

M2REG-MI-2425-ASM-SET 2-MATH

建議題解

1. (a) 當 $n = 1$,

$$\text{左式} = \sum_{r=2}^4 \frac{1}{r(r-1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} = \frac{3}{4}$$

$$\text{右式} = \frac{2+1}{4(2-1)} = \frac{3}{4} = \text{左式}$$

對 $n = 1$, 命題為真。

1

假設 $\sum_{r=2k}^{4k} \frac{1}{r(r-1)} = \frac{2k+1}{4k(2k-1)}$, 其中 $k \in \mathbb{Z}^+$ 。

1M

$$\begin{aligned} & \sum_{r=2k+2}^{4k+4} \frac{1}{r(r-1)} \\ &= \sum_{r=2k}^{4k} \frac{1}{r(r-1)} - \frac{1}{2k(2k-1)} - \frac{1}{(2k+1)(2k)} + \frac{1}{4k(4k+1)} + \frac{1}{(4k+1)(4k+2)} \\ & \quad + \frac{1}{(4k+2)(4k+3)} + \frac{1}{(4k+3)(4k+4)} \\ &= \frac{2k+1}{4k(2k-1)} - \frac{(2k+1) + (2k-1)}{2k(2k-1)(2k+1)} + \frac{(4k+2) + 4k}{4k(4k+1)(4k+2)} \\ & \quad + \frac{(4k+4) + (4k+2)}{(4k+2)(4k+3)(4k+4)} \\ &= \frac{2k+1}{4k(2k-1)} - \frac{2}{(2k-1)(2k+1)} + \frac{1}{4k(2k+1)} + \frac{1}{4(2k+1)(k+1)} \\ &= \frac{(2k+1)^2 - 2(4k)}{4k(2k-1)(2k+1)} + \frac{(k+1) + k}{4k(2k+1)(k+1)} \\ &= \frac{2k-1}{4k(2k+1)} + \frac{1}{4k(k+1)} \\ &= \frac{(2k-1)(k+1) + (2k+1)}{4k(2k+1)(k+1)} \\ &= \frac{2k+3}{4(k+1)(2k+1)} \end{aligned}$$

1M

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對 $n = k + 1$, 命題為真。

藉數學歸納法, $\forall n \in \mathbb{Z}^+$, 命題為真。

1

$$\begin{aligned} \text{(b)} \quad \sum_{r=30}^{120} \frac{1}{r(r-1)} &= \sum_{r=60}^{120} \frac{1}{r(r-1)} + \sum_{r=30}^{60} \frac{1}{r(r-1)} - \frac{1}{60(59)} \\ &= \frac{60+1}{120(60-1)} + \frac{30+1}{60(30-1)} - \frac{1}{3540} \\ &= \frac{91}{3480} \end{aligned}$$

1M

1A

2. (a) 當 $n = 1$,

$$\text{左式} = 1 \times 3^2 = 9, \text{右式} = \frac{1}{6}(1)(1+1)[6(1)^2 + 14(1) + 7] = 9 = \text{左式}$$

對 $n = 1$, 命題為真。

1

$$\text{假設 } 1 \times 3^2 + 2 \times 5^2 + \dots + k(2k+1)^2 = \frac{1}{6}k(k+1)(6k^2 + 14k + 7), \text{ 其中 } k \in \mathbb{Z}^+。$$

1

$$1 \times 3^2 + 2 \times 5^2 + 3 \times 7^2 + \dots + k(2k+1)^2 + (k+1)[2(k+1)+1]^2$$

$$= \frac{1}{6}k(k+1)(6k^2 + 14k + 7) + (k+1)(2k+3)^2$$

1

$$= \frac{1}{6}(k+1)[k(6k^2 + 14k + 7) + 6(2k+3)^2]$$

$$= \frac{1}{6}(k+1)[6k^3 + 38k^2 + 79k + 54]$$

$$= \frac{1}{6}(k+1)(k+2)(6k^2 + 26k + 27)$$

$$= \frac{1}{6}(k+1)[(k+1)+1][6(k+1)^2 + 14(k+1) + 7]$$

1

對 $n = k+1$, 命題為真。

藉數學歸納法, $\forall n \in \mathbb{Z}^+$, 命題為真。

1

(b) (i) $1 \times 3^2 + 2 \times 5^2 + 3 \times 7^2 + \dots + 20 \times 41^2$

$$= \sum_{r=1}^{20} r(2r+1)^2$$

1M

$$= \frac{1}{6}(20)(20+1)[6(20)^2 + 14(20) + 7]$$

1M

$$= 188\,090$$

1A

(ii) $11 \times 23^2 + 12 \times 25^2 + 13 \times 27^2 + \dots + 20 \times 41^2$

$$= \sum_{r=11}^{20} r(2r+1)^2$$

$$= \sum_{r=1}^{20} r(2r+1)^2 - \sum_{r=1}^{10} r(2r+1)^2$$

1M

$$= 188\,090 - \frac{1}{6}(10)(10+1)[6(10)^2 + 14(10) + 7]$$

1M

$$= 174\,395$$

1A

3. (a) 當 $n = 1$,

$$\text{左式} = (-1) = -1, \text{右式} = \frac{(-1)(2+1)-1}{4} = -1 = \text{左式}$$

對 $n = 1$, 命題為真。

1

$$\text{假設 } \sum_{k=1}^p (-1)^k k = \frac{(-1)^p (2p+1) - 1}{4}, \text{ 其中 } p \in \mathbb{Z}^+.$$

1

$$\begin{aligned} \sum_{k=1}^{p+1} (-1)^k k &= \sum_{k=1}^p (-1)^k k + (-1)^{p+1} (p+1) \\ &= \frac{(-1)^p (2p+1) - 1}{4} + (-1)^{p+1} (p+1) \\ &= \frac{(-1)^{p+1} [-(2p+1) + 4(p+1)] - 1}{4} \\ &= \frac{(-1)^{p+1} (2p+3) - 1}{4} \\ &= \text{右式} \end{aligned}$$

1

1

對 $n = p+1$, 命題為真。

藉數學歸納法, $\forall n \in \mathbb{Z}^+$, 命題為真。

1

$$\begin{aligned} \text{(b)} \quad \sum_{k=3}^{2017} (-1)^k (k+1) &= - \sum_{k=4}^{2018} (-1)^k k \\ &= - \sum_{k=1}^{2018} (-1)^k k + \sum_{k=1}^3 (-1)^k k \\ &= - \frac{(-1)^{2018} [2(2018) + 1] - 1}{4} + \frac{(-1)^3 [2(3) + 1] - 1}{4} \\ &= -1011 \end{aligned}$$

1M

1M

1A

4. (a) 當 $n = 1$,

$$\text{左式} = (2 - 1)^2 = 1, \text{右式} = \frac{1}{3}(1)(2 - 1)(2 + 1) = 1 = \text{左式}$$

對 $n = 1$, 命題為真。

1

$$\text{假設 } \sum_{k=1}^p (2k - 1)^2 = \frac{1}{3}p(2p - 1)(2p + 1), \text{ 其中 } p \in \mathbb{Z}^+.$$

1

$$\begin{aligned} \sum_{k=1}^{p+1} (2k - 1)^2 &= \sum_{k=1}^p (2k - 1)^2 + (2p + 1)^2 \\ &= \frac{1}{3}p(2p - 1)(2p + 1) + (2p + 1)^2 \\ &= \frac{1}{3}(2p + 1)[p(2p - 1) + 3(2p + 1)] \\ &= \frac{1}{3}(2p + 1)(2p^2 + 5p + 3) \\ &= \frac{1}{3}(p + 1)(2p + 1)(2p + 3) \end{aligned}$$

1

1

對 $n = p + 1$, 命題為真。

藉數學歸納法, $\forall n \in \mathbb{Z}^+$, 命題為真。

1

$$(b) \sum_{k=111}^{222} \left(k - \frac{1}{2}\right)^2 = \frac{1}{4} \sum_{k=1}^{222} (2k - 1)^2 - \frac{1}{4} \sum_{k=1}^{110} (2k - 1)^2$$

1M

$$= \frac{1}{4} \times \frac{1}{3} (222)(443)(445) - \frac{1}{4} \times \frac{1}{3} (110)(219)(221)$$

1M

$$= 3\,203\,340$$

1A

5. (a) 當 $n = 1$,

$$\text{左式} = 6 \times 10 = 60, \text{右式} = \frac{2}{3}(1)(2 + 21 + 67) = 60 = \text{左式}$$

對 $n = 1$, 命題為真。

1

假設 $6 \times 10 + 8 \times 12 + \dots + (2k + 4)(2k + 8) = \frac{2}{3}k(2k^2 + 21k + 67)$, 其中 $k \in \mathbb{Z}^+$ 。

1M

$$6 \times 10 + 8 \times 12 + \dots + (2k + 4)(2k + 8) + (2k + 6)(2k + 10)$$

$$= \frac{2}{3}k(2k^2 + 21k + 67) + (2k + 6)(2k + 10)$$

1M

$$= \frac{2}{3}[(2k^3 + 21k^2 + 67k) + 3(2k + 6)(k + 5)]$$

$$= \frac{2}{3}(2k^3 + 27k^2 + 115k + 90)$$

$$= \frac{2}{3}(k + 1)(2k^2 + 25k + 90)$$

1M

$$= \frac{2}{3}(k + 1)[2(k + 1)^2 + 21(k + 1) + 67]$$

對 $n = k + 1$, 命題為真。

1

藉數學歸納法, $\forall n \in \mathbb{Z}^+$, 命題為真。

1

$$(b) \sum_{r=2}^{30} (r+2)(r+4) = \frac{1}{4} \left[\sum_{r=1}^{30} (2r+4)(2r+8) - (6)(10) \right]$$

1M

$$= \frac{1}{4} \left[\frac{2}{3}(30)[2(30)^2 + 21(30) + 67] - 60 \right]$$

1M

$$= 12470$$

1A

6. (a) 當 $n = 1$,

$$\text{左式} = \sum_{k=1}^1 (-1)^k (2k-1)^2 = (-1)(1)^2 = -1$$

$$\text{右式} = \frac{(-1)^1 (2-1)(2+1) + 1}{2} = -1 = \text{左式}$$

對 $n = 1$, 命題為真。

1

$$\text{假設 } \sum_{k=1}^p (-1)^k (2k-1)^2 = \frac{(-1)^p (2p-1)(2p+1) + 1}{2}, \text{ 其中 } p \in \mathbb{Z}^+.$$

1

$$\begin{aligned} & \sum_{k=1}^{p+1} (-1)^k (2k-1)^2 \\ &= \sum_{k=1}^p (-1)^k (2k-1)^2 + (-1)^{p+1} (2p+1)^2 \\ &= \frac{(-1)^p (2p-1)(2p+1) + 1}{2} + (-1)^{p+1} (2p+1)^2 \\ &= \frac{(-1)^{p+1} (2p+1) [-(2p-1) + 2(2p+1)] + 1}{2} \\ &= \frac{(-1)^{p+1} (2p+1)(2p+3) + 1}{2} \end{aligned}$$

1

對 $n = p+1$, 命題為真。

藉數學歸納法, $\forall n \in \mathbb{Z}^+$, 命題為真。

1

(b) $\sum_{k=99}^{200} (-1)^{k+1} (2k-1)^2$

$$= - \sum_{k=1}^{200} (-1)^k (2k-1)^2 + \sum_{k=1}^{98} (-1)^k (2k-1)^2$$

1M

$$= - \frac{(-1)^{200} (400-1)(400+1) + 1}{2} + \frac{(-1)^{98} (196-1)(196+1) + 1}{2}$$

$$= -60\,792$$

1A

7. (a) 當 $n = 1$,

左式 $= (3 - 2)(3 - 1) = 2$, 右式 $= 1(3 - 1) = 2 =$ 左式

對 $n = 1$, 命題為真。

1

假設 $\sum_{k=1}^p (3k - 2)(3k - 1) = p(3p^2 - 1)$, 其中 $p \in \mathbb{Z}^+$ 。

1

$$\begin{aligned} \sum_{k=1}^{p+1} (3k - 2)(3k - 1) &= \sum_{k=1}^p (3k - 2)(3k - 1) + (3p + 1)(3p + 2) \\ &= p(3p^2 - 1) + (3p + 1)(3p + 2) \\ &= 3p^3 + 9p^2 + 8p + 2 \\ &= (p + 1)(3p^2 + 6p + 2) \\ &= (p + 1)[3(p + 1)^2 - 1] \end{aligned}$$

1

1

對 $n = p + 1$, 命題為真。

藉數學歸納法, $\forall n \in \mathbb{Z}^+$, 命題為真。

1

(b) (i) $2 + 5 + 8 + \dots + (3n - 1) = \sum_{k=1}^n (3k - 1)$

$$\begin{aligned} &= 3 \sum_{k=1}^n k - n \\ &= 3 \times \frac{n(n+1)}{2} - n \\ &= \frac{3n^2 + n}{2} \end{aligned}$$

1M

1A

(ii) $2^2 + 5^2 + 8^2 + \dots + (3n - 1)^2 = \sum_{k=1}^n (3k - 1)^2$

$$\begin{aligned} &= \sum_{k=1}^n (3k - 2)(3k - 1) + \sum_{k=1}^n (3k - 1) \\ &= n(3n^2 - 1) + \frac{3n^2 + n}{2} \\ &= \frac{n(6n^2 + 3n - 1)}{2} \end{aligned}$$

1M

1A