Solution	Marks
REG-2324-MOCK-SET 3-MATH-EP(M2)	
Suggested solutions	
1. $(1+ax)^2(1+2x)^n$	
$= (1+2ax+a^2x^2)\left[1+2C_1^n x+4C_2^n x^2+\ldots+2^n x^n\right]$	1M
$\begin{cases} 2n+2a=16 \\ 4C_2^n+4an+a^2=109 \end{cases}$	1M
$\dfrac{4n(n-1)}{2}+4n(8-n)+(8-n)^2=109$	1M
$-n^2+14n-45=0$	1A
$n=5 \quad \text{or} \quad 9$	1A
2. $f(2+h)=\sqrt{4(2+h)-(2+h)^2}$	
$=\sqrt{4-h^2}$	1A
$f'(2)=\lim\limits_{h\to 0}\dfrac{f(2+h)-f(2)}{h}$	
$=\lim\limits_{h\to 0}\dfrac{\sqrt{4-h^2}-\sqrt{8-4}}{h}$	1M
$=\lim\limits_{h\to 0}\dfrac{(4-h^2)-4}{h(\sqrt{4-h^2}+2)}$	1M
$=\lim\limits_{h\to 0}\dfrac{-h}{\sqrt{4-h^2}+2}$	
$=0$	1A
3. (a) $\displaystyle\int \ln x\,\mathrm{d}x = x\ln x-\int \mathrm{d}x$	1M+1A
$= x\ln x - x + \text{constant}$	1A
(b) Volume $=\pi\displaystyle\int_1^3 \frac{\ln y}{\ln 3}\,\mathrm{d}y$	1M+1A
$=\dfrac{\pi}{\ln 3}\Big[y\ln y-y\Big]_1^3$	1M
$=3\pi-\dfrac{2\pi}{\ln 3}$	1A

Solution	Marks
4. (a) $\overrightarrow{OP} = \frac{3}{3+1}\mathbf{a} + \frac{1}{3+1}\mathbf{b}$	1M
$= \frac{3}{4}\mathbf{a} + \frac{1}{4}\mathbf{b}$	1A
(b) (i) $30^2 = \left\|\overrightarrow{OP}\right\|^2$	
$900 = \frac{9}{16}\mathbf{a}\cdot\mathbf{a} + \frac{3}{8}\mathbf{a}\cdot\mathbf{b} + \frac{1}{16}\mathbf{b}\cdot\mathbf{b}$	1M
$= \frac{9}{16}(36)^3 + \frac{3}{8}\mathbf{a}\cdot\mathbf{b} + \frac{1}{16}(24)^2$	1A
$\mathbf{a}\cdot\mathbf{b} = 360$	1A
(ii) $\mathbf{a}\cdot\mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\|\cos\angle AOB$	
$360 = (36)(24)\cos\angle AOB$	1M
$\angle AOB = \cos^{-1}\frac{5}{12}$	1A
5. (a) $\det M = ab - 6ab$	
$= -5ab$	
$\neq 0$ $\quad (a > 0 \text{ and } b > 0)$	
Thus, M is non-singular.	1
(b) $\begin{pmatrix} 1 & 13 \\ -10 & 1 \end{pmatrix}\begin{pmatrix} a & 3b \\ 2a & b \end{pmatrix} = \begin{pmatrix} a & 3a \\ 3b & b \end{pmatrix}\begin{pmatrix} 1 & -10 \\ 13 & 1 \end{pmatrix}$	
Compare the top right element of the matrix.	
$13b + 3b = -10a + 2a$	1M
$a = -2b$	1A
(c) $M^T = \begin{pmatrix} a & 2a \\ 3b & b \end{pmatrix} = \begin{pmatrix} -2b & -4b \\ 3b & b \end{pmatrix}$	1A
$(M^T)^{-1} = \frac{1}{-2b^2 + 12b^2}\begin{pmatrix} b & 4b \\ -3b & -2b \end{pmatrix}$	1M
$= \begin{pmatrix} \frac{1}{10b} & \frac{2}{5b} \\ -\frac{3}{10b} & -\frac{1}{5b} \end{pmatrix}$	1A

Solution	Marks
6. (a) When $n = 1$,	
L.H.S. $= (-1)^1(1)^2 = -1$ and R.H.S. $= \dfrac{(-1)(1)(1+1)}{2} = -1 =$ L.H.S.	
It is true for $n = 1$.	1
Assume $\displaystyle\sum_{k=1}^{p}(-1)^k k^2 = \frac{(-1)^p p(p+1)}{2}$, where $p \in \mathbb{Z}^+$.	1
$\displaystyle\sum_{k=1}^{p+1}(-1)^k k^2 = \sum_{k=1}^{p}(-1)^k k^2 + (-1)^{p+1}(p+1)^2$	
$\displaystyle = \frac{(-1)^p p(p+1)}{2} + (-1)^{p+1}(p+1)^2$	1M
$\displaystyle = \frac{(-1)^{p+1}(p+1)}{2}[-p + 2(p+1)]$	1M
$\displaystyle = \frac{(-1)^{p+1}(p+1)(p+2)}{2}$	
It is true for $n = p + 1$.	
By M.I., it is true $\forall n \in \mathbb{Z}^+$.	1
(b) $\displaystyle\sum_{k=3}^{333}(-1)^{k+1}k^2 = -\left[\sum_{k=1}^{333}(-1)^k k^2 - \sum_{k=1}^{2}(-1)^k k^2\right]$	1M
$\displaystyle = -\left[\frac{-333(334)}{2} - (-1+4)\right]$	
$= 55\,614$	1A
7. (a) $\displaystyle\tan 3\beta = \frac{\tan 2\beta + \tan\beta}{1 - \tan 2\beta\tan\beta}$	1M
$\displaystyle = \frac{\frac{2\tan\beta}{1-\tan^2\beta} + \tan\beta}{1 - \frac{2\tan\beta}{1-\tan^2\beta}\cdot\tan\beta}$	
$\displaystyle = \frac{3\tan\beta - \tan^3\beta}{1 - 3\tan^2\beta}$	1A
(b) $\tan 3x = \cot x$	
$\tan 3x = \tan\left(\dfrac{\pi}{2} - x\right)$	1M
$3x = \dfrac{\pi}{2} - x$	
$x = \dfrac{\pi}{8}$	1A
(c) Take $\beta = \dfrac{\pi}{8}$, we have $\tan 3\beta = \cot\beta$.	
$\displaystyle\frac{3\tan\beta - \tan^3\beta}{1 - 3\tan^2\beta} = \frac{1}{\tan\beta}$	
$\tan^4\beta - 6\tan^2\beta + 1 = 0$	1A
$\displaystyle\tan^2\beta = \frac{6 \pm \sqrt{6^2 - 4(1)(1)}}{2(1)}$	1M
$= 3 \pm 2\sqrt{2}$	
Since $\tan\beta = \tan\dfrac{\pi}{8} < 1$, we have $\tan^2\beta = 3 - 2\sqrt{2}$.	1A

Solution	Marks
8. (a) $0 = \begin{vmatrix} k & 2 & -1 \\ 1 & k & 3 \\ 1 & 4 & 1 \end{vmatrix}$	1M
$= k^2 + 6 - 4 + k - 2 - 12k$	
$= k^2 - 11k$	
$k = 0$ or 11	1A
(b) Put $k = 0$, we have $\begin{cases} 2y - z = 0 \\ x + 3z = 0 \\ x + 4y + z = 0 \end{cases}$.	
Let $z = t$, where $t \in \mathbb{R}$.	1M
The solution set is $\left\{\left(-3t,\ \frac{t}{2},\ t\right) : t \in \mathbb{R}\right\}$.	1A
(c) The solution to the first 3 equations is $\left\{\left(-3t,\ \frac{t}{2},\ t\right) : t \in \mathbb{R}\right\}$.	
$t^2 + \frac{t}{2}(-3t + 2) = 0$	1M
$-\frac{t^2}{2} + t = 0$	
$t = 0$ or 2	1A
The solution is $(0,\ 0,\ 0)$ or $(-6,\ 1,\ 2)$.	1A

Solution	Marks
9. (a) $f'(x)=\dfrac{2x(x-3)-(x^2+a)}{(x-3)^2}$	1A
$=\dfrac{x^2-6x-a}{(x-3)^2}$	
$0=\dfrac{5^2-6(5)-a}{(5-3)^2}$	1M
$a=-5$	1A
(b) $f'(x)=\dfrac{x^2-6x+5}{(x-3)^2}=\dfrac{(x-1)(x-5)}{(x-3)^2}$	
When $f'(x)=0$, $x=1$ or 5.	1A
(see sign table below)	1M
Maximum point of C is $(1,\ 2)$.	1A
(c) $f(x)=\dfrac{x^2-5}{x-3}$	
$=x+3+\dfrac{4}{x-3}$	1M
Oblique asymptote is $y=x+3$. Vertical asymptote is $x=3$.	1A
(d) $\dfrac{x^2-5}{x-3}=0$	
$x=\pm\sqrt{5}$	1A
Area $=\displaystyle\int_{-\sqrt{5}}^{\sqrt{5}}\frac{x^2-5}{x-3}\,dx$	
$=\displaystyle\int_{-\sqrt{5}}^{\sqrt{5}}\left(x+3+\frac{4}{x-3}\right)dx$	1M
$=\left[\dfrac{x^2}{2}+3x+4\ln\lvert x-3\rvert\right]_{-\sqrt{5}}^{\sqrt{5}}$	1A
$=6\sqrt{5}+4\ln\dfrac{3-\sqrt{5}}{3+\sqrt{5}}$	
$=6\sqrt{5}+4\ln\dfrac{7-3\sqrt{5}}{2}$	1A

x	$x<1$	$1<x<3$	$3<x<5$	$x>5$
$f'(x)$	$+$	$-$	$-$	$+$

Solution	Marks
10. (a) (i) $\dfrac{\sin 3x}{\sin x} - (2\cos 2x + 1) = \dfrac{1}{\sin x}\,[\sin 3x - 2\sin x \cos 2x - \sin x]$	
$= \dfrac{1}{\sin x}\,[\sin 3x - (\sin 3x - \sin x) - \sin x]$	1M
$= 0$	
Thus, $\dfrac{\sin 3x}{\sin x} = 2\cos 2x + 1$.	1
(ii) Put $x = \dfrac{\pi}{4} + y$.	
$\dfrac{\sin 3\left(\frac{\pi}{4} + y\right)}{\sin\left(\frac{\pi}{4} + y\right)} = 2\cos 2\left(\dfrac{\pi}{4} + y\right) + 1$	
$\dfrac{\sin\frac{3\pi}{4}\cos 3y + \cos\frac{3\pi}{4}\sin 3y}{\sin\frac{\pi}{4}\cos y + \cos\frac{\pi}{4}\sin y} = -2\sin 2y + 1$	1A
$\dfrac{\cos 3y - \sin 3y}{\cos y + \sin y} = -2\sin 2y + 1$	
$\dfrac{\sin 3y - \cos 3y}{\sin x + \cos y} = 2\sin 2y - 1$	1A
(b) (i) Let $u = a - x$. Then $\mathrm{d}u = -\,\mathrm{d}x$.	
$\displaystyle\int_0^a f(x)\,\mathrm{d}x = -\int_a^0 f(a-u)\,\mathrm{d}u$	1M
$\displaystyle = \int_0^a f(a-u)\,\mathrm{d}u$	
$\displaystyle = \int_0^a f(a-x)\,\mathrm{d}x$	1A
(ii) Put $f(x) = \dfrac{\sin 3x}{\sin x + \cos x}$.	
$f\left(\dfrac{\pi}{2} - x\right) = \dfrac{\sin 3\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)} = \dfrac{-\cos 3x}{\cos x + \sin x}$	1
We have $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin 3x}{\sin x + \cos x}\,\mathrm{d}x = \int_0^{\frac{\pi}{2}} \frac{-\cos 3x}{\sin x + \cos x}\,\mathrm{d}x$.	1A
$\displaystyle 2\int_0^{\frac{\pi}{2}} \frac{\sin 3x}{\sin x + \cos x}\,\mathrm{d}x = \int_0^{\frac{\pi}{2}} \frac{\sin 3x}{\sin x + \cos x}\,\mathrm{d}x + \int_0^{\frac{\pi}{2}} \frac{-\cos 3x}{\sin x + \cos x}\,\mathrm{d}x$	1M
$\displaystyle \int_0^{\frac{\pi}{2}} \frac{\sin 3x}{\sin x + \cos x}\,\mathrm{d}x = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\sin 3x - \cos 3x}{\sin x + \cos x}\,\mathrm{d}x$	1A
(c) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin 3x}{\sin x + \cos x}\,\mathrm{d}x = \frac{1}{2}\int_0^{\frac{\pi}{2}} (2\sin 2x - 1)\,\mathrm{d}x$	1M
$\displaystyle = \frac{1}{2}\Big[-\cos 2x - x\Big]_0^{\frac{\pi}{2}}$	1A
$= 1 - \dfrac{\pi}{4}$	1A

Solution	Marks
11. (a) $M^2 = \begin{pmatrix} -3 & -28 \\ 4 & 29 \end{pmatrix} = aM + bI$	1A
Then $2a + b = -3$, $7a = -28$, $-a = 4$ and $-6a + b = 29$.	1M
Solving, we have $a = -4$ and $b = 5$.	1A
(b) When $n = 1$,	
R.H.S. $= (1 - (-5))M + (5 - 5)I = 6M =$ L.H.S.	
It is true when $n = 1$.	
Assume $6M^k = (1 - (-5)^k)M + (5 + (-5)^k)I$, where $k \in \mathbb{Z}^+$.	1
$6M^{k+1} = M(6M^k)$	
$= M\left[(1 - (-5)^k)M + (5 + (-5)^k)I\right]$	1
$= (1 - (-5)^k)M^2 + (5 + (-5)^k)M$	
$= (1 - (-5)^k)(-4M + 5I) + (5 + (-5)^k)M$	
$= (1 - (-5)^{k+1})M + (5 + (-5)^{k+1})I$	
It is true when $n = k + 1$.	1
By M.I., the statement is true $\forall n \in \mathbb{Z}^+$.	1
(c) By (b), $M^n = \left[\frac{1}{6}M + \frac{5}{6}I\right] + (-5)^n \left[\frac{1}{6}I - \frac{1}{6}M\right]$.	
Set $A = \frac{1}{6}M + \frac{5}{6}I = \frac{1}{6}\begin{pmatrix} 7 & 7 \\ -1 & -1 \end{pmatrix}$ and $B = \frac{1}{6}I - \frac{1}{6}M = \frac{1}{6}\begin{pmatrix} -1 & -7 \\ 1 & 7 \end{pmatrix}$.	1A
Note that $A^2 = \frac{1}{6}\begin{pmatrix} 7 & 7 \\ -1 & -1 \end{pmatrix} = A$, $B^2 = \frac{1}{6}\begin{pmatrix} -1 & -7 \\ 1 & 7 \end{pmatrix} = B$, $AB = BA = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.	1M
$M^n \left(A + \frac{1}{(-5)^n}B\right) = [A + (-5)^n B]\left[A + \frac{1}{(-5)^n}B\right]$	1M
$= A^2 + (-5)^n BA + \frac{1}{(-5)^n}AB + B^2$	
$= A + B$	
$= I$	1M
Thus, $(M^n)^{-1} = A + \frac{1}{(-5)^n}B$ for all positive integers n.	
Required matrices A and B exist.	1

Solution	Marks
12. (a) $\overrightarrow{AB} = 2\mathbf{i} + 4\mathbf{j} - \mathbf{k}$ and $\overrightarrow{AC} = -2\mathbf{i} + 4\mathbf{j} - \mathbf{k}$	1A
Area $= \frac{1}{2}\left\lvert\overrightarrow{AC} \times \overrightarrow{AB}\right\rvert$	
$= \frac{1}{2}\lvert -4\mathbf{j} - 16\mathbf{k}\rvert$	1M
$= \frac{1}{2}\sqrt{4^2 + 16^2}$	
$= 2\sqrt{17}$	1A
(b) $\overrightarrow{DA} = -7\mathbf{j} + \mathbf{k}$	
Let $\mathbf{n} = -\frac{1}{4}(\overrightarrow{AC} \times \overrightarrow{AB}) = \mathbf{j} + 4\mathbf{k}$, such that $\mathbf{n}$ is perpendicular to Π.	
$\overrightarrow{DP} = \frac{\overrightarrow{DA} \cdot \mathbf{n}}{\mathbf{n} \cdot \mathbf{n}}\mathbf{n}$	1M
$= \frac{(-7+4)}{1^2 + 4^2}(\mathbf{j} + 4\mathbf{k})$	1M
$= -\frac{3}{17}\mathbf{j} - \frac{12}{17}\mathbf{k}$	1A
(c) (i) Let $\overrightarrow{BQ} = t\overrightarrow{BC}$, where $0 < t < 1$.	
$\overrightarrow{BQ} = t(-4\mathbf{i}) = -4t\mathbf{i}$	
$\overrightarrow{DQ} = \overrightarrow{BQ} - \overrightarrow{BD}$	
$= (-4t\mathbf{i}) - (-2\mathbf{i} + 3\mathbf{j})$	
$= (-4t + 2)\mathbf{i} - 3\mathbf{j}$	
$0 = \overrightarrow{DQ} \cdot \overrightarrow{BC}$	1M
$= -4(-4t + 2)$	
$t = \frac{1}{2}$	
Thus, $\overrightarrow{DQ} = -3\mathbf{j}$.	1A
(ii) $\overrightarrow{BC} \cdot \overrightarrow{PQ} = \overrightarrow{BC} \cdot (\overrightarrow{PD} + \overrightarrow{DQ})$	1M
$= \overrightarrow{BC} \cdot \overrightarrow{PD} + \overrightarrow{BC} \cdot \overrightarrow{DQ}$	
$= 0 + 0$	
$= 0$	
Thus, $BC \perp PQ$.	1
(d) Required angle is $\angle DQP = 90° - \angle PDQ$.	
$\overrightarrow{DP} \cdot \overrightarrow{DQ} = (DP)(DQ)\cos\angle PDQ$	
$\left(-\frac{3}{17}\mathbf{j} - \frac{12}{17}\mathbf{k}\right) \cdot (-3\mathbf{j}) = \frac{3}{\sqrt{17}}(3)\cos\angle PDQ$	1M+1A
$\cos\angle PDQ = \frac{1}{\sqrt{17}}$	
Required angle $= 90° - \cos^{-1}\frac{1}{\sqrt{17}}$	
$\approx 14°$	1A

END OF PAPER