligned}$$

1M

$$\begin{cases} m - 2n = 0 \\ \frac{m(m-1)}{2} - 2mn + 2n(n-1) = -9 \end{cases}$$

1M

$$\frac{2n(2n-1)}{2} - 4n^2 + 2n(n-1) = -9$$

1M

$$9 - 3n = 0$$

$$n = 3$$

When $n = 3$, $m = 2(3) = 6$.

1A

Solution	Marks
3. (a) When $n = 1$, $\text{L.H.S.} = \sum_{k=1}^1 (-1)^k (2k - 1)^2 = (-1)(1)^2 = -1$ $\text{R.H.S.} = \frac{(-1)^1 (2 - 1)(2 + 1) + 1}{2} = -1 = \text{L.H.S.}$ It is true for $n = 1$. Assume $\sum_{k=1}^p (-1)^k (2k - 1)^2 = \frac{(-1)^p (2p - 1)(2p + 1) + 1}{2}$, where $p \in \mathbb{Z}^+$. $\begin{aligned} & \sum_{k=1}^{p+1} (-1)^k (2k - 1)^2 \\ &= \sum_{k=1}^p (-1)^k (2k - 1)^2 + (-1)^{p+1} (2p + 1)^2 \\ &= \frac{(-1)^p (2p - 1)(2p + 1) + 1}{2} + (-1)^{p+1} (2p + 1)^2 \\ &= \frac{(-1)^{p+1} (2p + 1) [-(2p - 1) + 2(2p + 1)] + 1}{2} \\ &= \frac{(-1)^{p+1} (2p + 1)(2p + 3) + 1}{2} \end{aligned}$ It is true when $n = p + 1$. By M.I., it is true $\forall n \in \mathbb{Z}^+$. (b) $\sum_{k=99}^{200} (-1)^{k+1} (2k - 1)^2$ $\begin{aligned} &= - \sum_{k=1}^{200} (-1)^k (2k - 1)^2 + \sum_{k=1}^{98} (-1)^k (2k - 1)^2 \\ &= - \frac{(-1)^{200} (400 - 1)(400 + 1) + 1}{2} + \frac{(-1)^{98} (196 - 1)(196 + 1) + 1}{2} \\ &= -60792 \end{aligned}$	1 1 1 1 1M 1A

Solution		Marks
4. (a)	$\frac{\sin x + \sin y}{\cos x + \cos y} = \frac{2 \sin \frac{x+y}{2} \cos \frac{x-y}{2}}{2 \cos \frac{x+y}{2} \cos \frac{x-y}{2}}$ $= \tan \frac{x+y}{2}$	1M 1
(b) (i)	$\frac{\tan \left[2 \left(\frac{\theta}{2} - \frac{\pi}{4} \right) \right]}{\tan \left(\frac{\theta}{2} - \frac{\pi}{4} \right)} = \frac{\tan \left(\theta - \frac{\pi}{2} \right)}{\tan \frac{\theta - \frac{\pi}{2}}{2}} = \frac{-\frac{1}{\tan \theta}}{\frac{\sin \theta + \sin \frac{-\pi}{2}}{\cos \theta + \cos \frac{-\pi}{2}}}$ $= -\frac{\cos^2 \theta}{\sin \theta (\sin \theta - 1)} = \frac{(\sin \theta + 1)(\sin \theta - 1)}{\sin \theta (\sin \theta - 1)}$ $= \frac{\sin \theta + 1}{\sin \theta} = \csc \theta + 1$	1M 1M 1
(ii)	$\frac{\tan \left[2 \left(\frac{\theta}{2} - \frac{\pi}{4} \right) \right]}{\tan \left(\frac{\theta}{2} - \frac{\pi}{4} \right)} = 3$ $\csc \theta + 1 = 3$ $\sin \theta = \frac{1}{2}$ $\theta = \frac{\pi}{6} \quad \text{or} \quad \frac{5\pi}{6}$	1M 1A
5. (a)	$\ell^2 = (OC + OP \cos \theta)^2 + (OB + OP \sin \theta)^2$ $= (1 + \cos \theta)^2 + (1 + \sin \theta)^2$ $= 2 + (\sin^2 \theta + \cos^2 \theta) + 2 \sin \theta + 2 \cos \theta$ $= 3 + 2(\sin \theta + \cos \theta)$	1A+1A 1
(b)	$\frac{d\ell^2}{dt} = 2(\cos \theta - \sin \theta) \frac{d\theta}{dt}$ $2\ell \frac{d\ell}{dt} = 2(\cos \theta - \sin \theta) \frac{d\theta}{dt}$ $\frac{d\ell}{dt} = \frac{\cos \theta - \sin \theta}{\ell} \frac{d\theta}{dt}$ When $\theta = \frac{\pi}{3}$, $\ell^2 = 3 + 2 \left(\sin \frac{\pi}{3} + \cos \frac{\pi}{3} \right)$ $\ell = \sqrt{4 + \sqrt{3}}$ $\frac{d\ell}{dt} = \frac{\cos \frac{\pi}{3} - \sin \frac{\pi}{3}}{\ell} (3)$ ≈ -0.46 Required rate is -0.46 cm/s.	1M 1A 1A

Solution	Marks
6. (a) $P(P^3 + P^2 + P + I) = P(\mathbf{0})$	1M
$P^4 + (P^3 + P^2 + P) = \mathbf{0}$	
$P^4 + (-I) = \mathbf{0}$	1M
$P^4 = I$	1
(b) $A^2 = \begin{pmatrix} 3 & 2 \\ -5 & -3 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ -5 & -3 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	1A
$A^3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ -5 & -3 \end{pmatrix} = \begin{pmatrix} -3 & -2 \\ 5 & 3 \end{pmatrix}$	
$A^3 + A^2 + A + I = \begin{pmatrix} -3 & -2 \\ 5 & 3 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 3 & 2 \\ -5 & -3 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $= \mathbf{0}$	1A
$A^{2021} = (A^4)^{505} A$	
$= I^{505} A$	1M
$= A$	
$= \begin{pmatrix} 3 & 2 \\ -5 & -3 \end{pmatrix}$	1A

Solution	Marks
7. (a) $\vec{AB} = 3\mathbf{i} + \mathbf{j} - \mathbf{k}$ $\vec{AC} = 4\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}$ $\vec{AB} \times \vec{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & -1 \\ 4 & 2 & -5 \end{vmatrix}$ $= -3\mathbf{i} + 11\mathbf{j} + 2\mathbf{k}$ (b) $\vec{AM} = (\alpha - 3)\mathbf{i} - 22\mathbf{j} + (\beta + 2)\mathbf{k}$ $\vec{AM}$ is parallel to $\vec{AB} \times \vec{AC}$. We have $\frac{\alpha - 3}{-3} = \frac{-22}{11} = \frac{\beta + 2}{2}$. Solving, we have $\alpha = 9$ and $\beta = -6$. (c) We have $\vec{AP} = \frac{t}{1+t}\vec{AB}$ and $\vec{AQ} = \frac{1}{1+t}\vec{AC}$. $\frac{3}{16} = \frac{\text{volume of } MAPQ}{\text{volume of } MABC}$ $= \frac{\frac{1}{3}(AM)(\text{area of } \triangle APQ)}{\frac{1}{3}(AM)(\text{area of } \triangle ABC)}$ $= \frac{ \vec{AP} \times \vec{AQ} }{ \vec{AB} \times \vec{AC} }$ $= \frac{\frac{t}{(1+t)^2} \vec{AB} \times \vec{AC} }{ \vec{AB} \times \vec{AC} }$ $= \frac{t}{(1+t)^2}$ $3(1+t)^2 = 16t$ $3t^2 - 10t + 3 = 0$ $t = 3 \quad \text{or} \quad \frac{1}{3}$	1A 1M 1A+1A 1M 1A 1A

REG-2324-MOCK-SET 4-MATH-EP(M2)-MS-6 6 **Dexter Mathematics @ Beacon College**

Solution	Marks
9. (a) Vertical asymptote is $x = -\frac{1}{2}$. $f(x) = \frac{x}{2} - \frac{9}{4} + \frac{9}{4(2x+1)}$ Thus, the oblique asymptote is $y = \frac{x}{2} - \frac{9}{4}$. (b) $f'(x) = \frac{1}{2} - \frac{9}{2(2x+1)^2}$ (c) $f''(x) = \frac{18}{(2x+1)^3}$. When $f'(x) = 0$, $\frac{1}{2} - \frac{9}{2(2x+1)^2} = 0$ $(2x+1)^2 = 9$ $x = -2 \quad \text{or} \quad 1$ When $x = -2$, $f(-2) = \frac{-2}{2} - \frac{9}{4} + \frac{9}{4(-4+1)} = -4$ and $f''(-2) = \frac{18}{(-4+1)^3} = -\frac{2}{3} < 0$ When $x = 1$, $f(1) = -1$ and $f''(1) = \frac{2}{3} > 0$. The maximum point is $(-2, -4)$ and the minimum point is $(1, -1)$. (d) $\frac{x^2 - 4x}{2x+1} = 0$ $x(x-4) = 0$ $x = 0 \quad \text{or} \quad 4$ Required area = $\int_0^4 -\left(\frac{x^2 - 4x}{2x+1}\right) dx$ $= \int_0^4 \left[-\frac{x}{2} + \frac{9}{4} - \frac{9}{4(2x+1)}\right] dx$ $= \left[-\frac{x^2}{4} + \frac{9x}{4} - \frac{9}{8} \ln 2x+1 \right]_0^4$ $= 5 - \frac{9}{8} \ln 9$	1A 1M 1A 1M+1A 1A 1M 1A+1A 1A 1M 1M 1A

Solution		Marks
10. (a) (i) (1)	$\begin{vmatrix} h & -2 & 2h-1 \\ 3 & -1 & 5 \\ 1 & -3 & 3-h \end{vmatrix} = \begin{vmatrix} h-6 & 0 & 2h-11 \\ 3 & -1 & 5 \\ -8 & 0 & -h-12 \end{vmatrix}$ $= (h-6)(-h-12) + 8(2h-11)$ $= h^2 - 10h + 16$ $= (h-2)(h-8)$ (E) has a unique solution if and only if $\begin{vmatrix} h & -2 & 2h-1 \\ 3 & -1 & 5 \\ 1 & -3 & 3-h \end{vmatrix} \neq 0$ So, $h \neq 2$ and $h \neq 8$. (2) $x = \frac{\begin{vmatrix} -2 & -2 & 2h-1 \\ -2k & -1 & 5 \\ k+1 & -3 & 3-h \end{vmatrix}}{(h-2)(h-8)} = \frac{18hk - 29k - 35}{(h-2)(h-8)}$ $y = \frac{\begin{vmatrix} h & -2 & 2h-1 \\ 3 & -2k & 5 \\ 1 & k+1 & 3-h \end{vmatrix}}{(h-2)(h-8)} = \frac{2h^2k - hk - 5h - 5k + 5}{(h-2)(h-8)}$ $z = \frac{\begin{vmatrix} h & -2 & -2 \\ 3 & -1 & -2k \\ 1 & -3 & k+1 \end{vmatrix}}{(h-2)(h-8)} = \frac{-7hk - h + 10k + 22}{(h-2)(h-8)}$ (ii) (1) The augmented matrix of (E) is $\left(\begin{array}{ccc c} 2 & -2 & 3 & -2 \\ 3 & -1 & 5 & -2k \\ 1 & -3 & 1 & k+1 \end{array} \right) \sim \left(\begin{array}{ccc c} 1 & -1 & \frac{3}{2} & -1 \\ 0 & 2 & \frac{1}{2} & 3-2k \\ 0 & -2 & -\frac{1}{2} & k+2 \end{array} \right) \sim \left(\begin{array}{ccc c} 1 & -1 & \frac{3}{2} & -1 \\ 0 & 2 & \frac{1}{2} & 3-2k \\ 0 & 0 & 0 & 5-k \end{array} \right)$ Since (E) is consistent, $k = 5$. (2) The augmented matrix of (E) is $\left(\begin{array}{ccc c} 1 & -1 & \frac{3}{2} & -1 \\ 0 & 2 & \frac{1}{2} & -7 \\ 0 & 0 & 0 & 0 \end{array} \right)$ Let $z = t$, where $t \in \mathbb{R}$. Then $y = \frac{-t-14}{4}$ and $x = \frac{-7t-18}{4}$ The solution set is $\left\{ \left(\frac{-7t-18}{4}, \frac{-t-14}{4}, t \right) : t \in \mathbb{R} \right\}$. (b) The system is equivalent to (E) when $h = 2$ and $k = 5$.	1A 1M 1 1M+1A 1A 1M 1A 1A

Solution	Marks
The solution is $\left\{\left(\frac{-7t-18}{4}, \frac{-t-14}{4}, t\right): t \in \mathbb{R}\right\}$. $2x + y^2 + z < 1$ $\frac{-7t-18}{2} + \left(\frac{-t-14}{4}\right)^2 + t < 1$ $\frac{t^2}{16} - \frac{3t}{4} + \frac{9}{4} < 0$ $\frac{1}{6}(t-6)^2 < 0$ The inequality has no solution as $\frac{1}{6}(t-6)^2$ is non-negative for all real values of t. Thus, there is no real solution of the system satisfying $2x + y^2 + z < 1$.	1M 1M 1A

Solution		Marks
11. (a) $\frac{d}{d\theta} \ln(\tan \theta) = \frac{\sec^2 \theta}{\tan \theta} = \frac{2}{2 \sin \theta \cos \theta} = 2 \csc 2\theta$		1M
$\int \csc 2\theta \, d\theta = \frac{1}{2} \ln(\tan \theta) + \text{constant}$		1A+1
(b) (i) $\sqrt{2} \sin\left(\frac{\pi}{4} + \theta\right) = \sqrt{2} \left[\sin \frac{\pi}{4} \cos \theta + \cos \frac{\pi}{4} \sin \theta \right]$		1M
$= \sin \theta + \cos \theta$		1
(ii) $\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sin \theta + \cos \theta} = \frac{1}{\sqrt{2}} \int_0^{\frac{\pi}{2}} \csc\left(\frac{\pi}{4} + \theta\right) d\theta$		
Let $u = \frac{\pi}{8} + \frac{\theta}{2}$. Then $du = \frac{1}{2} d\theta$.		
$\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sin \theta + \cos \theta} = \sqrt{2} \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \csc 2u \, du$		1M
$= \frac{1}{\sqrt{2}} \left[\ln(\tan u) \right]_{\frac{\pi}{8}}^{\frac{3\pi}{8}}$		
$= \frac{1}{\sqrt{2}} \left[\ln\left(\tan\left(\frac{\pi}{2} - \frac{\pi}{8}\right)\right) - \ln\left(\tan \frac{\pi}{8}\right) \right]$		1M
$= -\sqrt{2} \ln \tan \frac{\pi}{8}$		
$= -\sqrt{2} \ln(\sqrt{2} - 1)$		1A
(c) Let $\phi = \frac{\pi}{2} - \theta$. Then $d\phi = -d\theta$		1M
$\int_0^{\frac{\pi}{2}} \frac{\sin \theta}{\sin \theta + \cos \theta} d\theta = - \int_{\frac{\pi}{2}}^0 \frac{\sin\left(\frac{\pi}{2} - \phi\right)}{\sin\left(\frac{\pi}{2} - \phi\right) + \cos\left(\frac{\pi}{2} - \phi\right)} d\phi$		
$= \int_0^{\frac{\pi}{2}} \frac{\cos \phi}{\sin \phi + \cos \phi} d\phi = \int_0^{\frac{\pi}{2}} \frac{\cos \theta}{\sin \theta + \cos \theta} d\theta$		1
(d) $\int_0^{\frac{\pi}{2}} \frac{\sin \theta}{\sin \theta + \cos \theta} d\theta = \frac{1}{2} \left[\int_0^{\frac{\pi}{2}} \frac{\sin \theta}{\sin \theta + \cos \theta} d\theta + \int_0^{\frac{\pi}{2}} \frac{\cos \theta}{\sin \theta + \cos \theta} d\theta \right]$		1M
$= \frac{1}{2} \left[\theta \right]_0^{\frac{\pi}{2}} = \frac{\pi}{4}$		
$\int_0^{\frac{\pi}{2}} \frac{2 \sin \theta + 5}{\sin \theta + \cos \theta} d\theta = 2 \int_0^{\frac{\pi}{2}} \frac{\sin \theta}{\sin \theta + \cos \theta} d\theta + 5 \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sin \theta + \cos \theta}$		1M
$= \frac{\pi}{2} - 5\sqrt{2} \ln(\sqrt{2} - 1)$		1A

Solution		Marks
12. (a) (i)	$\overrightarrow{OA} \cdot \overrightarrow{OC} = c \left(\frac{\mathbf{a} \cdot \mathbf{a}}{ \mathbf{a} } + \frac{\mathbf{a} \cdot \mathbf{b}}{ \mathbf{b} } \right)$ $ \mathbf{a} \mathbf{c} \cos \angle AOC = c \mathbf{a} \left(1 + \frac{\mathbf{a} \cdot \mathbf{b}}{ \mathbf{a} \mathbf{b} } \right)$ $\angle AOC = \frac{c}{ \mathbf{c} } \left(1 + \frac{\mathbf{a} \cdot \mathbf{b}}{ \mathbf{a} \mathbf{b} } \right)$ $\overrightarrow{OB} \cdot \overrightarrow{OC} = c \left(\frac{\mathbf{a} \cdot \mathbf{b}}{ \mathbf{a} } + \mathbf{b} \right)$ $ \mathbf{b} \mathbf{c} \cos \angle BOC = c \mathbf{b} \left(\frac{\mathbf{a} \cdot \mathbf{b}}{ \mathbf{a} \mathbf{b} } + 1 \right)$ $\cos \angle BOC = \frac{c}{ \mathbf{c} } \left(1 + \frac{\mathbf{a} \cdot \mathbf{b}}{ \mathbf{a} \mathbf{b} } \right)$ $= \cos \angle AOC$ Since $0 \leq \angle AOC \leq \pi$ and $0 \leq \angle BOC \leq \pi$, we have $\angle AOC = \angle COB$.	1M 1M
(ii)	$\overrightarrow{BC} = c \left(\frac{\mathbf{a}}{ \mathbf{a} } + \frac{\mathbf{b}}{ \mathbf{b} } \right) - \mathbf{b}$ $= \frac{ \mathbf{b} \mathbf{a} + \mathbf{a} \mathbf{b}}{ \mathbf{a} + \mathbf{b} + \mathbf{a} - \mathbf{b} } - \mathbf{b}$ $= \frac{ \mathbf{b} \mathbf{a} - \mathbf{b} \mathbf{b} - \mathbf{a} - \mathbf{b} \mathbf{b}}{ \mathbf{a} + \mathbf{b} + \mathbf{a} - \mathbf{b} }$ $= \frac{ \mathbf{a} - \mathbf{b} -\mathbf{b} }{ \mathbf{a} + \mathbf{b} + \mathbf{a} - \mathbf{b} } \left(\frac{\mathbf{a} - \mathbf{b}}{ \mathbf{a} - \mathbf{b} } + \frac{-\mathbf{b}}{ -\mathbf{b} } \right)$ $= l \left(\frac{\mathbf{a} - \mathbf{b}}{ \mathbf{a} - \mathbf{b} } + \frac{-\mathbf{b}}{ -\mathbf{b} } \right) \quad \text{where } l = \frac{ \mathbf{a} - \mathbf{b} -\mathbf{b} }{ \mathbf{a} + \mathbf{b} + \mathbf{a} - \mathbf{b} }$ Note that $\overrightarrow{BA} = \mathbf{a} - \mathbf{b}$ and $\overrightarrow{BO} = -\mathbf{b}$. By (a)(i), we have $\angle ABC = \angle CBO$. Since $\angle ABC = \angle CBO$ and $\angle AOC = \angle COB$, C is the incentre of $\triangle OAB$.	1M 1A 1M 1
(b)	$\overrightarrow{PQ} = (3\mathbf{i} - 5\mathbf{k}) - (\mathbf{i} + \mathbf{j} - 3\mathbf{k}) = 2\mathbf{i} - \mathbf{j} - 2\mathbf{k} \text{ and } \overrightarrow{PR} = 4\mathbf{i} - 6\mathbf{j} + 12\mathbf{k}.$ Let $\mathbf{a} = \overrightarrow{PQ}$ and $\mathbf{b} = \overrightarrow{PR}$. $\mathbf{a} - \mathbf{b} = -2\mathbf{i} + 5\mathbf{j} - 14\mathbf{k}, \mathbf{a} = \sqrt{2^2 + 1^2 + 2^2} = 3, \mathbf{b} = 14 \text{ and } \mathbf{a} - \mathbf{b} = 15$ $\overrightarrow{PI} = \frac{(3)(14)}{3 + 14 + 15} \left(\frac{2\mathbf{i} - \mathbf{j} - 2\mathbf{k}}{3} + \frac{4\mathbf{i} - 6\mathbf{j} + 12\mathbf{k}}{14} \right)$ $= \frac{5}{4}\mathbf{i} - \mathbf{j} + \frac{1}{4}\mathbf{k}$ $\overrightarrow{OI} = \overrightarrow{OP} + \overrightarrow{PI} = \frac{9}{4}\mathbf{i} - \frac{11}{4}\mathbf{k}$	1M 1M 1A 1A