DAYM2-DIFF-2425-ASM-SET 2-MATH
建議題解

1. $y = \sqrt{(e^x)^3}$

$= e^{\frac{3x}{2}}$

$\frac{dy}{dx} = e^{\frac{3x}{2}}\left(\frac{3}{2}\right)$ 1M

$= \frac{3}{2}e^{\frac{3x}{2}}$ 1A

2. $y = \ln\left(4x^3 + 5x^2\right)$

$\frac{dy}{dx} = \frac{12x^2 + 10x}{4x^3 + 5x^2}$ 1M

$= \frac{12x + 10}{4x^2 + 5x}$ 1A

3. $y = \log_2(2 + e^{2x})$

$= \frac{\ln(2 + e^{2x})}{\ln 2}$

$\frac{dy}{dx} = \frac{1}{\ln 2} \cdot \frac{2e^{2x}}{2 + e^{2x}}$ 1M

$= \frac{2e^{2x}}{(2 + e^{2x})\ln 2}$ 1A

4. $y = \sqrt{x}e^{-3x}$

$= x^{\frac{1}{2}}e^{-3x}$

$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}}e^{-3x} + x^{\frac{1}{2}}e^{-3x}(-3)$ 1M

$= e^{-3x}\left(\frac{1}{2\sqrt{x}} - 3\sqrt{x}\right)$ 1A

5. $\ln x + \ln y = 2x^2 + 3x(x + y)$

$\ln x + \ln y = 5x^2 + 3xy$

$\frac{1}{x} + \frac{1}{y} \times \frac{dy}{dx} = 10x + 3y + 3x\frac{dy}{dx}$ 1M

$\left(\frac{1}{y} - 3x\right)\frac{dy}{dx} = 10x + 3y - \frac{1}{x}$

$\frac{1 - 3xy}{y} \times \frac{dy}{dx} = \frac{10x^2 + 3xy - 1}{x}$

$\frac{dy}{dx} = \frac{y(10x^2 + 3xy - 1)}{x(1 - 3xy)}$ 1A

6. $\ln\left(x^2y\right) = 4x^3 - 10y$

$\frac{1}{x^2y}\left(2xy + x^2\frac{dy}{dx}\right) = 12x^2 - 10\frac{dy}{dx}$ 1M

$\left(\frac{1}{y} + 10\right)\frac{dy}{dx} = 12x^2 - \frac{2}{x}$

$\left(\frac{1+10y}{y}\right)\frac{dy}{dx} = \frac{12x^3 - 2}{x}$

$\frac{dy}{dx} = \frac{2y(6x^3 - 1)}{x(1+10y)}$ 1A

7. $\ln y = 2\ln x + 3\ln(3x-1) - (-x) - \frac{1}{3}\ln\left(9 + 4x^2\right)$ 1M

$\frac{1}{y} \times \frac{dy}{dx} = \frac{2}{x} + \frac{3(3)}{3x-1} + 1 - \frac{1}{3}\left(\frac{8x}{9+4x^2}\right)$ 1M

$\frac{dy}{dx} = \frac{x^2e^x(3x-1)^3}{\sqrt[3]{9+4x^2}}\left[\frac{2}{x} + 1 + \frac{9}{3x-1} - \frac{8x}{3(9+4x^2)}\right]$ 1A

8. $\ln y = \frac{2}{3}\ln(3x+4) + \frac{4}{3}\ln(5+2x) - \frac{1}{3}\ln\left(2e^{3x} + 3x\right)$ 1M

$\frac{1}{y} \times \frac{dy}{dx} = \frac{2}{3}\left(\frac{3}{3x+4}\right) + \frac{4}{3}\left(\frac{2}{5+2x}\right) - \frac{1}{3}\left(\frac{6e^{3x}+3}{2e^{3x}+3x}\right)$ 1M

$\frac{dy}{dx} = \sqrt[3]{\frac{(3x+4)^2(5+2x)^4}{2e^{3x}+3x}}\left[\frac{2}{3x+4} + \frac{8}{3(5+2x)} - \frac{2e^{3x}+1}{2e^{3x}+3x}\right]$ 1A

9. $\ln y = \ln x \ln(x\tan x + e^x)$ 1M

$\frac{1}{y} \times \frac{dy}{dx} = \frac{1}{x}\ln(x\tan x + e^x) + \ln x \times \frac{\tan x + x\sec^2 x + e^x}{x\tan x + e^x}$ 1M

$\frac{dy}{dx} = (x\tan x + e^x)^{\ln x}\left[\frac{(x\sec^2 x + \tan x + e^x)\ln x}{x\tan x + e^x} + \frac{\ln(x\tan x + e^x)}{x}\right]$ 1A

10. $\sin xy = \ln\left(x^2 + 2\right)$

$\cos xy\left(y + x\frac{dy}{dx}\right) = \frac{2x}{x^2+2}$ 1M

$x\cos xy\frac{dy}{dx} + y\cos xy = \frac{2x}{x^2+2}$

$x\cos xy\frac{dy}{dx} = \frac{2x - (x^2+2)y\cos xy}{x^2+2}$

$\frac{dy}{dx} = \frac{2x - (x^2+2)y\cos xy}{(x^2+2)x\cos xy}$ 1A

11. $\ln(\sin y) = \cos xy + 3$

$\frac{\cos y}{\sin y} \times \frac{dy}{dx} = -\sin xy\left(y + x\frac{dy}{dx}\right)$ 1M

$(\cot y + x\sin xy)\frac{dy}{dx} = -y\sin xy$

$\frac{dy}{dx} = -\frac{y\sin xy}{\cot y + x\sin xy}$ 1A

12. $y^3 = 2\cos(x+y)$

$3y^2\dfrac{dy}{dx} = -2\sin(x+y)\left(1+\dfrac{dy}{dx}\right)$ 1M

$[3y^2+2\sin(x+y)]\dfrac{dy}{dx} = -2\sin(x+y)$

$\dfrac{dy}{dx} = -\dfrac{2\sin(x+y)}{3y^2+2\sin(x+y)}$ 1A

13. $\sin x^2y - \cos y = 3x^2$

$\cos x^2y\left(2xy + x^2\dfrac{dy}{dx}\right) + \sin y\dfrac{dy}{dx} = 6x$ 1M

$(x^2\cos x^2y + \sin y)\dfrac{dy}{dx} = 6x - 2xy\cos x^2y$

$\dfrac{dy}{dx} = \dfrac{6x - 2xy\cos x^2y}{x^2\cos x^2y + \sin y}$ 1A

14. $\dfrac{dy}{dx} = \dfrac{1}{2}(5x^2+2x)^{-\frac{1}{2}}(10x+2)$ 1M

$= (5x+1)(5x^2+2x)^{-\frac{1}{2}}$ 1A

$\dfrac{d^2y}{dx^2} = 5(5x^2+2x)^{-\frac{1}{2}} + (5x+1)\left(-\dfrac{1}{2}\right)(5x^2+2x)^{-\frac{3}{2}}(10x+2)$

$= (5x^2+2x)^{-\frac{3}{2}}[5(5x^2+2x) - (5x+1)^2]$

$= -(5x^2+2x)^{-\frac{3}{2}}$ 1A

15. $\dfrac{dy}{dx} = 2xe^{x^2} + x^2e^{x^2}(2x)$ 1M

$= 2e^{x^2}(x+x^3)$ 1A

$\dfrac{d^2y}{dx^2} = 2e^{x^2}(2x)(x+x^3) + 2e^{x^2}(1+3x^2)$

$= 2e^{x^2}[(2x^2+2x^4) + (1+3x^2)]$

$= 2e^{x^2}(2x^4+5x^2+1)$ 1A

16. $\dfrac{dy}{dx} = e^{-\frac{x}{2}}\left(-\dfrac{1}{2}\right)\cos\dfrac{x}{2} + e^{-\frac{x}{2}}\left(-\sin\dfrac{x}{2}\right)\left(\dfrac{1}{2}\right)$ 1M

$= -\dfrac{1}{2}e^{-\frac{x}{2}}\left(\cos\dfrac{x}{2} + \sin\dfrac{x}{2}\right)$ 1A

$\dfrac{d^2y}{dx^2} = -\dfrac{1}{2}e^{-\frac{x}{2}}\left(-\dfrac{1}{2}\right)\left(\cos\dfrac{x}{2} + \sin\dfrac{x}{2}\right) - \dfrac{1}{2}e^{-\frac{x}{2}}\left(-\dfrac{1}{2}\sin\dfrac{x}{2} + \dfrac{1}{2}\cos\dfrac{x}{2}\right)$

$= \dfrac{1}{4}e^{-\frac{x}{2}}\left[\left(\cos\dfrac{x}{2} + \sin\dfrac{x}{2}\right) + \left(\sin\dfrac{x}{2} - \cos\dfrac{x}{2}\right)\right]$

$= \dfrac{1}{2}e^{-\frac{x}{2}}\sin\dfrac{x}{2}$ 1A

17. $\dfrac{dy}{dx} = \cos x \ln(\cos x) + \sin x \cdot \dfrac{-\sin x}{\cos x}$ 1M

$= \cos x \ln(\cos x) - \sin x \tan x$ 1A

$\dfrac{d^2y}{dx^2} = -\sin x \ln(\cos x) + \cos x \cdot \dfrac{-\sin x}{\cos x} - \cos x \tan x - \sin x \sec^2 x$

$= -\sin x \left[\ln(\cos x) + 1 + 1 + \sec^2 x\right]$

$= -\sin x \left[\ln(\cos x) + \sec^2 x + 2\right]$ 1A

18. (a) $\dfrac{dy}{dx} = 5\left(\dfrac{2x+3}{2x-1}\right)^4 \cdot \dfrac{2(2x-1) - (2x+3)(2)}{(2x-1)^2}$ 1M

$= 5\left(\dfrac{2x+3}{2x-1}\right)^4 \cdot \dfrac{-8}{(2x-1)^2}$

$= \dfrac{-40(2x+3)^4}{(2x-1)^6}$ 1

(b) $\dfrac{dy}{dx} = -40(2x+3)^4(2x-1)^{-6}$

$\dfrac{d^2y}{dx^2} = -160(2x+3)^3(2)(2x-1)^{-6} - 40(2x+3)^4(-6)(2x-1)^{-7}(2)$ 1M

$= 160(2x+3)^3(2x-1)^{-7}\left[-2(2x-1) + 3(2x+3)\right]$

$= 160(2x+3)^3(2x-1)^{-7}(2x+11)$ 1A

19. (a) $y = \ln\left(1+x^2\right) - \ln\left(1-x^2\right)$

$\dfrac{dy}{dx} = \dfrac{2x}{1+x^2} - \dfrac{-2x}{1-x^2}$ 1M

$= \dfrac{2x(1-x^2) + 2x(1+x^2)}{(1-x^2)(1+x^2)}$

$= \dfrac{4x}{1-x^4}$ 1

(b) $\dfrac{d^2y}{dx^2} = \dfrac{4(1-x^4) - 4x(-4x^3)}{(1-x^4)^2}$ 1M

$= \dfrac{4(3x^4+1)}{(1-x^4)^2}$ 1A

20. $\dfrac{dy}{dx} = e^{-4x} + xe^{-4x}(-4)$ 1M

$= e^{-4x}(1-4x)$ 1A

$\dfrac{d^2y}{dx^2} = e^{-4x}(-4)(1-4x) + e^{-4x}(-4)$

$= -4e^{-4x}[(1-4x)+1]$

$= 8e^{-4x}(2x-1)$ 1A

$\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} - 4y$

$= \left[8e^{-4x}(2x-1)\right] + 3\left[e^{-4x}(1-4x)\right] - 4xe^{-4x}$

$= e^{-4x}\left[8(2x-1) + 3(1-4x) - 4x\right]$

$= -5e^{-4x}$ 1

21. $\dfrac{dy}{dx} = \dfrac{1}{x} \cdot \dfrac{1}{x^2} + \ln x(-2x^{-3})$ 1M

$= x^{-3}(1 - 2\ln x)$ 1A

$\dfrac{d^2y}{dx^2} = -3x^{-4}(1 - 2\ln x) + x^{-3}\left(-\dfrac{2}{x}\right)$

$= -x^{-4}\left[3(1 - 2\ln x) + 2\right]$

$= -x^{-4}(5 - 6\ln x)$ 1A

$x^2\dfrac{d^2y}{dx^2} + 5x\dfrac{dy}{dx} + 4y$

$= x^2\left[-x^{-4}(5 - 6\ln x)\right] + 5x\left[x^{-3}(1 - 2\ln x)\right] + 4x^{-2}\ln x$

$= x^{-2}\left[-(5 - 6\ln x) + 5(1 - 2\ln x) + 4\ln x\right]$

$= 0$ 1

22. $\dfrac{dy}{dx} = \dfrac{3}{2}x^{\frac{1}{2}}\ln x + x^{\frac{3}{2}} \cdot \dfrac{1}{x}$ 1M

$= \dfrac{1}{2}x^{\frac{1}{2}}(3\ln x + 2)$ 1A

$\dfrac{d^2y}{dx^2} = \dfrac{1}{4}x^{-\frac{1}{2}}(3\ln x + 2) + \dfrac{1}{2}x^{\frac{1}{2}}\left(\dfrac{3}{x}\right)$

$= \dfrac{1}{4}x^{-\frac{1}{2}}\left[(3\ln x + 2) + 6\right]$

$= \dfrac{1}{4}x^{-\frac{1}{2}}(3\ln x + 8)$ 1A

$4x^2\dfrac{d^2y}{dx^2} - 8x\dfrac{dy}{dx} + 9y$

$= 4x^2\left[\dfrac{1}{4}x^{-\frac{1}{2}}(3\ln x + 8)\right] - 8x\left[\dfrac{1}{2}x^{\frac{1}{2}}(3\ln x + 2)\right] + 9x^{\frac{3}{2}}\ln x$

$= x^{\frac{3}{2}}\left[(3\ln x + 8) - 4(3\ln x + 2) + 9\ln x\right]$

$= 0$ 1

23. $\dfrac{dy}{dx} = -2x^{-3}\cos x + x^{-2}(-\sin x)$ 1M

$= -x^{-3}(2\cos x + x\sin x)$ 1A

$\dfrac{d^2y}{dx^2} = 3x^{-4}(2\cos x + x\sin x) - x^{-3}\left[-2\sin x + \sin x + x\cos x\right]$

$= x^{-4}\left[3(2\cos x + x\sin x) - x(-2\sin x + \sin x + x\cos x)\right]$

$= x^{-4}(4x\sin x - x^2\cos x + 6\cos x)$ 1A

$x^2\dfrac{d^2y}{dx^2} + 4x\dfrac{dy}{dx} + (x^2 + 2)y$

$= x^2\left[x^{-4}(4x\sin x - x^2\cos x + 6\cos x)\right] + 4x\left[-x^{-3}(2\cos x + x\sin x)\right] + (x^2 + 2)x^{-2}\cos x$

$= x^{-2}\left[(4x\sin x - x^2\cos x + 6\cos x) - 4(2\cos x + x\sin x) + (x^2 + 2)\cos x\right]$

$= 0$ 1

24. $\dfrac{dy}{dx} = 9x^2 - 8x$ 1A

$\dfrac{d^2y}{dx^2} = 18x - 8$ 1A

$0 = x^2\dfrac{d^2y}{dx^2} + kx\dfrac{dy}{dx} + my$

$= x^2(18x - 8) + kx(9x^2 - 8x) + m(3x^3 - 4x^2)$

$= (18 + 9k + 3m)x^3 + (-8 - 8k - 4m)x^2$

可得 $18 + 9k + 3m = 0$ 及 $-8 - 8k - 4m = 0$。 1M

求解後，可得 $k = -4$ 及 $m = 6$。 1A

25. (a) $\dfrac{dy}{dx} = e^x \sin x + e^x \cos x$ 1M

$= e^x(\sin x + \cos x)$ 1A

$\dfrac{d^2y}{dx^2} = e^x(\sin x + \cos x) + e^x(\cos x - \sin x)$

$= 2e^x \cos x$ 1A

(b) $0 = \dfrac{d^2y}{dx^2} - k\dfrac{dy}{dx} + 2y$

$0 = 2e^x \cos x - ke^x(\sin x + \cos x) + 2e^x \sin x$

$= e^x[(2 - k)\cos x + (-k + 2)\sin x]$

可得 $2 - k = 0$。 1M

因此，$k = 2$。 1A

26. (a) (i) $\dfrac{dy}{dx} = k\left(\dfrac{x+1}{x-1}\right)^{k-1} \cdot \dfrac{1(x-1)-(x+1)(1)}{(x-1)^2}$ 1M

$= k\left(\dfrac{x+1}{x-1}\right)^{k}\left(\dfrac{x+1}{x-1}\right)^{-1} \cdot \dfrac{-2}{(x-1)^2}$

$= ky\dfrac{-2}{(x+1)(x-1)}$

$= \dfrac{2ky}{1-x^2}$ 1

(ii) $\dfrac{d^2y}{dx^2} = \dfrac{2k\frac{dy}{dx} - 2ky(-2x)}{(1-x^2)^2}$

$= \dfrac{2k}{1-x^2}\dfrac{dy}{dx} + \dfrac{4kxy}{(1-x^2)^2}$ 1A

$(x^2-1)\dfrac{d^2y}{dx^2} + 2(x+k)\dfrac{dy}{dx}$

$= (x^2-1)\left[\dfrac{2k}{1-x^2}\dfrac{dy}{dx} + \dfrac{4kxy}{(1-x^2)^2}\right] + 2(x+k)\left[\dfrac{2ky}{1-x^2}\right]$

$= -2k\left(\dfrac{2ky}{1-x^2}\right) - \dfrac{4kxy}{1-x^2} + \dfrac{4kxy}{1-x^2} + \dfrac{4k^2y}{1-x^2}$

$= 0$ 1

(b) $y = \left(\dfrac{0+1}{0-1}\right)^k$

$= (-1)^k$

$\left.\dfrac{dy}{dx}\right|_{x=0} = \dfrac{2k(-1)^k}{1-0}$

$= 2k(-1)^k$

$\left.\dfrac{d^2y}{dx^2}\right|_{x=0} = \dfrac{2k}{1}[2k(-1)^k] + 0$

$16 = 4k^2(-1)^k$ 1A

可得 k 為正偶數，且 $4k^2 = 16$。 1M

因此，$k = 2$。 1A