Solution	Marks

REG-2324-MOCK-SET 2-MATH-EP(M2)
Suggested solutions

1. $\dfrac{d}{dx}(\cot 5x) = \lim_{h\to 0}\dfrac{\cot(5x+5h)-\cot 5x}{h}$ 1M

$= \lim_{h\to 0}\dfrac{\tan 5x - \tan(5x+5h)}{h\tan 5x\tan(5x+5h)}$

$= \lim_{h\to 0}\dfrac{\tan[5x-(5x+5h)][1+\tan 5x\tan(5x+5h)]}{h\tan 5x\tan(5x+5h)}$ 1M

$= \lim_{h\to 0}\dfrac{\tan(-5h)}{-5h}\,\dfrac{-5[1+\tan 5x\tan(5x+5h)]}{\tan 5x\tan(5x+5h)}$ 1M

$= \dfrac{-5(1+\tan^2 5x)}{\tan^2 5x}$

$= -5\csc^2 5x$ 1A

2. (a) $(1+x)^2(4+ax)^5 = (1+2x+x^2)(4^5 + C_1^5 4^4 ax + C_2^5 4^3 a^2x^2 + C_3^5 4^2 a^3x^3 + \ldots)$ 1M

$= (1+2x+x^2)(1024 + 1280ax + 640a^2x^2 + 160a^3x^3 + \ldots)$

Consider the coefficient of x^3,

$1(160a^3) + 2(640a^2) + 1(1280a) = 3360$ 1M

$a^3 + 8a^2 + 8a - 21 = 0$

$(a+3)(a^2+5a-7) = 0$

$a = -3$ or $a^2+5a-7=0$

$a = -3$ or -6.14 (rejected) or 1.14 (rejected) 1A

(b) Coefficient of $x^2 = 1(640a^2) + 2(1280a) + 1(1024)$ 1M

$= -896$ 1A

3. (a) $\displaystyle\int x(\ln x)^2\,dx = \frac{x^2}{2}(\ln x)^2 - \int\left(\frac{x^2}{2}\right)(2)\ln x\left(\frac{1}{x}\right)dx$ 1M+1A

$\displaystyle = \frac{x^2(\ln x)^2}{2} - \int x\ln x\,dx$

$\displaystyle = \frac{x^2(\ln x)^2}{2} - \frac{x^2\ln x}{2} + \frac{1}{2}\int x^2\times\frac{1}{x}\,dx$ 1M

$\displaystyle = \frac{x^2(\ln x)^2}{2} - \frac{x^2\ln x}{2} + \frac{x^2}{4} + \text{constant}$ 1A

(b) Volume $\displaystyle = \pi\int_1^e x(\ln x)^2\,dx$ 1M

$\displaystyle = \pi\left[\frac{x^2(\ln x)^2}{2} - \frac{x^2\ln x}{2} + \frac{x^2}{4}\right]_1^e$

$\displaystyle = \frac{\pi(e^2-1)}{4}$ 1A

Solution	Marks
4. (a) Area of $\triangle OPQ = \frac{1}{2}(u)(ue^{2u}) = \frac{1}{2}u^2e^{2u}$.	1A
(b) Let v be the y-coordinate of P.	
$v = ue^{2u}$	
$\frac{dv}{dt} = e^{2u}\frac{du}{dt} + 2ue^{2u}\frac{du}{dt}$	1M
When $u = 1$,	
$-3 = (e^2 + 2e^2)\frac{du}{dt}$	
$\frac{du}{dt} = -\frac{1}{e^2}$	
Let A be the area of $\triangle OPQ$.	
$A = \frac{1}{2}u^2e^{2u}$	
$\frac{dA}{dt} = (u^2e^{2u} + ue^{2u})\frac{du}{dt}$	1M+1M
When $u = 1$,	
$\frac{dA}{dt} = (e^2 + e^2)\frac{du}{dt}$	
$= 2e^2\left(-\frac{1}{e^2}\right)$	
$= -2$	
Required rate is -2/s.	1A

Solution	Marks
5. (a) When $n = 1$,	
L.H.S. $= \sin\theta$ and R.H.S. $= \dfrac{\sin^2\theta}{\sin\theta} = \sin\theta =$ L.H.S.	
It is true when $n = 1$.	1
Assume $\displaystyle\sum_{k=1}^{p}\sin(2k-1)\theta = \frac{\sin^2 p\theta}{\sin\theta}$, where $p \in \mathbb{Z}^+$.	1
$\displaystyle\sum_{k=1}^{p+1}\sin(2k-1)\theta = \sum_{k=1}^{p}\sin(2k-1)\theta + \sin(2p+1)\theta$	
$\displaystyle= \frac{\sin^2 p\theta}{\sin\theta} + \sin(2p+1)\theta$	1
$\displaystyle= \frac{1}{\sin\theta}[\sin^2 p\theta + \sin(2p+1)\theta\sin\theta]$	
$\displaystyle= \frac{1}{\sin\theta}\left(\frac{1-\cos(2p\theta)}{2} + \frac{\cos(2p\theta)-\cos(2p+2)\theta}{2}\right)$	1M
$\displaystyle= \frac{1}{\sin\theta} \times \frac{1-\cos(2p+2)\theta}{2}$	
$\displaystyle= \frac{\sin^2(p+1)\theta}{\sin\theta}$	
It is true for $n = p + 1$.	
By M.I., the statement is true $\forall n \in \mathbb{Z}^+$.	1
(b) $\displaystyle\sum_{k=135}^{234}\sin\frac{(2k-1)\pi}{9} = \sin\phi$	
$\displaystyle\sum_{k=1}^{234}\sin\frac{(2k-1)\pi}{9} - \sum_{k=1}^{134}\sin\frac{(2k-1)\pi}{9} = \sin\phi$	1M
$\displaystyle\frac{\sin^2\frac{234\pi}{9}}{\sin\frac{\pi}{9}} - \frac{\sin^2\frac{134\pi}{9}}{\sin\frac{\pi}{9}} = \sin\phi$	
$\displaystyle 0 - \frac{\sin^2\left(15\pi - \frac{\pi}{9}\right)}{\sin\frac{\pi}{9}} = \sin\phi$	1A
$\displaystyle\sin\phi = -\sin\frac{\pi}{9}$	
$\displaystyle\phi = -\frac{\pi}{9}$	1A

Solution	Marks
6. (a) Let $x = a\tan\theta$. Then $\mathrm{d}x = a\sec^2\theta\,\mathrm{d}\theta$.	
$\displaystyle\int \frac{1}{x^2+a^2}\,\mathrm{d}x = \int \frac{a\sec^2\theta}{a^2\tan^2\theta + a^2}\,\mathrm{d}\theta$	1M+1A
$\displaystyle = \frac{1}{a}\theta + \text{constant}$	
$\displaystyle = \frac{1}{a}\tan^{-1}\frac{x}{a} + \text{constant}$	1
(b) (i) $\displaystyle t = \tan\frac{\theta}{2}$	
$\displaystyle 1 = \sec^2\frac{\theta}{2} \times \frac{1}{2}\frac{\mathrm{d}\theta}{\mathrm{d}t}$	1A
$\displaystyle \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{2}{1+\tan^2\frac{\theta}{2}}$	
$\displaystyle = \frac{2}{1+t^2}$	1
(ii) Let $t = \tan\frac{\theta}{2}$. Then $\mathrm{d}\theta = \frac{2}{1+t^2}\,\mathrm{d}t$.	
$\displaystyle \cos\theta = 2\cos^2\frac{\theta}{2} - 1 = \frac{2}{1+t^2} - 1 = \frac{1-t^2}{1+t^2}.$	
$\displaystyle \int \frac{1}{2+\cos\theta}\,\mathrm{d}\theta = \int \frac{\frac{2}{1+t^2}}{2+\frac{1-t^2}{1+t^2}}\,\mathrm{d}t$	1M
$\displaystyle = \int \frac{2}{2+2t^2+1-t^2}\,\mathrm{d}t$	
$\displaystyle = 2\int \frac{1}{t^2+3}\,\mathrm{d}t$	
$\displaystyle = \frac{2}{\sqrt{3}}\tan^{-1}\frac{t}{\sqrt{3}} + \text{constant}$	1M
$\displaystyle = \frac{2}{\sqrt{3}}\tan^{-1}\frac{\tan\frac{\theta}{2}}{\sqrt{3}} + \text{constant}$	1A
7. (a) $A^2 = \begin{pmatrix} 4 & -7 \\ 3 & -5 \end{pmatrix}$ and $A^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	1M+1A
Thus, the smallest value of n is 3.	1A
(b) (i) $A + A^2 + A^3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	1A
(ii) $\displaystyle\sum_{k=1}^{335} A^k = (A + A^2 + A^3) + A^3(A + A^2 + A^3) + \ldots + A^{333}(A + A^2)$	1M
$= A + A^2$	
$= -A^3 = -I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	1M+1A

Solution	Marks
8. (a) (i) $\begin{vmatrix} 1 & 1 & 1 \\ 1 & -2 & 2 \\ h & 2 & -2 \end{vmatrix} = 4 + 2h + 2 + 2h + 2 - 4$	
$= 4h + 4$	1A
(E) has a unique solution if and only if $4h + 4 \neq 0$, i.e., $h \neq -1$.	1A
Thus, $h < -1$ or $h > -1$.	
(ii) $x = \dfrac{\begin{vmatrix} 2 & 1 & 1 \\ 3 & -2 & 2 \\ k & 2 & -2 \end{vmatrix}}{4h + 4}$	1M
$= \dfrac{4k + 12}{4h + 4}$	
$= \dfrac{k + 3}{h + 1}$	1A
(b) Put $h = -1$, the augmented matrix of (E) becomes	
$\left(\begin{array}{ccc\|c} 1 & 1 & 1 & 2 \\ 1 & -2 & 2 & 3 \\ -1 & 2 & -2 & k \end{array}\right) \sim \left(\begin{array}{ccc\|c} 1 & 1 & 1 & 2 \\ 0 & -3 & 1 & 1 \\ 0 & 3 & -1 & k+2 \end{array}\right) \sim \left(\begin{array}{ccc\|c} 1 & 1 & 1 & 2 \\ 0 & -3 & 1 & 1 \\ 0 & 0 & 0 & k+3 \end{array}\right)$	1M
If (E) is consistent, then $k = -3$.	1A
Let $z = t$, then $y = \dfrac{t - 1}{3}$ and $x = \dfrac{7 - 4t}{3}$.	
The solution set is $\left\{ \left(\dfrac{7 - 4t}{3}, \dfrac{t - 1}{3}, t \right) : t \in \mathbb{R} \right\}$.	1A

Solution	Marks
9. (a) Vertical asymptotes are $x = -1$ and $x = 3$	1A+1A
Horizontal asymptote is $y = 2$.	1A
(b) $f'(x) = -\dfrac{1}{(x+1)^2} + \dfrac{1}{(x-3)^2}$	1A
When $f'(x) = 0$,	
$-\dfrac{1}{(x+1)^2} + \dfrac{1}{(x-3)^2} = 0$	1M
$(x-3)^2 = (x+1)^2$	
$-8x + 8 = 0$	
$x = 1$	
(sign table below)	1M
The minimum point is $(1,\ f(1))$, i.e., $(1,\ 3)$.	1A
(c) (i) $f''(x) = \dfrac{2}{(x+1)^3} - \dfrac{2}{(x-3)^3}$	1A
(ii) When $f''(x) = 0$,	
$\dfrac{2}{(x+1)^3} = \dfrac{2}{(x-3)^3}$	1M
$(x-3)^3 = (x+1)^3$	
$x - 3 = x + 1$	
which has no solution.	1M
The claim is agreed.	1A
(d) $S = \displaystyle\int_4^k \left[2 - \left(2 + \frac{1}{x+1} - \frac{1}{x-3}\right)\right] \mathrm{d}x$	1M
$= \Big[-\ln\lvert x+1\rvert + \ln\lvert x-3\rvert\Big]_4^k$	1M
$= \ln\dfrac{k-3}{k+1} + \ln 5$	
Since $k > 4$, $0 < \dfrac{k-3}{k+1} < 1$ and so $\ln\dfrac{k-3}{k+1} < 0$.	
Thus, $S = \ln\dfrac{k-3}{k+1} + \ln 5 < \ln 5$.	1

x	$x < -1$	$-1 < x < 1$	$1 < x < 3$	$x > 3$
$f'(x)$	$-$	$-$	$+$	$+$

Solution	Marks
10. (a) $\det B = \cos^2\theta + \sin^2\theta = 1$	
$B^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$	1M+1A
$C = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$	
$= \begin{pmatrix} 4\cos\theta + \sin\theta & \cos\theta + 4\sin\theta \\ -4\sin\theta + \cos\theta & -\sin\theta + 4\cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$	1M
$= \begin{pmatrix} 4\cos^2\theta + 2\sin\theta\cos\theta + 4\sin^2\theta & -\sin^2\theta + \cos^2\theta \\ \cos^2\theta - \sin^2\theta & 4\sin^2\theta - 2\sin\theta\cos\theta + 4\cos^2\theta \end{pmatrix}$	
$= \begin{pmatrix} 4 + \sin 2\theta & \cos 2\theta \\ \cos 2\theta & 4 - \sin 2\theta \end{pmatrix}$	1
(b) (i) If C is diagonal,	
$\cos 2\theta = 0$	1A
$2\theta = \frac{\pi}{2}$	
So, $C = \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}$	1A
(ii) $C^n = \lambda C + \mu I$	
$\begin{pmatrix} 5^n & 0 \\ 0 & 3^n \end{pmatrix} = \begin{pmatrix} 5\lambda + \mu & 0 \\ 0 & 3\lambda + \mu \end{pmatrix}$	1M
So, $\begin{cases} 5\lambda + \mu = 5^n \\ 3\lambda + \mu = 3^n \end{cases}$	1M
Solving, we have $\lambda = \frac{5^n - 3^n}{2}$ and $\mu = \frac{5(3^n) - 3(5^n)}{2}$.	1A
(iii) $(B^{-1}AB)^n = \lambda B^{-1}AB + \mu I$	
$B^{-1}A^nB = \lambda B^{-1}AB + \mu I$	1M
$A^n = B(\lambda B^{-1}AB + \mu I)B^{-1}$	
$= \lambda A + \mu I$	1A
$= \frac{1}{2}\begin{pmatrix} 5^n + 3^n & 5^n - 3^n \\ 5^n - 3^n & 5^n + 3^n \end{pmatrix}$	1A

Solution	Marks
11. (a) Let $u = a + b - x$. Then $\mathrm{d}u = -\,\mathrm{d}x$.	
$\int_a^b f(a+b-x)\,\mathrm{d}x = -\int_b^a f(u)\,\mathrm{d}u$	1M
$= \int_a^b f(u)\,\mathrm{d}u$	1M
$= \int_a^b f(x)\,\mathrm{d}x$	1
(b) $\int_0^{\frac{\pi}{4}} \ln \sin 2x\,\mathrm{d}x = \int_0^{\frac{\pi}{4}} \ln \sin \left[2\left(\frac{\pi}{4} - x\right)\right]\mathrm{d}x$	1M
$= \int_0^{\frac{\pi}{4}} \ln \cos 2x\,\mathrm{d}x$	1
(c) Let $2x = t$. Then $2\,\mathrm{d}x = \mathrm{d}t$.	1M
$\int_0^{\frac{\pi}{4}} \ln \sin 4x\,\mathrm{d}x = \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln \sin 2t\,\mathrm{d}t$	
$= \frac{1}{2}\left[\int_0^{\frac{\pi}{4}} \ln \sin 2t\,\mathrm{d}t + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln \sin 2t\,\mathrm{d}t\right]$	1M
$= \frac{1}{2}\left[\int_0^{\frac{\pi}{4}} \ln \sin 2t\,\mathrm{d}t + \int_0^{\frac{\pi}{4}} \ln \sin \left[2\left(u + \frac{\pi}{4}\right)\right]\mathrm{d}u\right]$ where $u = t - \frac{\pi}{4}$	1M
$= \frac{1}{2}\left[\int_0^{\frac{\pi}{4}} \ln \sin 2t\,\mathrm{d}t + \int_0^{\frac{\pi}{4}} \ln \cos 2u\,\mathrm{d}u\right]$	
$= \frac{1}{2}\left[\int_0^{\frac{\pi}{4}} \ln \sin 2x\,\mathrm{d}x + \int_0^{\frac{\pi}{4}} \ln \sin 2x\,\mathrm{d}x\right]$	
$= \int_0^{\frac{\pi}{4}} \ln \sin 2x\,\mathrm{d}x$	1
(d) $\int_0^{\frac{\pi}{4}} \ln \sin 4x\,\mathrm{d}x = \int_0^{\frac{\pi}{4}} (\ln 2 + \ln \sin 2x + \ln \cos 2x)\,\mathrm{d}x$	
$= \ln 2\Big[x\Big]_0^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}} \ln \sin 2x\,\mathrm{d}x + \int_0^{\frac{\pi}{4}} \ln \cos 2x\,\mathrm{d}x$	1M
$= \frac{\pi \ln 2}{4} + 2\int_0^{\frac{\pi}{4}} \ln \sin 4x\,\mathrm{d}x$	1M
$\int_0^{\frac{\pi}{4}} \ln \sin 4x\,\mathrm{d}x = -\frac{\pi \ln 2}{4}$	1A

Solution	Marks
12. (a) (i) $\overrightarrow{AB} = -\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}$ and $\overrightarrow{AC} = -7\mathbf{i} + 3\mathbf{j} + \mathbf{k}$.	
$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 5 & -3 \\ -7 & 3 & 1 \end{vmatrix}$	1M
$= 14\mathbf{i} + 22\mathbf{j} + 32\mathbf{k}$	1A
(ii) $\overrightarrow{DA} = \mathbf{i} - 7\mathbf{j} + 4\mathbf{k}$	
$\overrightarrow{DE} = \dfrac{\overrightarrow{DA} \cdot (14\mathbf{i} + 22\mathbf{j} + 32\mathbf{k})}{\lvert 14\mathbf{i} + 22\mathbf{j} + 32\mathbf{k} \rvert^2}(14\mathbf{i} + 22\mathbf{j} + 32\mathbf{k})$	1M
$= -\dfrac{7}{71}\mathbf{i} - \dfrac{11}{71}\mathbf{j} - \dfrac{16}{71}\mathbf{k}$	1A
(b) (i) $\overrightarrow{BC} = -6\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$	
Let $\overrightarrow{BF} = t\overrightarrow{BC}$.	
$\overrightarrow{DF} = (1-t)\overrightarrow{DB} + t\overrightarrow{DC}$	1M
$= -6t\mathbf{i} + (-2-2t)\mathbf{j} + (1+4t)\mathbf{k}$	
$0 = \overrightarrow{DF} \cdot \overrightarrow{BC}$	1M
$= (-6t)(-6) + (-2-2t)(-2) + (1+4t)(4)$	
$t = -\dfrac{1}{7}$	
Thus, $\overrightarrow{DF} = \dfrac{6}{7}\mathbf{i} - \dfrac{12}{7}\mathbf{j} + \dfrac{3}{7}\mathbf{k}$.	1A
(ii) $\overrightarrow{EF} = \overrightarrow{DF} - \overrightarrow{DE}$	1M
$= \dfrac{475}{497}\mathbf{i} - \dfrac{775}{497}\mathbf{j} + \dfrac{325}{497}\mathbf{k}$	
$\overrightarrow{BC} \cdot \overrightarrow{EF} = -6\left(\dfrac{475}{497}\right) - 2\left(-\dfrac{775}{497}\right) + 4\left(\dfrac{325}{497}\right)$	
$= 0$	
Thus, $\overrightarrow{BC}$ is perpendicular to $\overrightarrow{EF}$.	1A
(c) Required angle is $\angle DFE$.	1M
$\overrightarrow{DF} \cdot \overrightarrow{EF} = (DF)(EF)\cos\angle DFE$	1M
$\dfrac{1875}{497} = \dfrac{3\sqrt{21}}{7} \cdot \dfrac{25\sqrt{1491}}{497}\cos\angle DFE$	
$\cos\angle DFE = \dfrac{25}{3\sqrt{71}}$	
$\angle DFE = \cos^{-1}\dfrac{25}{3\sqrt{71}}$	1A

END OF PAPER