1. (a)

$$f'(u) = (m+1)(1+u)^m(1-u)^n - n(1+u)^{m+1}(1-u)^{n-1} \quad \text{1M}$$

$$(m+1)\int_{-1}^{1}(1+u)^m(1-u)^n\,\mathrm{d}u = \int_{-1}^{1} f'(u)\,\mathrm{d}u + n\int_{-1}^{1}(1+u)^{m+1}(1-u)^{n-1}\,\mathrm{d}u \quad \text{1M}$$

$$(m+1)\int_{-1}^{1}(1+u)^m(1-u)^n\,\mathrm{d}u = \Big[(1+u)^{m+1}(1-u)^n\Big]_{-1}^{1} + n\int_{-1}^{1}(1+u)^{m+1}(1-u)^{n-1}\,\mathrm{d}u \quad \text{1M}$$

$$\int_{-1}^{1}(1+u)^n(1-u)^n\,\mathrm{d}u = \frac{n}{m+1}\int_{-1}^{1}(1+u)^{m+1}(1-u)^{n-1}\,\mathrm{d}u \quad \text{1}$$

(b) Let $t = \tan x$. Then $\mathrm{d}t = \sec^2 x\,\mathrm{d}x$.

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{(1+\tan x)^2(\cos 4x+1)}{\cos^6 x}\,\mathrm{d}x$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec^6 x(1+\tan x)^2\left[(2\cos^2 2x - 1)+1\right]\mathrm{d}x \quad \text{1M}$$

$$= 2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec^6 x(1+\tan x)^2(2\cos^2 x - 1)^2\,\mathrm{d}x$$

$$= 2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec^2 x(1+\tan x)^2(2-\sec^2 x)^2\,\mathrm{d}x$$

$$= 2\int_{-1}^{1}(1+t)^2[2-(1+t^2)]^2\,\mathrm{d}t \quad \text{1M}$$

$$= 2\int_{-1}^{1}(1+t)^2[(1+t)(1-t)]^2\,\mathrm{d}t$$

$$= 2\int_{-1}^{1}(1+t)^4(1-t)^2\,\mathrm{d}t$$

$$= \frac{4}{5}\int_{-1}^{1}(1+t)^5(1-t)\,\mathrm{d}t \quad \text{1M}$$

$$= \frac{4}{5}\times\frac{1}{6}\int_{-1}^{1}(1+t)^6\,\mathrm{d}t \quad \text{1M}$$

$$= \frac{2}{15}\left[\frac{(1+t)^7}{7}\right]_{-1}^{1}$$

$$= \frac{256}{105} \quad \text{1A}$$

(c) Let $f(x) = \dfrac{\tan x(\cos 4x+1)}{\cos^6 x}$ and $g(x) = \dfrac{(1+\tan^2 x)(\cos 4x+1)}{\cos^6 x}$.

$$f(-x) = \frac{\tan(-x)(\cos(-4x)+1)}{\cos^6(-x)} = -\frac{\tan x(\cos 4x+1)}{\cos^6 x} = -f(x) \quad \text{1M}$$

$$g(-x) = \frac{(1+\tan^2(-x))(\cos(-4x)+1)}{\cos^6(-x)} = \frac{(1+\tan^2 x)(\cos 4x+1)}{\cos^6 x} = g(x)$$

Thus, $f(x)$ is an odd function and $g(x)$ is an even function.

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{(1+\tan x)^2(\cos 4x+1)}{\cos^6 x}\,dx = \frac{256}{105}$$

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{(1+\tan^2 x)(\cos 4x+1)}{\cos^6 x}\,dx + 2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\tan x(\cos 4x+1)}{\cos^6 x}\,dx = \frac{256}{105} \qquad \text{1M}$$

$$2\int_{0}^{\frac{\pi}{4}} \frac{(1+\tan^2 x)(\cos 4x+1)}{\cos^6 x}\,dx = \frac{256}{105} \qquad \text{1M}$$

$$2\int_{0}^{\frac{\pi}{4}} \sec^8 x(\cos 4x+1)\,dx = \frac{256}{105}$$

$$\int_{0}^{\frac{\pi}{4}} \sec^8 x(\cos 4x+1)\,dx = \frac{128}{105} \qquad \text{1A}$$

2. (a) (i) Let $u = a - x$. Then $du = -dx$. 1M

$$\int_0^a \ln\cos(a-x)\,dx = -\int_a^0 \ln u\,du$$
$$= \int_0^a \ln u\,du$$
$$= \int_0^a \ln x\,dx \qquad \text{1}$$

(ii) $$\int_0^a \ln\cos(a-x)\,dx - \int_0^a \ln x\,dx = 0 \qquad \text{1M}$$

$$\int_0^a \ln\frac{\cos a\cos x + \sin a\sin x}{\cos x}\,dx = 0 \qquad \text{1M}$$

$$\int_0^a \ln(\cos a + \sin a\tan x)\,dx = 0 \qquad \text{1}$$

(b) (i) $$\int_0^{\frac{\pi}{4}} \frac{\theta\sec^2\theta}{1+\tan\theta}\,d\theta$$

$$= \Big[\theta\ln(1+\tan\theta)\Big]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \ln(1+\tan\theta)\,d\theta \qquad \text{1M}$$

$$= \frac{\pi}{4}\ln 2 - \int_0^{\frac{\pi}{4}} \ln(1+\tan\theta)\,d\theta \qquad \text{1}$$

(ii) Put $a = \dfrac{\pi}{4}$ into (a)(ii).

$$\int_0^{\frac{\pi}{4}} \ln\left(\cos\frac{\pi}{4} + \sin\frac{\pi}{4}\tan x\right)dx = 0 \qquad \text{1M}$$

$$\int_0^{\frac{\pi}{4}} \ln\frac{1+\tan x}{\sqrt{2}}\,dx = 0$$

$$\int_0^{\frac{\pi}{4}} \left[\ln(1+\tan x) - \frac{1}{2}\ln 2\right]dx = 0 \qquad \text{1M+1M}$$

$$\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx = \left(\frac{1}{2}\ln 2\right)\Big[x\Big]_0^{\frac{\pi}{4}}$$
$$= \frac{\pi}{8}\ln 2$$

$$\int_0^{\frac{\pi}{4}} \frac{\theta \sec^2\theta}{1+\tan\theta}\,\mathrm{d}\theta = \frac{\pi}{4} - \int_0^{\frac{\pi}{4}} \ln(1+\tan\theta)\,\mathrm{d}\theta$$
$$= \frac{\pi}{4}\ln 2 - \frac{\pi}{8}\ln 2$$
$$= \frac{\pi}{8}\ln 2 \qquad \text{1A}$$

3. (a) Let $u = a - x$. Then $\mathrm{d}u = -\,\mathrm{d}x$. 1M

$$\int_0^a f(a-x)\,\mathrm{d}x = -\int_a^0 f(u)\,\mathrm{d}u$$
$$= \int_0^a f(u)\,\mathrm{d}u \qquad \text{1M}$$
$$= \int_0^a f(x)\,\mathrm{d}x \qquad \text{1}$$

(b) (i) $I = \displaystyle\int_0^{\pi} \frac{(\pi-x)\sin(\pi-x)}{1+\cos^2(\pi-x)}\,\mathrm{d}x$ 1M

$$= \int_0^{\pi} \frac{(\pi-x)\sin x}{1+\cos^2 x}\,\mathrm{d}x \qquad \text{1}$$

(ii) Use the result of (b)(i).

$$I = \frac{1}{2}\left[\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,\mathrm{d}x + \int_0^{\pi} \frac{(\pi-x)\sin x}{1+\cos^2 x}\,\mathrm{d}x\right] \qquad \text{1M}$$
$$= \frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\,\mathrm{d}x$$

Let $u = \pi - x$. Then $\mathrm{d}u = -\,\mathrm{d}x$. 1M

$$I = \frac{\pi}{2}\left[\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}\,\mathrm{d}x + \int_{\frac{\pi}{2}}^{\pi} \frac{\sin x}{1+\cos^2 x}\,\mathrm{d}x\right]$$
$$= \frac{\pi}{2}\left[\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}\,\mathrm{d}x - \int_{\frac{\pi}{2}}^{0} \frac{\sin(\pi-u)}{1+\cos^2(\pi-u)}\,\mathrm{d}u\right]$$
$$= \frac{\pi}{2}\left[\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}\,\mathrm{d}x + \int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}\,\mathrm{d}x\right]$$
$$= \pi\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}\,\mathrm{d}x \qquad \text{1}$$

(iii) Let $u = \cos x$. Then $\mathrm{d}u = -\sin x\,\mathrm{d}x$. 1M

$$I = \pi\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}\,\mathrm{d}x$$
$$= -\pi\int_1^{-1} \frac{\mathrm{d}u}{1+u^2}$$
$$= \pi\int_{-1}^{1} \frac{\mathrm{d}u}{1+u^2}$$

Let $u = \tan\theta$. Then $\mathrm{d}u = \sec^2\theta\,\mathrm{d}\theta$. 1M

$$I = \pi\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sec^2\theta}{1+\tan^2\theta}\,\mathrm{d}\theta$$

$$= \pi\Big[\theta\Big]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$= \frac{\pi^2}{4}$$ 1

4. (a) Let $x = 3\tan\theta$. Then $\mathrm{d}x = 3\sec^2\theta\,\mathrm{d}\theta$. 1M

$$\int_0^1 \frac{1}{x^2+9}\,\mathrm{d}x = \int_0^{\tan^{-1}\frac{1}{3}} \frac{3\sec^2\theta}{9\tan^2\theta+9}\,\mathrm{d}\theta$$

$$= \frac{1}{3}\Big[\theta\Big]_0^{\tan^{-1}\frac{1}{3}}$$ 1M

$$= \frac{1}{3}\tan^{-1}\frac{1}{3}$$ 1A

(b) (i) $$\frac{1-\tan^2\theta}{1+\tan^2\theta} = \frac{1-\frac{\sin^2\theta}{\cos^2\theta}}{1+\frac{\sin^2\theta}{\cos^2\theta}} \times \frac{\cos^2\theta}{\cos^2\theta}$$

$$= \frac{\cos^2\theta-\sin^2\theta}{\cos^2\theta+\sin^2\theta}$$

$$= \cos 2\theta$$ 1

(ii) Let $t = \tan\theta$. Then $\mathrm{d}t = \sec^2\theta\,\mathrm{d}\theta = (1+t^2)\,\mathrm{d}\theta$. 1M

$$\int_0^{\frac{\pi}{4}} \frac{1}{4\cos 2\theta+5}\,\mathrm{d}\theta = \int_0^1 \frac{1}{4\left(\frac{1-t^2}{1+t^2}\right)+5}\cdot\frac{1}{1+t^2}\,\mathrm{d}t$$ 1M

$$= \int_0^1 \frac{1}{t^2+9}\,\mathrm{d}t$$

$$= \frac{1}{3}\tan^{-1}\frac{1}{3}$$ 1A

(c) Let $u = \frac{\pi}{4} - \theta$. Then $\mathrm{d}u = -\,\mathrm{d}\theta$. 1M

$$\int_0^{\frac{\pi}{4}} \frac{1}{4\cos 2\theta+5}\,\mathrm{d}\theta = -\int_{\frac{\pi}{4}}^{0} \frac{1}{4\cos\left(\frac{\pi}{4}-2u\right)+5}\,\mathrm{d}u$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{4\sin 2u+5}\,\mathrm{d}u$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{4\sin 2\theta+5}\,\mathrm{d}\theta$$ 1

(d) $\displaystyle\int_0^{\frac{\pi}{4}} \frac{4\sin 2\theta + 4\cos 2\theta + 10}{(4\sin 2\theta + 5)(4\cos 2\theta + 5)}\,\mathrm{d}\theta$

$\displaystyle= \int_0^{\frac{\pi}{4}} \frac{1}{\cos 2\theta + 5}\,\mathrm{d}\theta + \int_0^{\frac{\pi}{4}} \frac{1}{4\sin 2\theta + 5}\,\mathrm{d}\theta$ 1M

$\displaystyle= 2\int_0^{\frac{\pi}{4}} \frac{1}{4\cos 2\theta + 5}\,\mathrm{d}\theta$

$\displaystyle= 2\left(\frac{1}{3}\tan^{-1}\frac{1}{3}\right)$ 1M

$\displaystyle= \frac{2}{3}\tan^{-1}\frac{1}{3}$ 1A

5. (a) $\displaystyle\int_0^1 xe^{ax}\,\mathrm{d}x = \left[\frac{xe^{ax}}{a}\right]_0^1 - \frac{1}{a}\int_0^1 e^{ax}\,\mathrm{d}x$ 1M+1M

$\displaystyle= \frac{e^a}{a} - \left[\frac{e^{ax}}{a^2}\right]_0^1$

$\displaystyle= \frac{ae^a - e^a + 1}{a^2}$ 1

(b) Let $u = \ln(x+1)$. Then $\mathrm{d}u = \dfrac{1}{x+1}\,\mathrm{d}x$. 1M

$\displaystyle\int_0^{e-1} \frac{\ln(x+1)}{(1+x)^r}\,\mathrm{d}x = \int_0^1 \frac{u}{(e^u)^{r-1}}\,\mathrm{d}u$ 1M

$\displaystyle= \int_0^1 ue^{(1-r)u}\,\mathrm{d}u$

$\displaystyle= \frac{(1-r)e^{1-r} - e^{1-r} + 1}{(1-r)^2}$ 1M

$\displaystyle= \frac{1 - re^{1-r}}{(1-r)^2}$ 1A

(c) Let $v = (e-1)\tan x$. Then $\mathrm{d}v = (e-1)\sec^2 x\,\mathrm{d}x$. 1M

$\displaystyle\int_0^{\frac{\pi}{4}} \frac{\ln[1+(e-1)\tan x]}{(1+e\tan x - \tan x)^3}\,\mathrm{d}x + \int_0^{\frac{\pi}{4}} \frac{[\ln[1+(e-1)\tan x]]\tan^2 x}{(1+e\tan x - \tan x)^3}\,\mathrm{d}x$

$\displaystyle= \int_0^{\frac{\pi}{4}} \frac{\ln[1+(e-1)\tan x]\sec^2 x}{[1+(e-1)\tan x]^3}\,\mathrm{d}x$

$\displaystyle= \frac{1}{e-1}\int_0^{e-1} \frac{\ln(1+v)}{(1+v)^3}\,\mathrm{d}v$

$\displaystyle= \frac{1}{e-1} \times \frac{1-3e^{1-3}}{(1-3)^2}$ 1M

$\displaystyle= \frac{e^2-3}{4e^2(e-1)}$ 1A

(d) Let $w = \dfrac{\pi}{2} - 2x$. Then $\mathrm{d}w = -2\,\mathrm{d}x$. 1M

$$\int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \frac{[\ln[1+(e-1)\cot 2x]]\csc^2 2x}{(1+e\cot 2x-\cot 2x)^3}\,\mathrm{d}x$$

$$= -\frac{1}{2}\int_{\frac{\pi}{4}}^{0} \frac{\ln\left[1+(e-1)\cot\left(\frac{\pi}{2}-w\right)\right]\csc^2\left(\frac{\pi}{2}-w\right)}{\left[1+e\cot\left(\frac{\pi}{2}-w\right)-\cot\left(\frac{\pi}{2}-w\right)\right]^3}\,\mathrm{d}w$$

$$= \frac{1}{2}\int_{0}^{\frac{\pi}{4}} \frac{\ln[1+(e-1)\tan w]\sec^2 w}{(1+e\tan w-\tan w)^3}\,\mathrm{d}w$$

$$= \frac{1}{2}\times\frac{e^2-3}{4e^2(e-1)}$$ 1M

$$= \frac{e^2-3}{8e^2(e-1)}$$ 1A

6. (a) Let $u = \frac{\pi}{4}$. Then $\mathrm{d}u = -\,\mathrm{d}x$. 1M

$$\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln\left[\cos\left(\frac{\pi}{4}-x\right)\right]\mathrm{d}x = -\int_{\frac{\pi}{6}}^{\frac{\pi}{12}} \ln\cos(u)\,\mathrm{d}u$$

$$= \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln\cos(u)\,\mathrm{d}u$$ 1M

$$= \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln(\cos x)\,\mathrm{d}x$$ 1

(b) $$\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln\left[\cos\left(\frac{\pi}{4}-x\right)\right]\mathrm{d}x - \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln(\cos x)\,\mathrm{d}x = 0$$

$$\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln\frac{\cos\frac{\pi}{4}\cos x+\sin\frac{\pi}{4}\sin x}{\cos x}\,\mathrm{d}x = 0$$ 1M

$$\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln\frac{1+\tan x}{\sqrt{2}}\,\mathrm{d}x = 0$$

$$\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln(\tan x+1)\,\mathrm{d}x = \left(\frac{1}{2}\ln 2\right)\Big[x\Big]_{\frac{\pi}{12}}^{\frac{\pi}{6}}$$ 1M

$$\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln(\tan x+1)\,\mathrm{d}x = \frac{\pi}{24}\ln 2$$ 1A

(c) (i) $$\tan\frac{\pi}{12} = \tan\left(\frac{\pi}{3}-\frac{\pi}{4}\right)$$

$$= \frac{\tan\frac{\pi}{3}-\tan\frac{\pi}{4}}{1+\tan\frac{\pi}{3}\tan\frac{\pi}{4}}$$ 1M

$$= \frac{\sqrt{3}-1}{1+\sqrt{3}}\times\frac{\sqrt{3}-1}{\sqrt{3}-1}$$

$$= 2-\sqrt{3}$$ 1

(ii) Note that $\dfrac{\mathrm{d}}{\mathrm{d}x}\ln(\tan x+1) = \dfrac{\sec^2 x}{\tan x+1}$. 1M

$$\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \frac{x\sec^2 x}{\tan x+1}\,dx$$

$$=\Big[x\ln(\tan x+1)\Big]_{\frac{\pi}{12}}^{\frac{\pi}{6}} - \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln(\tan x+1)\,dx \qquad \text{1M}$$

$$=\frac{\pi}{6}\ln\left(\tan\frac{\pi}{6}+1\right)-\frac{\pi}{12}\ln\left(\tan\frac{\pi}{12}+1\right)-\frac{\pi}{24}\ln 2 \qquad \text{1M}$$

$$=\frac{\pi}{24}\ln\frac{(\sqrt{3}+3)^4}{3^4}-\frac{\pi}{24}\ln\left(3-\sqrt{3}\right)^2-\frac{\pi}{24}\ln 2$$

$$=\frac{\pi}{24}\ln\left[\frac{(3+\sqrt{3})^4}{3^4(2)(3-\sqrt{3})^2}\times\frac{(3+\sqrt{3})^2}{(3+\sqrt{3})^2}\right] \qquad \text{1M}$$

$$=\frac{\pi}{24}\ln\left[\frac{(3+\sqrt{3})^6}{(3^6)(2^3)}\right]$$

$$=\frac{\pi}{4}\ln\frac{3+\sqrt{3}}{3\sqrt{2}}$$

$$=\frac{\pi}{4}\ln\frac{\sqrt{6}+3\sqrt{2}}{6} \qquad \text{1}$$

7. (a) $\sqrt{2}\cos\left(\frac{\pi}{4}-\theta\right)=\sqrt{2}\left(\cos\frac{\pi}{4}\cos\theta+\sin\frac{\pi}{4}\sin\theta\right)$ 1M

$$=\sqrt{2}\left(\frac{1}{\sqrt{2}}\cos\theta+\frac{1}{\sqrt{2}}\sin\theta\right)$$

$$=\sin\theta+\cos\theta \qquad \text{1}$$

(b) Let $u=\frac{\pi}{4}-x$. Then $du=-\,dx$. 1M

$$\int_0^{\frac{\pi}{4}} g\left[\cos\left(\frac{\pi}{4}-x\right)\right]dx=-\int_{\frac{\pi}{4}}^{0} g(\cos u)\,du$$

$$=\int_0^{\frac{\pi}{4}} g(\cos u)\,du \qquad \text{1M}$$

$$=\int_0^{\frac{\pi}{4}} g(\cos x)\,dx \qquad \text{1}$$

(c) $\int_0^{\frac{\pi}{4}} \ln(1+\tan\theta)\,d\theta$

$$=\int_0^{\frac{\pi}{4}} \ln\frac{\cos\theta+\sin\theta}{\cos\theta}\,d\theta$$

$$=\int_0^{\frac{\pi}{4}} [\ln(\cos\theta+\sin\theta)-\ln(\cos\theta)]\,d\theta \qquad \text{1M}$$

$$=\int_0^{\frac{\pi}{4}} \ln\left[\sqrt{2}\cos\left(\frac{\pi}{4}-\theta\right)\right]d\theta-\int_0^{\frac{\pi}{4}}\ln(\cos\theta)\,d\theta \qquad \text{1M}$$

$$=\frac{1}{2}\ln 2\int_0^{\frac{\pi}{4}} d\theta+\int_0^{\frac{\pi}{4}}\ln\left[\cos\left(\frac{\pi}{4}-\theta\right)\right]d\theta-\int_0^{\frac{\pi}{4}}\ln(\cos\theta)\,d\theta$$

$$=\frac{1}{2}\ln 2\Big[\theta\Big]_0^{\frac{\pi}{4}}+\int_0^{\frac{\pi}{4}}\ln(\cos\theta)\,d\theta-\int_0^{\frac{\pi}{4}}\ln(\cos\theta)\,d\theta \qquad \text{1M}$$

$$=\frac{\pi}{8}\ln 2 \qquad \text{1A}$$

(d) Let $x = \tan\theta$. Then $\mathrm{d}x = \sec^2\theta\,\mathrm{d}\theta$. 1M

$$\int_0^1 \frac{\ln(1+x)^2}{1+x^2}\,\mathrm{d}x = \int_0^{\frac{\pi}{4}} \frac{\ln(1+\tan\theta)^2}{1+\tan^2\theta}\sec^2\theta\,\mathrm{d}\theta$$

$$= 2\int_0^{\frac{\pi}{4}} \ln(1+\tan\theta)\,\mathrm{d}\theta$$

$$= 2\left(\frac{\pi}{8}\ln 2\right) \qquad \text{1M}$$

$$= \frac{\pi}{4}\ln 2 \qquad \text{1A}$$

8. (a) $\dfrac{1+\sin 2t}{1+\cos 2t} = \dfrac{1+2\sin t\cos t}{1+(2\cos^2 t-1)}$ 1M

$$= \frac{1+2\sin t\cos t}{2\cos^2 t}$$

$$= \frac{1}{2}\sec^2 t + \tan t \qquad \text{1}$$

(b) $\displaystyle\int e^x f'(x)\,\mathrm{d}x = e^x f(x) - \int e^x f(x)\,\mathrm{d}x$ 1M

$$\int e^x[f(x)+f'(x)]\,\mathrm{d}x = e^x f(x) + \text{constant} \qquad \text{1}$$

(c) Let $f(x) = \tan\dfrac{x}{2}$. Then $f'(x) = \dfrac{1}{2}\sec^2\dfrac{x}{2}$. 1M

$$\int_0^{\frac{\pi}{2}} \frac{e^x(1+\sin x)}{1+\cos x}\,\mathrm{d}x = \int_0^{\frac{\pi}{2}} e^x\left(\frac{1}{2}\sec^2\frac{x}{2} + \tan\frac{x}{2}\right)\mathrm{d}x \qquad \text{1M}$$

$$= \left[e^x\tan\frac{x}{2}\right]_0^{\frac{\pi}{2}} \qquad \text{1M}$$

$$= e^{\frac{\pi}{2}} \qquad \text{1A}$$

(d) Let $u = \dfrac{\pi}{2} - x$. Then $\mathrm{d}u = -\,\mathrm{d}x$. 1M

$$\int_0^{\frac{\pi}{2}} \frac{1+\cos x}{e^x(1+\sin x)}\,\mathrm{d}x = -\int_{\frac{\pi}{2}}^0 \frac{1+\cos\left(\frac{\pi}{2}-x\right)}{e^{\frac{\pi}{2}-u}\left(1+\sin\left(\frac{\pi}{2}-u\right)\right)}\,\mathrm{d}u$$

$$= e^{-\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{e^u(1+\sin u)}{1+\cos u}\,\mathrm{d}u$$

$$= e^{-\frac{\pi}{2}}\cdot e^{\frac{\pi}{2}} \qquad \text{1M}$$

$$= 1 \qquad \text{1A}$$

9. (a) $\dfrac{1-\cos 2x}{1+\cos 2x} = \dfrac{1-(1-2\sin^2 x)}{1+(2\cos^2 x-1)}$ 1M

$$= \frac{2\sin^2 x}{2\cos^2 x}$$

$$= \tan^2 x \qquad \text{1}$$

(b) Let $u = a - x$. Then $\mathrm{d}u = -\,\mathrm{d}x$.

$$\int_0^a f(a-x)\,\mathrm{d}x = -\int_a^0 f(u)\,\mathrm{d}u$$
$$= \int_0^a f(u)\,\mathrm{d}u$$
$$= \int_0^a f(x)\,\mathrm{d}x \qquad 1$$

(c) Let $u = \cos x$. Then $\mathrm{d}u = -\sin x\,\mathrm{d}x$.

$$\int \tan x\,\mathrm{d}x = \int \frac{\sin x}{\cos x}\,\mathrm{d}x$$
$$= -\int \frac{\mathrm{d}u}{u} \qquad 1\mathrm{M}$$
$$= -\ln|u| + \text{constant}$$
$$= -\ln|\cos x| + \text{constant} \qquad 1$$

(d) $$\int x\sec^2 x\,\mathrm{d}x = x\tan x - \int \tan x\,\mathrm{d}x \qquad 1\mathrm{M}$$
$$= x\tan x - (-\ln|\cos x|) + \text{constant}$$
$$= x\tan x + \ln|\cos x| + \text{constant} \qquad 1\mathrm{A}$$

(e) $$\int_0^{\frac{\pi}{4}} \frac{(\cos x - \sin x + \sqrt{x})(\cos x - \sin x - \sqrt{x})}{1+\sin 2x}\,\mathrm{d}x$$
$$= \int_0^{\frac{\pi}{4}} \frac{(\cos x - \sin x)^2 - x}{1+\sin 2x}\,\mathrm{d}x$$
$$= \int_0^{\frac{\pi}{4}} \frac{\sin^2 x - 2\sin x\cos x + \cos^2 x - x}{1+\sin 2x}\,\mathrm{d}x$$
$$= \int_0^{\frac{\pi}{4}} \frac{1 - \sin 2x - x}{1+\sin 2x}\,\mathrm{d}x \qquad 1\mathrm{M}$$
$$= \int_0^{\frac{\pi}{4}} \frac{1 - \sin\left(\frac{\pi}{2} - 2x\right) - \left(\frac{\pi}{4} - x\right)}{1+\sin\left(\frac{\pi}{2} - 2x\right)}\,\mathrm{d}x \qquad 1\mathrm{M}$$
$$= \int_0^{\frac{\pi}{4}} \frac{1-\cos 2x}{1+\cos 2x}\,\mathrm{d}x + \int_0^{\frac{\pi}{4}} \frac{x - \frac{\pi}{4}}{1+\cos 2x}\,\mathrm{d}x$$
$$= \int_0^{\frac{\pi}{4}} \tan^2 x\,\mathrm{d}x + \int_0^{\frac{\pi}{4}} \frac{x - \frac{\pi}{4}}{2\cos^2 x}\,\mathrm{d}x \qquad 1\mathrm{M}$$
$$= \int_0^{\frac{\pi}{4}} (\sec^2 x - 1)\,\mathrm{d}x + \frac{1}{2}\int_0^{\frac{\pi}{4}} x\sec^2 x\,\mathrm{d}x - \frac{\pi}{8}\int_0^{\frac{\pi}{4}} \sec^2 x\,\mathrm{d}x \qquad 1\mathrm{M}$$
$$= \Big[\tan x - x\Big]_0^{\frac{\pi}{4}} + \frac{1}{2}\Big[x\tan x + \ln(\cos x)\Big]_0^{\frac{\pi}{4}} - \frac{\pi}{8}\Big[\tan x\Big]_0^{\frac{\pi}{4}} \qquad 1\mathrm{M}$$
$$= 1 - \frac{\pi}{4} - \frac{1}{4}\ln 212 \qquad 1\mathrm{A}$$

10. (a) (i) $$\frac{\mathrm{d}}{\mathrm{d}\theta}\ln(\csc\theta + \cot\theta) = \frac{-(\sin\theta)^{-2}\cos\theta - (\tan\theta)^{-2}\sec^2\theta}{\csc\theta + \cot\theta} \qquad 1\mathrm{M}$$
$$= \frac{-\csc\theta\cot\theta - \csc^2\theta}{\csc\theta + \cot\theta}$$
$$= -\csc\theta$$

Thus, $\displaystyle\int \csc\theta\,\mathrm{d}\theta = -\ln(\csc\theta+\cot\theta)+\text{constant.}$ 1

(ii) $\displaystyle\int \frac{1}{\sin x\cos x}\,\mathrm{d}x = 2\int \csc 2x\,\mathrm{d}x$ 1M

$\displaystyle= -\ln(\csc 2x+\cot 2x)+\text{constant}$ 1A

(b) Let $u=\dfrac{\pi}{2}-x$. Then $\mathrm{d}u=-\,\mathrm{d}x$. 1M

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} f(x)\ln\left(1+e^{\sin x-\cos x}\right)\mathrm{d}x = -\int_{\frac{\pi}{3}}^{\frac{\pi}{6}} f\left(\frac{\pi}{2}-u\right)\ln\left[1+e^{\sin\left(\frac{\pi}{2}-u\right)-\cos\left(\frac{\pi}{2}-u\right)}\right]\mathrm{d}u$$

$$= -\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} f(u)\ln\left(\frac{e^{\sin u-\cos u}+1}{e^{\sin u-\cos u}}\right)\mathrm{d}u \quad \text{1M}$$

$$= -\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} f(u)\ln\left(1+e^{\sin x-\cos x}\right)\mathrm{d}x$$

$$+\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} f(x)(\sin x-\cos x)\,\mathrm{d}x \quad \text{1M}$$

$$2\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} f(x)\ln\left(1+e^{\sin x-\cos x}\right)\mathrm{d}x = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} f(x)(\sin x-\cos x)\,\mathrm{d}x \quad \text{1M}$$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} f(x)\ln\left(1+e^{\sin x-\cos x}\right)\mathrm{d}x = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} f(x)(\sin x-\cos x)\,\mathrm{d}x \quad \text{1}$$

(c) Put $f(x)=\dfrac{\sin x-\cos x}{\sin x\cos x}$. 1M

$$f\left(\frac{\pi}{2}-x\right)=\frac{\sin\left(\frac{\pi}{2}-x\right)-\cos\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)\cos\left(\frac{\pi}{2}-x\right)}$$

$$=\frac{\cos x-\sin x}{\cos x\sin x}$$

$$=-f(x)$$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{(\sin x-\cos x)\ln\left(1+e^{\sin x-\cos x}\right)}{\sin x\cos x}\,\mathrm{d}x$$

$$=\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{(\sin x-\cos x)^2}{\sin x\cos x}\,\mathrm{d}x \quad \text{1M}$$

$$=\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1-2\sin x\cos x}{\sin x\cos x}\,\mathrm{d}x$$

$$=\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{\sin x\cos x}\,\mathrm{d}x-\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\mathrm{d}x$$

$$=\frac{1}{2}\Big[-\ln(\csc 2x+\cot 2x)\Big]_{\frac{\pi}{6}}^{\frac{\pi}{3}}-\Big[x\Big]_{\frac{\pi}{6}}^{\frac{\pi}{3}} \quad \text{1M}$$

$$=\frac{1}{2}\left(\ln 3-\frac{\pi}{3}\right) \quad \text{1A}$$

11. (a) (i) $\displaystyle\frac{\sin 3x}{\sin x}-(2\cos 2x+1)=\frac{1}{\sin x}\left[\sin 3x-2\sin x\cos 2x-\sin x\right]$

$$=\frac{1}{\sin x}\left[\sin 3x-(\sin 3x-\sin x)-\sin x\right] \quad \text{1M}$$

$$=0$$

Thus, $\dfrac{\sin 3x}{\sin x} = 2\cos 2x + 1$. 1

(ii) Put $x = \dfrac{\pi}{4} + y$.

$$\frac{\sin 3\left(\frac{\pi}{4}+y\right)}{\sin\left(\frac{\pi}{4}+y\right)} = 2\cos 2\left(\frac{\pi}{4}+y\right) + 1$$

$$\frac{\sin\frac{3\pi}{4}\cos 3y + \cos\frac{3\pi}{4}\sin 3y}{\sin\frac{\pi}{4}\cos y + \cos\frac{\pi}{4}\sin y} = -2\sin 2y + 1 \qquad \text{1A}$$

$$\frac{\cos 3y - \sin 3y}{\cos y + \sin y} = -2\sin 2y + 1$$

$$\frac{\sin 3y - \cos 3y}{\sin x + \cos y} = 2\sin 2y - 1 \qquad \text{1A}$$

(b) (i) Let $u = a - x$. Then $\mathrm{d}u = -\,\mathrm{d}x$.

$$\int_0^a f(x)\,\mathrm{d}x = -\int_a^0 f(a-u)\,\mathrm{d}u \qquad \text{1M}$$

$$= \int_0^a f(a-u)\,\mathrm{d}u$$

$$= \int_0^a f(a-x)\,\mathrm{d}x \qquad \text{1A}$$

(ii) Put $f(x) = \dfrac{\sin 3x}{\sin x + \cos x}$.

$$f\left(\frac{\pi}{2}-x\right) = \frac{\sin 3\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)} = \frac{-\cos 3x}{\cos x + \sin x} \qquad 1$$

We have $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin 3x}{\sin x + \cos x}\,\mathrm{d}x = \int_0^{\frac{\pi}{2}} \frac{-\cos 3x}{\sin x + \cos x}\,\mathrm{d}x$. 1A

$$2\int_0^{\frac{\pi}{2}} \frac{\sin 3x}{\sin x + \cos x}\,\mathrm{d}x = \int_0^{\frac{\pi}{2}} \frac{\sin 3x}{\sin x + \cos x}\,\mathrm{d}x + \int_0^{\frac{\pi}{2}} \frac{-\cos 3x}{\sin x + \cos x}\,\mathrm{d}x \qquad \text{1M}$$

$$\int_0^{\frac{\pi}{2}} \frac{\sin 3x}{\sin x + \cos x}\,\mathrm{d}x = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\sin 3x - \cos 3x}{\sin x + \cos x}\,\mathrm{d}x \qquad \text{1A}$$

(c) $$\int_0^{\frac{\pi}{2}} \frac{\sin 3x}{\sin x + \cos x}\,\mathrm{d}x = \frac{1}{2}\int_0^{\frac{\pi}{2}} (2\sin 2x - 1)\,\mathrm{d}x \qquad \text{1M}$$

$$= \frac{1}{2}\Big[-\cos 2x - x\Big]_0^{\frac{\pi}{2}} \qquad \text{1A}$$

$$= 1 - \frac{\pi}{4} \qquad \text{1A}$$

12. (a) $$\int g(x)\,\mathrm{d}x = \sin^2 x\left(\frac{\sin 2x}{2}\right) - \frac{1}{2}\int (2\sin x\cos x)\sin 2x\,\mathrm{d}x \qquad \text{1M}$$

$$= \frac{1}{2}\sin^2 x\sin 2x - \frac{1}{2}\int \sin^2 2x\,\mathrm{d}x \qquad 1$$

(b) $\displaystyle\int_0^{\pi} g(x)\,\mathrm{d}x = \frac{1}{2}\left[\sin^2 x \sin 2x\right]_0^{\pi} - \frac{1}{2}\int_0^{\pi} \sin^2 2x\,\mathrm{d}x$

$$= 0 - \frac{1}{4}\int_0^{\pi} (1 - \cos 4x)\,\mathrm{d}x \qquad \text{1M}$$

$$= -\frac{1}{4}\left[x - \frac{\sin 4x}{4}\right]_0^{\pi}$$

$$= -\frac{\pi}{4} \qquad \text{1A}$$

(c) Let $u = \pi - x$. Then $\mathrm{d}u = -\,\mathrm{d}x$. 1M

$$\int_0^{\pi} xg(x)\,\mathrm{d}x = -\int_{\pi}^{0} (\pi - u)g(\pi - u)\,\mathrm{d}u$$

$$= \int_0^{\pi} (\pi - u)\sin^2(\pi - u)\cos(2\pi - 2u)\,\mathrm{d}u \qquad \text{1M}$$

$$= \int_0^{\pi} (\pi - x)\sin^2 x \cos 2x\,\mathrm{d}x$$

$$= \pi\int_0^{\pi} g(x)\,\mathrm{d}x - \int_0^{\pi} xg(x)\,\mathrm{d}x$$

$$= \frac{\pi}{2}\int_0^{\pi} g(x)\,\mathrm{d}x \qquad \text{1M}$$

$$= \frac{\pi}{2}\left(-\frac{\pi}{4}\right)$$

$$= -\frac{\pi^2}{8} \qquad \text{1A}$$

(d) Note that $(-x)g(-x) = -x\sin^2(-x)\cos(-2x) = -x\sin^2\cos 2x = -xg(x)$. 1M

Thus, $xg(x)$ is an odd function.

Let $u = x - \pi$. Then $\mathrm{d}u = \mathrm{d}x$. 1M

$$\int_{-\pi}^{2\pi} xg(x)\,\mathrm{d}x = \int_{-\pi}^{\pi} xg(x)\,\mathrm{d}x + \int_{\pi}^{2\pi} xg(x)\,\mathrm{d}x$$

$$= 0 + \int_{\pi}^{2\pi} xg(x)\,\mathrm{d}x$$

$$= \int_0^{\pi} (u + \pi)\sin^2(u + \pi)\cos(2u + 2\pi)\,\mathrm{d}u$$

$$= \int_0^{\pi} (\pi + x)g(x)\,\mathrm{d}x \qquad \text{1M}$$

$$= \pi\int_0^{\pi} g(x)\,\mathrm{d}x + \int_0^{\pi} xg(x)\,\mathrm{d}x$$

$$= \pi\left(-\frac{\pi}{4}\right) + \left(-\frac{\pi^2}{8}\right)$$

$$= -\frac{3\pi^2}{8} \qquad \text{1A}$$

13. (a) $\sin x + \sin x \tan^2 \frac{x}{2} = 2 \sin \frac{x}{2} \cos \frac{x}{2} \left(1 + \frac{\sin^2 \frac{x}{2}}{\cos^2 \frac{x}{2}} \right)$

$= 2 \tan \frac{x}{2} \left(\cos^2 \frac{x}{2} + \sin^2 \frac{x}{2} \right)$

$= 2 \tan \frac{x}{2}$ 1

(b) Note that $x^2 + ax + a^2 = \left(x + \frac{a}{2} \right)^2 + \frac{3a^2}{4}$.

Let $x + \frac{a}{2} = \frac{\sqrt{3}a}{2} \tan \theta$. Then $dx = \frac{\sqrt{3}a}{2} \sec^2 \theta \, d\theta$. 1M

$$\int_0^a \frac{dx}{x^2 + ax + a^2} = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\frac{\sqrt{3}a}{2} \sec^2 \theta}{\frac{3a^2}{4} \tan^2 \theta + \frac{3a^2}{4}} \, d\theta$$ 1M

$$= \frac{2}{\sqrt{3}a} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} d\theta$$

$$= \frac{2}{\sqrt{3}a} \Big[\theta \Big]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= \frac{\pi}{3\sqrt{3}a}$$ 1A

$$= \frac{\sqrt{3}\pi}{9a}$$

(c) Use the result of (a).

$$\sin x + \sin x \tan^2 \frac{x}{2} = 2 \tan \frac{x}{2}$$

$$\sin x = \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$$

Let $t = \tan \frac{x}{2}$. Then $dt = \frac{1}{2} \sec^2 \frac{x}{2} \, dx = \frac{1 + t^2}{2} \, dx$. 1M

$$\int_0^{\frac{\pi}{2}} \frac{dx}{\sin x + 2} = \int_0^1 \frac{1}{\frac{2t}{1+t^2} + 2} \cdot \frac{2}{1 + t^2} \, dt$$ 1M

$$= \int_0^1 \frac{dt}{t^2 + t + 1}$$

$$= \frac{\sqrt{3}\pi}{9(1)}$$

$$= \frac{\sqrt{3}\pi}{9}$$ 1A

(d) $\frac{d}{dx} \left(\frac{\cos x}{\sin x + 2} \right) = \frac{(-\sin x)(\sin x + 2) - \cos x (\cos x)}{(\sin x + 2)^2}$ 1M

$$= \frac{-2 \sin x - 1}{(\sin x + 2)^2}$$

$$= \frac{-2(\sin x + 2) + 3}{(\sin x + 2)^2}$$

$$= -\frac{2}{\sin x + 2} + \frac{3}{(\sin x + 2)^2}$$

Integrate both sides with respect to x.

$$\left[\frac{\cos x}{\sin x+2}\right]_0^{\frac{\pi}{2}} = -2\int_0^{\frac{\pi}{2}} \frac{dx}{\sin x+2} + 3\int_0^{\frac{\pi}{2}} \frac{dx}{(\sin x+2)^2} \qquad \text{1M}$$

$$\frac{1}{2} - 0 = -2\left(\frac{\sqrt{3}\pi}{9}\right) + 3\int_0^{\frac{\pi}{2}} \frac{dx}{(\sin x+2)^2}$$

$$\int_0^{\frac{\pi}{2}} \frac{dx}{(\sin x+2)^2} = \frac{2\sqrt{3}\pi}{27} - \frac{1}{6}$$

Note that $\dfrac{1}{[\cos(-x)+2]^2} = \dfrac{1}{(\cos x+2)^2}$.

Thus, $\dfrac{1}{(\cos x+2)^2}$ is an even function.

Let $u = \dfrac{\pi}{2} - x$. Then $du = -\,dx$. 1M

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{dx}{(\cos +2)^2} = 2\int_0^{\frac{\pi}{2}} \frac{dx}{(\cos x+2)^2} \qquad \text{1M}$$

$$= -2\int_{-\frac{\pi}{2}}^{0} \frac{du}{\left[\cos\left(\frac{\pi}{2}-u\right)+2\right]^2}$$

$$= 2\int_0^{\frac{\pi}{2}} \frac{du}{(\sin u+2)^2} \qquad \text{1M}$$

$$= 2\left(\frac{2\sqrt{3}\pi}{27} - \frac{1}{6}\right)$$

$$= \frac{4\sqrt{3}\pi}{27} - \frac{1}{3} \qquad \text{1A}$$

14. (a) Let $u = -x$. Then $du = -\,dx$. 1M

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx$$

$$= -\int_a^0 f(-u)\,du + \int_0^a f(x)\,dx$$

$$= \int_0^a f(-u)\,du + \int_0^a f(x)\,dx \qquad \text{1M}$$

$$= \int_0^a [f(x)+f(-x)]\,dx \qquad \text{1}$$

(b) $\tan\left(\dfrac{\pi}{8}+\dfrac{\pi}{8}\right) = \dfrac{2\tan\frac{\pi}{8}}{1-\tan^2\frac{\pi}{8}}$ 1M

$$1 = \frac{2\tan\frac{\pi}{8}}{1-\tan^2\frac{\pi}{8}}$$

$$1 - \tan^2\frac{\pi}{8} = 2\tan\frac{\pi}{8}$$

$$1 - 2\tan\frac{\pi}{8} = \tan^2\frac{\pi}{8} \qquad \text{1A}$$

(c) $\displaystyle\int_{-1}^{1} \frac{\tan\frac{\pi}{8}}{(1+e^x)\left(x^2+1-2\tan\frac{\pi}{8}\right)}\,\mathrm{d}x$

$\displaystyle = \int_{-1}^{1} \frac{\tan\frac{\pi}{8}}{(1+e^x)\left(x^2+\tan^2\frac{\pi}{8}\right)}\,\mathrm{d}x$ 1M

$\displaystyle = \int_{0}^{1} \left[\frac{\tan\frac{\pi}{8}}{(1+e^x)\left(x^2+\tan^2\frac{\pi}{8}\right)} + \frac{\tan\frac{\pi}{8}}{(1+e^{-x})\left((-x)^2+\tan^2\frac{\pi}{8}\right)}\right]\mathrm{d}x$ 1M+1M

$\displaystyle = \int_{0}^{1} \left[\frac{\tan\frac{\pi}{8}}{(1+e^x)\left(x^2+\tan^2\frac{\pi}{8}\right)} + \frac{e^x\tan\frac{\pi}{8}}{(e^x+1)\left((-x)^2+\tan^2\frac{\pi}{8}\right)}\right]\mathrm{d}x$

$\displaystyle = \int_{0}^{1} \frac{\tan\frac{\pi}{8}}{x^2+\tan^2\frac{\pi}{8}}\,\mathrm{d}x$ 1A

Let $x = \tan\dfrac{\pi}{8}\tan\theta$. Then $\mathrm{d}x = \tan\dfrac{\pi}{8}\sec^2\theta\,\mathrm{d}\theta$. 1M

$\displaystyle\int_{-1}^{1} \frac{\tan\frac{\pi}{8}}{(1+e^x)\left(x^2+1-2\tan\frac{\pi}{8}\right)}\,\mathrm{d}x$

$\displaystyle = \int_{0}^{\frac{3\pi}{8}} \frac{\tan\frac{\pi}{8}\cdot\tan\frac{\pi}{8}\sec^2\theta}{\tan^2\frac{\pi}{8}\tan^2\theta+\tan^2\frac{\pi}{8}}\,\mathrm{d}\theta$ 1A+1A

$\displaystyle = \int_{0}^{\frac{3\pi}{8}} \mathrm{d}\theta$

$\displaystyle = \Big[\theta\Big]_{0}^{\frac{3\pi}{8}}$

$\displaystyle = \frac{3\pi}{8}$ 1A

15. (a) (i) Let $u = \sqrt{3}\tan\theta$. Then $\mathrm{d}u = \sqrt{3}\sec^2\theta\,\mathrm{d}\theta$. 1M

$$\int_0^1 \frac{\mathrm{d}u}{3+u^2} = \int_0^{\frac{\pi}{6}} \frac{\sqrt{3}\sec^2\theta}{3+3\tan^2\theta}\,\mathrm{d}\theta$$

$\displaystyle = \frac{\sqrt{3}}{3}\int_0^{\frac{\pi}{6}}\mathrm{d}\theta$ 1M

$\displaystyle = \frac{\sqrt{3}}{3}\Big[\theta\Big]_0^{\frac{\pi}{6}}$

$\displaystyle = \frac{\sqrt{3}\pi}{18}$ 1A

(ii) $\displaystyle\int_0^1 \frac{u^2}{3+u^2}\,\mathrm{d}u = \int_0^1\left(1-\frac{3}{3+u^2}\right)\mathrm{d}u$

$\displaystyle = \int_0^1\mathrm{d}u - 3\int_0^1\frac{\mathrm{d}u}{3+u^2}$ 1M

$\displaystyle = \Big[u\Big]_0^1 - 3\left(\frac{\sqrt{3}\pi}{18}\right)$

$\displaystyle = 1-\frac{\sqrt{3}\pi}{6}$ 1A

(b) (i) Let $u = a - x$. Then $\mathrm{d}u = -\,\mathrm{d}x$. 1M

$$\int_{\frac{a}{2}}^{a} f(x)\,\mathrm{d}x = -\int_{\frac{a}{2}}^{0} f(a-u)\,\mathrm{d}u$$

$$= \int_{0}^{\frac{a}{2}} f(a-u)\,\mathrm{d}u$$

$$= \int_{0}^{\frac{a}{2}} f(a-x)\,\mathrm{d}x$$ 1

(ii) $$\int_{0}^{a} f(x)\,\mathrm{d}x = \int_{0}^{\frac{a}{2}} f(x)\,\mathrm{d}x + \int_{\frac{a}{2}}^{a} f(x)\,\mathrm{d}x$$

$$= \int_{0}^{\frac{a}{2}} f(x)\,\mathrm{d}x + \int_{0}^{\frac{a}{2}} f(a-x)\,\mathrm{d}x\,e$$

$$= \int_{0}^{\frac{a}{2}} [f(x) + f(a-x)]\,\mathrm{d}x$$ 1

(c) $$\int_{0}^{\pi} \frac{x\sin^3 x}{3+\cos^2 x}\,\mathrm{d}x$$

$$= \int_{0}^{\frac{\pi}{2}} \left[\frac{x\sin^3 x}{3+\cos^2 x} + \frac{(\pi-x)\sin^3(\pi-x)}{3+\cos^2(\pi-x)}\right]\mathrm{d}x$$ 1M

$$= \int_{0}^{\frac{\pi}{2}} \left[\frac{x\sin^3 x}{3+\cos^2 x} + \frac{(\pi-x)\sin^3 x}{3+\cos^2 x}\right]\mathrm{d}x$$

$$= \pi\int_{0}^{\frac{\pi}{2}} \frac{\sin^3 x}{3+\cos^2 x}\,\mathrm{d}x$$

Let $u = \cos x$. Then $\mathrm{d}u = -\sin x\,\mathrm{d}x$. 1M

$$\int_{0}^{\pi} \frac{x\sin^3 x}{3+\cos^2 x}\,\mathrm{d}x$$

$$= \pi\int_{1}^{0} \frac{(1-u^2)(-1)}{3+u^2}\,\mathrm{d}u$$

$$= \pi\left[\int_{0}^{1} \frac{\mathrm{d}u}{3+u^2}\,\mathrm{d}u - \int_{0}^{1} \frac{u^2}{3+u^2}\,\mathrm{d}u\right]$$

$$= \pi\left[\frac{\sqrt{3}\pi}{18} - \left(1 - \frac{\sqrt{3}\pi}{6}\right)\right]$$ 1M

$$= \frac{2\sqrt{3}\pi^2}{9} - \pi$$ 1A

16. (a) (i) Let $u = x - \dfrac{\pi}{2}$. Then $\mathrm{d}u = \mathrm{d}x$.

$$\int_{\frac{\pi}{2}}^{\pi} \frac{\sin^4 x}{\sin^4 x + \cos^4 x}\,\mathrm{d}x = \int_{0}^{\frac{\pi}{2}} \frac{\sin^4\left(u+\frac{\pi}{2}\right)}{\sin^4\left(u+\frac{\pi}{2}\right) + \cos^4\left(u+\frac{\pi}{2}\right)}\,\mathrm{d}u$$ 1M

$$= \int_{0}^{\frac{\pi}{2}} \frac{\cos^4 x}{\sin^4 x + \cos^4 x}\,\mathrm{d}x$$ 1

(ii) $\displaystyle\int_0^{\pi} \frac{\sin^4 x}{\sin^4 x+\cos^4 x}\,\mathrm{d}x$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin^4 x}{\sin^4 x+\cos^4 x}\,\mathrm{d}x + \int_{\frac{\pi}{2}}^{\pi} \frac{\sin^4 x}{\sin^4 x+\cos^4 x}\,\mathrm{d}x \qquad \text{1M}$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin^4 x}{\sin^4 x+\cos^4 x}\,\mathrm{d}x + \int_0^{\frac{\pi}{2}} \frac{\cos^4 x}{\sin^4 x+\cos^4 x}\,\mathrm{d}x \qquad \text{1M}$$

$$= \int_0^{\frac{\pi}{2}} \mathrm{d}x \qquad \text{1M}$$

$$= \Big[x\Big]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2} \qquad \text{1A}$$

(b) Let $u = \lambda - x$. Then $\mathrm{d}u = -\,\mathrm{d}x$. 1M

$$\int_0^{\lambda} xf(x)\,\mathrm{d}x = -\int_{\lambda}^{0} (\lambda-u)f(\lambda-u)\,\mathrm{d}u$$

$$= \int_0^{\lambda} (\lambda-u)f(u)\,\mathrm{d}u \qquad \text{1M}$$

$$= \lambda\int_0^{\lambda} f(x)\,\mathrm{d}x - \int_0^{\lambda} xf(x)\,\mathrm{d}x$$

$$= \frac{\lambda}{2}\int_0^{\lambda} f(x)\,\mathrm{d}x \qquad \text{1}$$

(c) Put $f(x) = \dfrac{4\sin^2 x-\sin^2 2x}{\sin^4 x+\cos^4 x}$.

We have $f(\pi-x) = \dfrac{4\sin^2(\pi-x)-\sin^2(2\pi-2x)}{\sin^4(\pi-x)+\cos^4(\pi-x)} = \dfrac{4\sin^2 x-\sin^2 2x}{\sin^4 x+\cos^4 x} = f(x)$.

$$\int_0^{\pi} \frac{4x\sin^2 x - x\sin^2 2x}{\sin^4 x+\cos^4 x}\,\mathrm{d}x = \int_0^{\pi} x\left(\frac{4\sin^2 x-\sin^2 2x}{\sin^4 x+\cos^4 x}\right)\mathrm{d}x$$

$$= \frac{\pi}{2}\int_0^{\pi} \frac{4\sin^2 x-\sin^2 2x}{\sin^4 x+\cos^4 x}\,\mathrm{d}x \qquad \text{1M}$$

$$= \frac{\pi}{2}\int_0^{\pi} \frac{4\sin^2 x-(2\sin x\cos x)^2}{\sin^4 x+\cos^4 x}\,\mathrm{d}x \qquad \text{1M}$$

$$= \frac{\pi}{2}\int_0^{\pi} \frac{4\sin^2 x(1-\cos^2 x)}{\sin^4 x+\cos^4 x}\,\mathrm{d}x$$

$$= 2\pi\int_0^{\pi} \frac{\sin^4 x}{\sin^4 x+\cos^4 x}\,\mathrm{d}x$$

$$= 2\pi\left(\frac{\pi}{2}\right) \qquad \text{1M}$$

$$= \pi^2 \qquad \text{1A}$$

17. (a) Let $t = \pi - x$. Then $\mathrm{d}t = -\,\mathrm{d}x$. 1M

$$\int_0^{\pi} x f(\sin x)\,\mathrm{d}x = -\int_{\pi}^{0} (\pi - t) f[\sin(\pi - t)]\,\mathrm{d}t$$

$$= \int_0^{\pi} (\pi - t) f(\sin t)\,\mathrm{d}t \quad \text{1M}$$

$$= \pi \int_0^{\pi} f(\sin x)\,\mathrm{d}x - \int_0^{\pi} x f(\sin x)\,\mathrm{d}x$$

$$= \frac{\pi}{2} \int_0^{\pi} f(\sin x)\,\mathrm{d}x \quad \text{1}$$

(b) Let $u = \pi - x$. Then $\mathrm{d}u = -\,\mathrm{d}x$. 1M

$$\int_0^{\pi} x f(\sin x)\,\mathrm{d}x = \frac{\pi}{2} \int_0^{\pi} f(\sin x)\,\mathrm{d}x$$

$$= \frac{\pi}{2} \left[\int_0^{\frac{\pi}{2}} f(\sin x)\,\mathrm{d}x + \int_{\frac{\pi}{2}}^{\pi} f(\sin x)\,\mathrm{d}x \right] \quad \text{1M}$$

$$= \frac{\pi}{2} \int_0^{\frac{\pi}{2}} f(\sin x)\,\mathrm{d}x - \frac{\pi}{2} \int_{\frac{\pi}{2}}^{0} f[\sin(\pi - u)]\,\mathrm{d}u$$

$$= \frac{\pi}{2} \int_0^{\frac{\pi}{2}} f(\sin x)\,\mathrm{d}x + \frac{\pi}{2} \int_0^{\frac{\pi}{2}} f(\sin x)\,\mathrm{d}x \quad \text{1M}$$

$$= \pi \int_0^{\frac{\pi}{2}} f(\sin x)\,\mathrm{d}x \quad \text{1}$$

(c) Note that $\dfrac{\sin x}{1 + \cos^2 x} = \dfrac{\sin x}{1 + (1 - \sin^2 x)} = \dfrac{\sin x}{2 - \sin^2 x}$.

Put $f(\sin x) = \dfrac{\sin x}{2 - \sin^2 x}$.

$$\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x}\,\mathrm{d}x = \pi \int_0^{\frac{\pi}{2}} \frac{\sin x}{1 + \cos^2 x}\,\mathrm{d}x \quad \text{1M}$$

Let $u = \cos x$. Then $\mathrm{d}u = -\sin x\,\mathrm{d}x$. 1M

$$\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x}\,\mathrm{d}x = -\pi \int_1^{0} \frac{\mathrm{d}u}{1 + u^2}$$

$$= \pi \int_0^{1} \frac{\mathrm{d}u}{1 + u^2}$$

Let $u = \tan\theta$. Then $\mathrm{d}u = \sec^2\theta\,\mathrm{d}\theta$. 1M

$$\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x}\,\mathrm{d}x = \pi \int_0^{\frac{\pi}{4}} \frac{\sec^2\theta}{1 + \tan^2\theta}\,\mathrm{d}\theta$$

$$= \pi \int_0^{\frac{\pi}{4}} \mathrm{d}\theta$$

$$= \pi \Big[\theta \Big]_0^{\frac{\pi}{4}}$$

$$= \frac{\pi^2}{4} \quad \text{1A}$$

(d) Put $f(x) = \dfrac{x}{2 - x^2}$. 1M

Since $f(-x) = \dfrac{-x}{2-(-x)^2} = \dfrac{-x}{2-x^2} = -f(x)$, $f(x)$ is an odd function.

Note that $(-x)f(\sin(-x)) = -x(-f(\sin x)) = xf(\sin x) = \dfrac{x\sin x}{1+\cos^2 x}$.

$$\begin{aligned}\int_{-\pi}^{\pi} xf(\sin x)\,\mathrm{d}x &= 2\int_0^{\pi} xf(\sin x)\,\mathrm{d}x \\ &= 2\left(\frac{\pi^2}{4}\right) \\ &= \frac{\pi^2}{2} \\ &\neq 0\end{aligned}$$

The claim is disagreed. 1A

18. (a) Let $u = \pi - x$. Then $\mathrm{d}u = -\,\mathrm{d}x$. 1M

$$\begin{aligned}\int_0^{\pi} xf(\sin x)\,\mathrm{d}x &= -\int_{\pi}^{0} (\pi-u)f(\sin(\pi-u))\,\mathrm{d}u \\ &= \int_0^{\pi}(\pi-u)f(\sin u)\,\mathrm{d}u \qquad \text{1M} \\ &= \pi\int_0^{\pi} f(\sin u)\,\mathrm{d}u - \int_0^{\pi} uf(\sin u)\,\mathrm{d}u \\ &= \pi\int_0^{\pi} f(\sin x)\,\mathrm{d}x - \int_0^{\pi} xf(\sin x)\,\mathrm{d}x \\ 2\int_0^{\pi} xf(\sin x)\,\mathrm{d}x &= \pi\int_0^{\pi} f(\sin x)\,\mathrm{d}x \\ \int_0^{\pi} xf(\sin x)\,\mathrm{d}x &= \frac{\pi}{2}\int_0^{\pi} f(\sin x)\,\mathrm{d}x \qquad \text{1}\end{aligned}$$

(b) (i) Let $v = -x$. Then $\mathrm{d}v = -\,\mathrm{d}x$. 1M

$$\begin{aligned}\int_{-\pi}^{0} \frac{g(x)}{1+e^x}\,\mathrm{d}x &= -\int_{\pi}^{0} \frac{g(-v)}{1+e^{-v}}\,\mathrm{d}v \\ &= \int_0^{\pi} \frac{e^v g(v)}{e^v(1+e^{-v})}\,\mathrm{d}v \\ &= \int_0^{\pi} \frac{e^v g(v)}{1+e^v}\,\mathrm{d}v \\ &= \int_0^{\pi} \frac{e^x g(x)}{1+e^x}\,\mathrm{d}x \qquad \text{1}\end{aligned}$$

(ii)
$$\begin{aligned}\int_{-\pi}^{\pi} \frac{g(x)}{1+e^x}\,\mathrm{d}x &= \int_{-\pi}^{0} \frac{g(x)}{1+e^x}\,\mathrm{d}x + \int_0^{\pi} \frac{g(x)}{1+e^x}\,\mathrm{d}x \\ &= \int_0^{\pi} \frac{e^x g(x)}{1+e^x}\,\mathrm{d}x + \int_0^{\pi} \frac{g(x)}{1+e^x}\,\mathrm{d}x \\ &= \int_0^{\pi} \frac{(1+e^x)g(x)}{1+e^x}\,\mathrm{d}x \\ &= \int_0^{\pi} g(x)\,\mathrm{d}x \qquad \text{1}\end{aligned}$$

(c) Put $g(x)=\dfrac{\sqrt{2+\cos x}}{\sqrt{2+\cos x}+\sqrt{2-\cos x}}$.

$$g(-x)=\frac{\sqrt{2+\cos(-x)}}{\sqrt{2+\cos(-x)}+\sqrt{2-\cos(-x)}}=\frac{\sqrt{2+\cos x}}{\sqrt{2+\cos x}+\sqrt{2-\cos x}}=g(x)$$ 1M

$g(x)$ is a continuous even function on $[-\pi,\ \pi]$.

$$\int_{-\pi}^{\pi}\frac{\sqrt{2+\cos x}}{(1+e^x)(\sqrt{2+\cos x}+\sqrt{2-\cos x})}\,\mathrm{d}x$$

$$=\int_0^{\pi} g(x)\,\mathrm{d}x$$

$$=\Big[xg(x)\Big]_0^{\pi}-\int_0^{\pi} xg'(x)\,\mathrm{d}x$$ 1M

$$g'(x)=\frac{\frac{-\sin x}{2\sqrt{2+\cos x}}(\sqrt{2+\cos x}+\sqrt{2-\cos x})-(\sqrt{2+\cos x})\left(\frac{-\sin x}{2\sqrt{2+\cos x}}+\frac{\sin x}{2\sqrt{2-\cos x}}\right)}{(\sqrt{2+\cos x}+\sqrt{2-\cos x})^2}$$ 1M

$$=\frac{-\sin x}{2\sqrt{3+\sin^2 x}+(3+\sin^2 x)}$$ 1A

Take $f(x)=\dfrac{-x}{2\sqrt{3+x^2}+3+x^2}$ such that it is continuous on $[0,\ 1]$ and $g'(x)=f(\sin x)$.

$$\int_{-\pi}^{\pi}\frac{\sqrt{2+\cos x}}{(1+e^x)(\sqrt{2+\cos x}+\sqrt{2-\cos x})}\,\mathrm{d}x$$

$$=\Big[xg(x)\Big]_0^{\pi}-\int_0^{\pi} xf(\sin x)\,\mathrm{d}x$$

$$=\pi g(\pi)-\frac{\pi}{2}\int_0^{\pi} f(\sin x)\,\mathrm{d}x$$ 1M

$$=\pi g(\pi)-\frac{\pi}{2}\int_0^{\pi} g'(x)\,\mathrm{d}x$$

$$=\pi g(\pi)-\frac{\pi}{2}\Big[g(x)\Big]_0^{\pi}$$

$$=\frac{\pi}{2}\left(\frac{\sqrt{2-1}}{\sqrt{2-1}+\sqrt{2-1}}+\frac{\sqrt{2+1}}{\sqrt{2+1}+\sqrt{2-1}}\right)$$

$$=\frac{\pi}{2}$$ 1A

19. (a) Let $u=k-x$. Then $\mathrm{d}u=-\,\mathrm{d}x$. 1M

$$\int_0^k xf(x)\,\mathrm{d}x=-\int_k^0 (k-u)f(\pi-u)\,\mathrm{d}u$$

$$=\int_0^k (k-u)f(u)\,\mathrm{d}u$$

$$=\int_0^k (k-x)f(x)\,\mathrm{d}x$$ 1

Put $k = \pi$.

$$\int_0^{\pi} x f(x)\,\mathrm{d}x = \int_0^{\pi} (\pi - x) f(x)\,\mathrm{d}x$$

$$= \frac{1}{2}\left[\int_0^{\pi} (\pi - x) f(x)\,\mathrm{d}x + \int_0^{\pi} x f(x)\,\mathrm{d}x\right] \qquad \text{1M}$$

$$= \frac{\pi}{2}\int_0^{\pi} f(x)\,\mathrm{d}x \qquad 1$$

(b) Let $u = \cos x$. Then $\mathrm{d}u = -\sin x\,\mathrm{d}x$. 1M

$$\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\,\mathrm{d}x = -\int_1^{-1} \frac{\mathrm{d}u}{1+u^2} \qquad \text{1A}$$

$$= \int_{-1}^{1} \frac{\mathrm{d}u}{1+u^2}$$

Let $u = \tan\theta$. Then $\mathrm{d}u = \sec^2\theta\,\mathrm{d}\theta$. 1M

$$\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\,\mathrm{d}x = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sec^2\theta}{1+\tan^2\theta}\,\mathrm{d}\theta \qquad \text{1A}$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \mathrm{d}\theta$$

$$= \Big[\theta\Big]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$= \frac{\pi}{2} \qquad \text{1A}$$

(c) $\dfrac{\cos\left(\frac{\pi}{6} - x\right) - \cos\left(\frac{\pi}{6} + x\right)}{3 + \cos 2x}$

$$= \frac{-2\sin\frac{\pi}{6}\sin(-x)}{3 + (2\cos^2 x - 1)} \qquad \text{1M}$$

$$= \frac{1}{2}\cdot\frac{\sin x}{1+\cos^2 x}$$

Put $f(x) = \dfrac{\sin x}{1+\cos^2 x}$. 1M

Then $f(\pi - x)\dfrac{\sin(\pi - x)}{1+\cos^2(\pi - x)} = \dfrac{\sin x}{1+\cos^2 x} = f(x)$.

$$\int_0^{\pi} \frac{x\left[\cos\left(\frac{\pi}{6} - x\right) - \cos\left(\frac{\pi}{6} + x\right)\right]}{3 + \cos 2x}\,\mathrm{d}x$$

$$= \frac{1}{2}\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,\mathrm{d}x$$

$$= \frac{1}{2}\cdot\frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\,\mathrm{d}x$$

$$= \frac{\pi}{4}\cdot\frac{\pi}{2}$$

$$= \frac{\pi^2}{8} \qquad \text{1A}$$

20. (a) Let $u = \pi - x$. Then $\mathrm{d}u = -\,\mathrm{d}x$. 1M

$$\int_0^{\pi} f(\pi - x)\,\mathrm{d}x = -\int_{\pi}^{0} f(u)\,\mathrm{d}u$$

$$= \int_0^{\pi} f(u)\,\mathrm{d}u \qquad \text{1M}$$

$$= \int_0^{\pi} f(x)\,\mathrm{d}x \qquad \text{1}$$

(b) $$\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,\mathrm{d}x = \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1+\cos^2(\pi - x)}\,\mathrm{d}x \qquad \text{1M}$$

$$= \int_0^{\pi} \frac{(\pi - x)\sin x}{1+\cos^2 x}\,\mathrm{d}x$$

$$= \frac{1}{2}\left[\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,\mathrm{d}x + \int_0^{\pi} \frac{(\pi - x)\sin x}{1+\cos^2 x}\,\mathrm{d}x\right] \qquad \text{1M}$$

$$= \frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\,\mathrm{d}x \qquad \text{1A}$$

(c) (i) Let $x = \tan\theta$. Then $\mathrm{d}x = \sec^2\theta\,\mathrm{d}\theta$. 1M

$$\int \frac{1}{1+x^2}\,\mathrm{d}x = \int \frac{\sec^2\theta}{1+\tan^2\theta}\,\mathrm{d}\theta$$

$$= \int \mathrm{d}\theta$$

$$= \theta + \text{constant}$$

$$= \tan^{-1} x + \text{constant} \qquad \text{1A}$$

(ii) $$\int_0^{\pi} \frac{(x+\cos^5 x)\sin x}{1+\cos^2 x}\,\mathrm{d}x$$

$$= \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,\mathrm{d}x + \int_0^{\pi} \frac{\cos^5 x}{1+\cos^2 x}\,\mathrm{d}x$$

$$= \frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\,\mathrm{d}x + \int_0^{\pi} \frac{\cos^5 x\sin x}{1+\cos^2 x}\,\mathrm{d}x \qquad \text{1M}$$

Since $\dfrac{\cos^5(-x)\sin(-x)}{1+\cos^2(-x)} = \dfrac{-\cos^5 x\sin x}{1+\cos^2 x}$, it is an odd function.

Let $u = \cos x$. Then $\mathrm{d}u = -\sin x\,\mathrm{d}x$. 1M

$$\int_0^{\pi} \frac{(x+\cos^5 x)\sin x}{1+\cos^2 x}\,\mathrm{d}x$$

$$= -\frac{\pi}{2}\int_1^{-1} \frac{\mathrm{d}u}{1+u^2} + 0 \qquad \text{1M}$$

$$= \frac{\pi}{2}\int_{-1}^{1} \frac{\mathrm{d}u}{1+u^2}$$

$$= \frac{\pi}{2}\Big[\tan^{-1} x\Big]_{-1}^{1} \qquad \text{1M}$$

$$= \frac{\pi^2}{4} \qquad \text{1A}$$

21. (a) $\int_0^{\pi} \cos^4 x \, \mathrm{d}x$

$= \int_0^{\pi} \left(\frac{1+\cos 2x}{2}\right)^2 \mathrm{d}x$ 1M

$= \int_0^{\pi} \left(\frac{1}{4} + \frac{\cos 2x}{2} + \frac{\cos^2 2x}{8}\right) \mathrm{d}x$

$= \int_0^{\pi} \left(\frac{1}{4} + \frac{\cos 2x}{2} + \frac{1+\cos 4x}{16}\right) \mathrm{d}x$ 1M

$= \left[\frac{3x}{8} + \frac{\sin 2x}{4} + \frac{\sin 4x}{32}\right]_0^{\pi}$

$= \frac{3\pi}{8}$ 1A

(b) Let $u = k - x$. Then $\mathrm{d}u = -\,\mathrm{d}x$. 1M

$$\int_0^k f(x)g(x)\,\mathrm{d}x = -\int_k^0 f(k-u)g(k-u)\,\mathrm{d}u$$

$$= \int_0^k f(u)g(k-u)\,\mathrm{d}u$$

$$= \int_0^k f(x)g(k-x)\,\mathrm{d}x$$

$$= \frac{1}{2}\left[\int_0^k f(x)g(x)\,\mathrm{d}x + \int_0^k f(x)g(k-x)\,\mathrm{d}x\right]$$ 1M

$$= \frac{1}{2}\int_0^k f(x)[g(x)+g(k-x)]\,\mathrm{d}x$$

$$= \frac{a}{2}\int_0^k f(x)\,\mathrm{d}x$$ 1

(c) (i) Put $f(x) = \cos^4 x$ and $g(x) = \frac{1}{1+e^{\sin x}}$. 1M

$f(2\pi - x) = \cos^4(2\pi - x) = \cos^4 x = f(x)$

$$g(x) + g(2\pi - x) = \frac{1}{1+e^{\sin x}} + \frac{1}{1+e^{\sin(2\pi - x)}}$$

$$= \frac{1}{1+e^{\sin x}} + \frac{1}{1+e^{-\sin x}} \times \frac{e^{\sin x}}{e^{\sin x}}$$

$$= \frac{1}{1+e^{\sin x}} + \frac{e^{\sin x}}{1+e^{\sin x}}$$

$$= 1$$ 1M

Let $u = 2\pi - x$. Then $du = -\,dx$.

$$\int_0^{2\pi} \frac{\cos^4 x}{1+e^{\sin x}}\,dx = \frac{1}{2}\int_0^{2\pi} \cos^4 x\,dx \qquad \text{1M}$$

$$= \frac{1}{2}\left(\int_0^{\pi} \cos^4 x\,dx + \int_{\pi}^{2\pi} \cos^4 x\,dx\right)$$

$$= \frac{1}{2}\int_0^{\pi} \cos^4 x\,dx - \frac{1}{2}\int_{\pi}^{0} \cos^4(2\pi - u)\,du$$

$$= \frac{1}{2}\int_0^{\pi} \cos^4 x\,dx + \frac{1}{2}\int_0^{\pi} \cos^4 u\,du$$

$$= \frac{1}{2}\left(\frac{3\pi}{8}\right) + \frac{1}{2}\left(\frac{3\pi}{8}\right)$$

$$= \frac{3\pi}{8} \qquad \text{1A}$$

(ii) $$\int_0^{2\pi} \frac{e^{\sin x}\cos^4 x}{1+e^{\sin x}}\,dx = \int_0^{2\pi}\left[\frac{(1+e^{\sin x})\cos^4 x}{1+e^{\sin x}} - \frac{\cos^4 x}{1+e^{\sin x}}\right]dx$$

$$= \int_0^{2\pi} \cos^4 x\,dx - \int_0^{2\pi} \frac{\cos^4 x}{1+e^{\sin x}}\,dx \qquad \text{1M}$$

$$= 2\left(\frac{3\pi}{8}\right) - \frac{3\pi}{8}$$

$$= \frac{3\pi}{8} \qquad \text{1A}$$

22. (a) $$\frac{2}{3\cos 2x + 5} = \frac{2}{3(\cos^2 x - 1) + 5}$$

$$= \frac{2}{6\cos^2 x + 2}$$

$$= \frac{\sec^2 x}{3 + \sec^2 x} \qquad \text{1}$$

(b) Let $u = \tan x$. Then $du = \sec^2 x\,dx$. 1M

$$\int_0^{\frac{\pi}{4}} \frac{2}{3\cos 2x + 5}\,dx = \int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{3 + \sec^2 x}\,dx$$

$$= \int_0^{1} \frac{1}{4+u^2}\,du$$

Let $u = 2\tan\theta$. Then $du = 2\sec^2\theta\,d\theta$. 1M

$$\int_0^{\frac{\pi}{4}} \frac{2}{3\cos 2x + 5}\,dx = \int_0^{\tan^{-1}\frac{1}{2}} \frac{2\sec^2\theta}{4 + 4\tan^2\theta}\,d\theta$$

$$= \frac{1}{2}\Big[\theta\Big]_0^{\tan^{-1}\frac{1}{2}}$$

$$= \frac{1}{2}\tan^{-1}\frac{1}{2} \qquad \text{1A}$$

(c) Let $u = -x$. Then $\mathrm{d}u = -\,\mathrm{d}x$. 1M

$$\int_{-\alpha}^{\alpha} f(x)\ln(e^x+1)\,\mathrm{d}x = \int_{-\alpha}^{0} f(x)\ln(e^x+1)\,\mathrm{d}x + \int_{0}^{\alpha} f(x)\ln(e^x+1)\,\mathrm{d}x \quad \text{1M}$$

$$= -\int_{\alpha}^{0} f(-u)\ln(e^{-u}+1)\,\mathrm{d}u + \int_{0}^{\alpha} f(x)\ln(e^x+1)\,\mathrm{d}x$$

$$= -\int_{0}^{\alpha} f(x)\ln\left(\frac{1+e^x}{e^x}\right)\mathrm{d}x + \int_{0}^{\alpha} f(x)\ln(e^x+1)\,\mathrm{d}x$$

$$= -\int_{0}^{\alpha} f(x)\ln(1+e^x)\,\mathrm{d}x + \int_{0}^{\alpha} f(x)\ln e^x\,\mathrm{d}x + \int_{0}^{\alpha} f(x)\ln(e^x+1)\,\mathrm{d}x$$

$$= \int_{0}^{\alpha} x f(x)\,\mathrm{d}x \quad \text{1}$$

(d) Put $f(x) = \dfrac{6\sin 2x}{(3\cos 2x+5)^2}$. 1M

Then $f(-x) = \dfrac{6\sin(-2x)}{(3\cos(-2x)+5)^2} = \dfrac{-6\sin 2x}{(3\cos 2x+5)^2} = -f(x)$.

Note that $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{2}{3\cos 2x+5}\right) = \dfrac{6\sin 2x}{(3\cos 2x+5)^2}$.

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{6\sin 2x}{(3\cos 2x+5)^2}\ln(e^x+1)\,\mathrm{d}x$$

$$= \int_{0}^{\frac{\pi}{4}} \frac{6x\sin 2x}{(3\cos 2x+5)^2}\,\mathrm{d}x \quad \text{1M}$$

$$= \left[\frac{x}{3\cos 2x+5}\right]_0^{\frac{\pi}{4}} - \int_{0}^{\frac{\pi}{4}} \frac{1}{3\cos 2x+5}\,\mathrm{d}x \quad \text{1M}$$

$$= \frac{\frac{\pi}{4}}{3(0)+5} - \frac{1}{2}\left(\frac{1}{2}\tan^{-1}\frac{1}{2}\right) \quad \text{1M}$$

$$= \frac{\pi}{20} - \frac{1}{4}\tan^{-1}\frac{1}{2} \quad \text{1A}$$

23. (a) (i) $$\frac{1}{1+2\sin^2 x} = \frac{1}{1+2(1-\cos^2 x)} \div \frac{\cos^2 x}{\cos^2 x}$$

$$= \frac{\sec^2 x}{3\sec^2 x - 2} \quad \text{1}$$

(ii) Let $u = \tan x$. Then $\mathrm{d}u = \sec^2 x\,\mathrm{d}x$. 1M

$$\int_{0}^{\frac{\pi}{4}} \frac{1}{1+2\sin^2 x}\,\mathrm{d}x = \int_{0}^{\frac{\pi}{4}} \frac{\sec^2 x}{3\sec^2 x - 2}\,\mathrm{d}x$$

$$= \int_{0}^{1} \frac{\mathrm{d}u}{3(u^2+1)-2}$$

$$= \int_{0}^{1} \frac{\mathrm{d}u}{1+3u^2}$$

Let $\sqrt{3}u = \tan\theta$. Then $\sqrt{3}\,\mathrm{d}u = \sec^2\theta\,\mathrm{d}\theta$. 1M

$$\begin{aligned}\int_0^{\frac{\pi}{4}} \frac{1}{1+2\sin^2 x}\,\mathrm{d}x &= \frac{\sqrt{3}}{3}\int_0^{\frac{\pi}{3}} \frac{\sec^2\theta}{1+\tan^2\theta}\,\mathrm{d}\theta \\ &= \frac{\sqrt{3}}{3}\Big[\theta\Big]_0^{\frac{\pi}{3}} \\ &= \frac{\sqrt{3}\pi}{9}\end{aligned}$$ 1A

(b) Let $u = -x$. Then $\mathrm{d}u = -\,\mathrm{d}x$. 1M

$$\begin{aligned}\int_{-m}^{m} g(x)h(x)\,\mathrm{d}x &= \int_{-m}^{0} g(x)h(x)\,\mathrm{d}x + \int_0^m g(x)h(x)\,\mathrm{d}x \quad \text{1M}\\ &= -\int_m^0 g(-u)h(-u)\,\mathrm{d}u + \int_0^m g(x)h(x)\,\mathrm{d}x \\ &= -\int_0^m g(-u)h(u)\,\mathrm{d}u + \int_0^m g(x)h(x)\,\mathrm{d}x \\ &= -\int_0^m g(-x)h(x)\,\mathrm{d}x + \int_0^m g(x)h(x)\,\mathrm{d}x \\ &= \int_0^m [g(x)-g(-x)]h(x)\,\mathrm{d}x \quad \text{1M}\\ &= \int_0^m xh(x)\,\mathrm{d}x \quad \text{1}\end{aligned}$$

(c) Put $g(x) = \ln(e^x+1)$ and $h(x) = \dfrac{\sin 2x}{(1+2\sin^2 x)^2}$. 1M

$h(-x) = \dfrac{\sin(-2x)}{(1+2\sin^2(-2x))^2} = \dfrac{-\sin 2x}{(1+2\sin^2 2x)^2} = -h(x)$.

$$\begin{aligned}g(x)-g(-x) &= \ln(e^x+1) - \ln(e^{-x}+1) \\ &= \ln(e^x+1) - \ln\left(\frac{1+e^x}{e^x}\right) \\ &= \ln e^x \\ &= x\end{aligned}$$

$$\begin{aligned}&\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\ln(e^x+1)\sin 2x}{(1+2\sin^2 x)^2}\,\mathrm{d}x \\ &= \int_0^{\frac{\pi}{4}} \frac{x\sin 2x}{(1+2\sin^2 x)^2}\,\mathrm{d}x \quad \text{1M}\\ &= \left[\frac{-x}{2(1+\sin^2 x)}\right]_0^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}} \frac{1}{2(1+2\sin^2 x)}\,\mathrm{d}x \quad \text{1M}\\ &= \frac{-\frac{\pi}{4}}{2\left(1+2\sin^2\frac{\pi}{4}\right)} + \frac{1}{2}\left(\frac{\sqrt{3}\pi}{9}\right) \quad \text{1M}\\ &= \frac{(8\sqrt{3}-9)\pi}{144} \quad \text{1A}\end{aligned}$$

24. (a) Let $x = \sin\theta$. Then $\mathrm{d}x = \cos\theta\,\mathrm{d}\theta$. 1M

$$\int_0^1 x^2\sqrt{1-x^2}\,\mathrm{d}x = \int_0^{\frac{\pi}{2}} \sin^2\theta\sqrt{1-\sin^2 x}\cos\theta\,\mathrm{d}\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin^2\theta\cos^2\theta\,\mathrm{d}\theta$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{2}} \sin^2 2\theta\,\mathrm{d}\theta \qquad \text{1M}$$

$$= \frac{1}{8}\int_0^{\frac{\pi}{2}} (1-\cos 4\theta)\,\mathrm{d}\theta \qquad \text{1M}$$

$$= \frac{1}{8}\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{16} \qquad \text{1A}$$

(b) (i) Let $u = k - x$. Then $\mathrm{d}u = -\,\mathrm{d}x$. 1M

$$\int_0^k f(x)g(x)\,\mathrm{d}x = -\int_k^0 f(k-u)g(k-u)\,\mathrm{d}u$$

$$= \int_0^k f(u)g(k-u)\,\mathrm{d}u$$

$$= \int_0^k f(x)g(k-x)\,\mathrm{d}x \qquad \text{1}$$

(ii) $$\int_0^k f(x)g(x)\,\mathrm{d}x = \frac{1}{2}\left[\int_0^k f(x)g(x)\,\mathrm{d}x + \int_0^k f(x)g(k-x)\,\mathrm{d}x\right] \qquad \text{1M}$$

$$= \frac{1}{2}\int_0^k f(x)[g(x)+g(k-x)]\,\mathrm{d}x$$

$$= \frac{a}{2}\int_0^k f(x)\,\mathrm{d}x \qquad \text{1}$$

(c) Put $f(x) = \sin^2 x\cos^2 x$ and $g(x) = \dfrac{1}{1+e^{\sin 8x}}$. 1M

$$f\left(\frac{\pi}{2}-x\right) = \sin^2\left(\frac{\pi}{2}-x\right)\cos^2\left(\frac{\pi}{2}-x\right) = \cos^2 x\sin^2 x = f(x)$$

$$g(x) + g\left(\frac{\pi}{2}-x\right) = \frac{1}{1+e^{\sin 8x}} + \frac{1}{1+e^{\sin(4\pi-8x)}}$$

$$= \frac{1}{1+e^{\sin x}} + \frac{1}{1+e^{-\sin 8x}}\cdot\frac{e^{\sin 8x}}{e^{\sin 8x}}$$

$$= 1$$

$$\int_0^{\frac{\pi}{2}} \frac{\sin^2 x\cos^2 x}{1+e^{\sin 8x}}\,\mathrm{d}x = \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin^2 x\cos^2 x\,\mathrm{d}x \qquad \text{1M}$$

$$= \frac{1}{2}\left(\frac{\pi}{16}\right) \qquad \text{1M}$$

$$= \frac{\pi}{32} \qquad \text{1A}$$

25. (a) Let $u = a - x$. Then $\mathrm{d}u = -\,\mathrm{d}x$. 1M

When $x = 0$, $u = a$; when $x = a$, $u = 0$.

$$\begin{aligned}\int_0^a f(x)\,\mathrm{d}x &= -\int_a^0 f(a-u)\,\mathrm{d}u \\ &= \int_0^a f(a-u)\,\mathrm{d}u && \text{1M} \\ &= \int_0^a f(a-x)\,\mathrm{d}x && \text{1}\end{aligned}$$

(b) $\ln(1+\tan x)$ is a continuous function on $\left[0,\ \frac{\pi}{4}\right]$. By (a),

$$\begin{aligned}\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,\mathrm{d}x &= \int_0^{\frac{\pi}{4}} \ln\left[1+\tan\left(\frac{\pi}{4}-x\right)\right]\mathrm{d}x && \text{1M} \\ &= \int_0^{\frac{\pi}{4}} \ln\left(1+\frac{1-\tan x}{1+\tan x}\right)\mathrm{d}x && \text{1M} \\ &= \int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan x}\right)\mathrm{d}x && \text{1}\end{aligned}$$

(c) By (b),

$$\begin{aligned}\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,\mathrm{d}x &= \int_0^{\frac{\pi}{4}} \ln 2\,\mathrm{d}x - \int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,\mathrm{d}x && \text{1M} \\ 2\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,\mathrm{d}x &= \int_0^{\frac{\pi}{4}} \ln 2\,\mathrm{d}x && \text{1M} \\ \int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,\mathrm{d}x &= \frac{\ln 2}{2}\Big[x\Big]_0^{\frac{\pi}{4}} \\ &= \frac{\pi \ln 2}{8} && \text{1}\end{aligned}$$

(d)

$$\begin{aligned}\int_0^{\frac{\pi}{4}} \frac{x\sec^2 x}{1+\tan x}\,\mathrm{d}x &= \Big[x\ln(1+\tan x)\Big]_0^{\frac{\pi}{4}} - \int_0^{4} \ln(1+\tan x)\,\mathrm{d}x && \text{1M} \\ &= \left(\frac{\pi\ln 2}{4}-0\right) - \frac{\pi\ln 2}{8} && \text{1M} \\ &= \frac{\pi\ln 2}{8} && \text{1A}\end{aligned}$$

26. (a) $x^2 + 2x + 3 = (x+1)^2 + 2$.

Let $x + 1 = \sqrt{2}\tan\theta$. $\mathrm{d}x = \sqrt{2}\sec^2\theta\,\mathrm{d}\theta$.

When $x = 0$, $\theta = \tan^{-1} \dfrac{\sqrt{2}}{2}$; when $x = 1$, $\theta = \tan^{-1} \sqrt{2}$.

$$\int_0^1 \frac{1}{x^2 + 2x + 3}\, dx = \int_0^1 \frac{1}{(x+1)^2 + 2}\, dx \qquad \text{1M}$$

$$= \int_{\tan^{-1} \frac{\sqrt{2}}{2}}^{\tan^{-1} \sqrt{2}} \frac{\sqrt{2}\sec^2\theta}{2\sec^2\theta}\, d\theta$$

$$= \frac{\sqrt{2}}{2}\Big[\theta\Big]_{\tan^{-1} \frac{\sqrt{2}}{2}}^{\tan^{-1} \sqrt{2}} \qquad \text{1M}$$

$$= \frac{\sqrt{2}}{2}\left[\tan^{-1}\sqrt{2} - \tan^{-1}\frac{\sqrt{2}}{2}\right]$$

$$= \frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{\sqrt{2}}{4}\right) \qquad \text{1A}$$

(b) (i) $$\frac{2\tan\theta}{1+\tan^2\theta} = \frac{2\tan\theta\cos^2\theta}{\sec^2\theta\cos^2\theta}$$

$$= \frac{2\sin\theta\cos\theta}{1}$$

$$= \sin 2\theta \qquad \text{1}$$

$$\frac{1-\tan^2\theta}{1+\tan^2\theta} = \frac{(1-\tan^2\theta)\cos^2\theta}{\sec^2\theta\cos^2\theta}$$

$$= \cos^2\theta - \sin^2\theta$$

$$= \cos 2\theta \qquad \text{1}$$

(ii) Let $t = \tan\theta$. Then $dt = \sec^2\theta\, d\theta = (1+\tan^2\theta)\, d\theta$ 1M

$$\int_0^{\frac{\pi}{4}} \frac{1}{\sin 2\theta + \cos 2\theta + 2}\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{1+\tan^2\theta}{2\tan\theta + (1-\tan^2\theta) + 2(1+\tan^2\theta)}\, d\theta \qquad \text{1M}$$

$$= \int_0^1 \frac{1}{t^2+2t+3}\, dt$$

$$= \frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{\sqrt{2}}{4}\right) \qquad \text{1M}$$

(c) Let $y = \dfrac{\pi}{4} - \theta$. Then, we have $\mathrm{d}y = -\,\mathrm{d}\theta$. 1M

$$\int_0^{\frac{\pi}{4}} \frac{\sin 2\theta + 1}{\sin 2\theta + \cos 2\theta + 2}\,\mathrm{d}\theta = -\int_{\frac{\pi}{4}}^{0} \frac{\sin\left(\frac{\pi}{2} - 2y\right) + 1}{\sin\left(\frac{\pi}{2} - 2y\right) + \cos\left(\frac{\pi}{2} - 2y\right) + 2}\,\mathrm{d}y$$

$$= \int_0^{\frac{\pi}{4}} \frac{\cos 2y + 1}{\sin 2y + \cos 2y + 2}\,\mathrm{d}y$$

$$= \int_0^{\frac{\pi}{4}} \frac{\cos 2\theta + 1}{\sin 2\theta + \cos 2\theta + 2}\,\mathrm{d}\theta \qquad 1$$

(d) $$\int_0^{\frac{\pi}{4}} \frac{8\sin 2\theta + 9}{\sin 2\theta + \cos 2\theta + 2}\,\mathrm{d}\theta$$

$$= 8\int_0^{\frac{\pi}{4}} \frac{\sin 2\theta + 1}{\sin 2\theta + \cos 2\theta + 2}\,\mathrm{d}\theta + \int_0^{\frac{\pi}{4}} \frac{1}{\sin 2\theta + \cos 2\theta + 2}\,\mathrm{d}\theta$$

$$= 4\int_0^{\frac{\pi}{4}} \frac{\sin 2\theta + 1}{\sin 2\theta + \cos 2\theta + 2}\,\mathrm{d}\theta + 4\int_0^{\frac{\pi}{4}} \frac{\cos 2\theta + 1}{\sin 2\theta + \cos 2\theta + 2}\,\mathrm{d}\theta$$

$$+ \int_0^{\frac{\pi}{4}} \frac{1}{\sin 2\theta + \cos 2\theta + 2}\,\mathrm{d}\theta \qquad 1\mathrm{M}$$

$$= 4\int_0^{\frac{\pi}{4}} \mathrm{d}\theta + \int_0^{\frac{\pi}{4}} \frac{1}{\sin 2\theta + \cos 2\theta + 2}\,\mathrm{d}\theta \qquad 1\mathrm{M}$$

$$= 4\left(\frac{\pi}{4} - 0\right) + \frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{\sqrt{2}}{4}\right)$$

$$= \pi + \frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{\sqrt{2}}{4}\right) \qquad 1\mathrm{M}$$

27. (a) $$\frac{1}{2 + \cos 2x} = \frac{1}{2 + (2\cos^2 x - 1)} \times \frac{\sec^2 x}{\sec^2 x}$$

$$= \frac{\sec^2 x}{2 + \sec^2 x} \qquad 1$$

(b) $$\int_0^{\frac{\pi}{4}} \frac{1}{2 + \cos 2x}\,\mathrm{d}x = \int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{2 + \sec^2 x}\,\mathrm{d}x \qquad 1\mathrm{M}$$

$$= \int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{\tan^2 x + 3}\,\mathrm{d}x$$

Let $\tan x = \sqrt{3}\tan u$. Then $\sec^2 x\,\mathrm{d}x = \sqrt{3}\sec^2 u\,\mathrm{d}u$. 1M

When $x = 0$, $u = 0$; when $x = \dfrac{\pi}{4}$, $u = \dfrac{\pi}{6}$.

$$\int_0^{\frac{\pi}{4}} \frac{1}{2 + \cos 2x}\,\mathrm{d}x = \sqrt{3}\int_0^{\frac{\pi}{6}} \frac{\sec^2 u}{3\sec^2 u}\,\mathrm{d}u$$

$$= \sqrt{3}\left[\frac{u}{3}\right]_0^{\frac{\pi}{6}}$$

$$= \frac{\sqrt{3}\pi}{18} \qquad 1\mathrm{A}$$

(c) Let $u = -x$. Then $du = -dx$.

$$\int_{-a}^{a} f(x)\ln(1+e^x)\,dx = \int_{-a}^{0} f(x)\ln(1+e^x)\,dx + \int_{0}^{a} f(x)\ln(1+e^x)\,dx \qquad \text{1M}$$

$$= -\int_{a}^{0} f(-u)\ln(1+e^{-u})\,du + \int_{0}^{a} f(x)\ln(1+e^x)\,dx \qquad \text{1M}$$

$$= -\int_{0}^{a} f(u)\ln[e^{-u}(e^u+1)]\,du + \int_{0}^{a} f(x)\ln(1+e^x)\,dx \qquad \text{1M}$$

$$= \int_{0}^{a} f(x)\ln\frac{1+e^x}{e^{-x}(e^x+1)}\,dx$$

$$= \int_{0}^{a} xf(x)\,dx \qquad \text{1}$$

(d) Let $f(x) = \dfrac{\sin 2x}{(2+\cos 2x)^2}$.

$$f(-x) = \frac{\sin(-2x)}{(2+\cos(-2x))^2} = -\frac{\sin 2x}{(2+\cos 2x)^2} = -f(x). \qquad \text{1M}$$

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sin 2x}{(2+\cos 2x)^2}\ln(1+e^x)\,dx = \int_{0}^{\frac{\pi}{4}} \frac{x\sin 2x}{(2+\cos 2x)^2}\,dx \qquad \text{1M}$$

$$= \left[\frac{x}{2(2+\cos 2x)}\right]_0^{\frac{\pi}{4}} - \frac{1}{2}\int_0^{\frac{\pi}{4}} \frac{1}{2+\cos 2x}\,dx \qquad \text{1M}$$

$$= \frac{\pi}{16} - \frac{1}{2}\times\frac{\sqrt{3}\pi}{18} \qquad \text{1M}$$

$$= \frac{\pi}{16} - \frac{\sqrt{3}\pi}{36} \qquad \text{1A}$$

28. (a) $$\int g(x)\,dx = \cos^2 x\cdot\frac{\sin 2x}{2} - \frac{1}{2}\int 2\cos x(-\sin x)\sin 2x\,dx \qquad \text{1M}$$

$$= \frac{\sin 2x\cos^2 x}{2} + \frac{1}{2}\int \sin^2 2x\,dx \qquad \text{1}$$

(b) $$\int_0^{\pi} g(x)\,dx = \left[\frac{\sin 2x\cos^2 x}{2}\right]_0^{\pi} + \frac{1}{2}\int_0^{\pi}\sin^2 2x\,dx$$

$$= 0 + \frac{1}{4}\int_0^{\pi}(1-\cos 4x)\,dx \qquad \text{1M}$$

$$= \frac{1}{4}\left[x - \frac{\sin 4x}{4}\right]_0^{\pi}$$

$$= \frac{\pi}{4} \qquad \text{1A}$$

(c) Let $u = \pi - x$. Then $du = -dx$.

$g(\pi - u) = \cos^2(\pi - u)\cos(2\pi - 2u) = \cos^2 u \cos 2u = g(u)$.

$$\int_0^{\pi} xg(x)\,\mathrm{d}x = -\int_{\pi}^{0} (\pi - u)g(\pi - u)\,\mathrm{d}u \qquad \text{1M}$$

$$= \int_0^{\pi} (\pi - u)g(u)\,\mathrm{d}u \qquad \text{1M}$$

$$= \pi\int_0^{\pi} g(x)\,\mathrm{d}x - \int_0^{\pi} xg(x)\,\mathrm{d}x$$

$$2\int_0^{\pi} xg(x)\,\mathrm{d}x = \pi\int_0^{\pi} g(x)\,\mathrm{d}x \qquad \text{1M}$$

$$\int_0^{\pi} xg(x)\,\mathrm{d}x = \frac{\pi}{2}\cdot\frac{\pi}{4}$$

$$= \frac{\pi^2}{8} \qquad \text{1A}$$

(d) $(-x)g(-x) = -x\cos^2(-x)\cos(-2x) = -x\cos^2 x\cos 2x = -xg(x)$

Thus, $xg(x)$ is an odd function. 1M

$$\int_{-\pi}^{\pi} xg(x)\,\mathrm{d}x = \int_{-\pi}^{\pi} xg(x)\,\mathrm{d}x + \int_{\pi}^{2\pi} xg(x)\,\mathrm{d}x$$

$$= 0 + \int_{\pi}^{2\pi} xg(x)\,\mathrm{d}x \qquad \text{1M}$$

Let $u = x - \pi$. Then $\mathrm{d}u = \mathrm{d}x$.

$$\int_{-\pi}^{2\pi} xg(x)\,\mathrm{d}x = \int_0^{\pi} (u+\pi)\cos^2(u+\pi)\cos(2u+2\pi)\,\mathrm{d}u \qquad \text{1M}$$

$$= \int_0^{\pi} (u+\pi)g(u)\,\mathrm{d}u$$

$$= \int_0^{\pi} xg(x)\,\mathrm{d}x + \pi\int_0^{\pi} g(x)\,\mathrm{d}x$$

$$= \frac{\pi^2}{8} + \pi\cdot\frac{\pi}{4}$$

$$= \frac{3\pi^2}{8} \qquad \text{1A}$$

29. (a) $$\int_0^1 x^2e^{ax}\,\mathrm{d}x = \left[\frac{x^2e^{ax}}{a}\right]_0^1 - \frac{2}{a}\int_0^1 xe^{ax}\,\mathrm{d}x \qquad \text{1M}$$

$$= \frac{e^a}{a} - \left[\frac{2xe^{ax}}{a^2}\right]_0^1 + \frac{2}{a^2}\int_0^1 e^{ax}\,\mathrm{d}x \qquad \text{1M}$$

$$= \frac{e^a}{a} - \frac{2e^a}{a^2} + \frac{2}{a^3}\left[e^{ax}\right]_0^1$$

$$= \frac{(a^2 - 2a + 2)e^a - 2}{a^3} \qquad \text{1}$$

(b) Let $u = \ln(1+x)$. Then $du = \dfrac{1}{1+x}\,dx$. 1M

$$\int_0^{e-1} x(\ln(1+x))^2\,dx = \int_0^{e-1} \frac{x(1+x)}{1+x}(\ln(1+x))^2\,dx$$

$$= \int_0^1 (e^u - 1)e^u \cdot u^2\,du \quad \text{1M}$$

$$= \int_0^1 u^2 e^{2u}\,du - \int_0^1 u^2 e^u\,du$$

$$= \frac{[(2)^2 - 2(2) + 2]e^2 - 2}{2^3} - \frac{(1-2+2)e - 2}{1^3} \quad \text{1M}$$

$$= \frac{e^2 - 4e + 7}{4} \quad \text{1A}$$

(c) Let $u = (e-1)\cos x$. Then $du = (1-e)\sin x\,dx$. 1M

$$\int_0^{\frac{\pi}{2}} (\ln(1+(e-1)\cos x))^2 \sin 2x\,dx$$

$$= \int_0^{\frac{\pi}{2}} (\ln(1+(e-1)\cos x))^2 \cdot 2\sin x\cos x\,dx$$

$$= \int_{e-1}^0 (\ln(1+u))^2 \cdot \frac{2}{1-e} \cdot \frac{u}{e-1}\,du \quad \text{1M}$$

$$= \frac{2}{(e-1)^2}\int_0^{e-1} u\,(\ln(1+u))^2\,du$$

$$= \frac{e^2 - 4e + 7}{2(e-1)^2} \quad \text{1M}$$

(d) Let $u = x - \dfrac{\pi}{2}$. Then $du = dx$. 1M

$$\int_{\frac{\pi}{2}}^{\pi} (\ln(1+(e-1)\sin x))^2 \sin 2x\,dx$$

$$= \int_0^{\frac{\pi}{2}} \left[\ln\left(1+(e-1)\sin\left(\frac{\pi}{2}+u\right)\right)\right]^2 \sin(\pi + 2u)\,du \quad \text{1M}$$

$$= -\int_0^{\frac{\pi}{2}} (\ln(1+(e-1)\cos u))^2 \sin 2u\,du$$

$$= -\frac{e^2 - 4e + 7}{2(e-1)^2} \quad \text{1M}$$

30. (a) (i) Let $u = -x$. Then $du = -\,dx$. 1M

When $x = -1$, $u = 1$; when $x = 0$, $u = 0$.

$$\int_{-1}^0 f(x)\,dx = -\int_1^0 f(-u)\,du$$

$$= \int_0^1 f(-u)\,du \quad \text{1M}$$

$$= \int_0^1 f(-x)\,dx \quad \text{1}$$

(ii) If $f(x)$ is an odd function, then $f(-x) = -f(x)$ and

$$\int_{-1}^{1} f(x)\,\mathrm{d}x = \int_{-1}^{0} f(x)\,\mathrm{d}x + \int_{0}^{1} f(x)\,\mathrm{d}x$$

$$= \int_{0}^{1} -f(x)\,\mathrm{d}x + \int_{0}^{1} f(x)\,\mathrm{d}x \qquad \text{1M}$$

$$= 0 \qquad 1$$

(b) (i) $h(-x) = g(-x) - \dfrac{1}{2}$

$$= 1 - g(x) - \frac{1}{2} \qquad \text{1M}$$

$$= -g(x) + \frac{1}{2}$$

$$= -h(x)$$

Thus, $h(x)$ is an odd function on $\mathbb{R}$.

(ii) $\displaystyle\int_{-1}^{1} x^2 g(x)\,\mathrm{d}x = \int_{-1}^{1} x^2 \left[h(x) + \frac{1}{2}\right]\mathrm{d}x$ 1M

$$= \int_{-1}^{1} x^2 h(x)\,\mathrm{d}x + \frac{1}{2}\int_{-1}^{1} x^2\,\mathrm{d}x$$

$$= 0 + \frac{1}{2}\left[\frac{x^3}{3}\right]_{-1}^{1} \qquad \text{1M}$$

$$= \frac{1}{3} \qquad 1$$

(c) Let $g(x) = \dfrac{1}{1 + e^{\sin x}}$. Then $g(x)$ is a function with derivatives of any order on $\mathbb{R}$.

$g(-x) + g(x) = \dfrac{1}{e^{-\sin x}} + \dfrac{1}{1 + e^{\sin x}} = \dfrac{e^{\sin x}}{1 + e^{\sin x}} + \dfrac{1}{1 + e^{\sin x}} = 1$ 1M

Thus, $\displaystyle\int_{-1}^{1} \frac{x^2}{1 + e^{\sin x}}\,\mathrm{d}x = \frac{1}{3}$. 1A

31. (a) (i) Let $u = a - x$. Then $\mathrm{d}u - \mathrm{d}x$. 1M

When $x = 0$, $u = a$; when $x = a$, $u = 0$.

$$\int_{0}^{a} \ln\cos(a - x)\,\mathrm{d}x = -\int_{a}^{0} \ln\cos u\,\mathrm{d}u$$

$$= \int_{0}^{a} \ln\cos u\,\mathrm{d}u \qquad \text{1M}$$

$$= \int_{0}^{a} \ln\cos x\,\mathrm{d}x \qquad 1$$

(ii) By (a),

$$0 = \int_{0}^{a} \ln\cos(a - x)\,\mathrm{d}x - \int_{0}^{a} \ln\cos x\,\mathrm{d}x \qquad \text{1M}$$

$$= \int_{0}^{a} [\ln\cos(a - x) - \ln\cos x]\,\mathrm{d}x$$

$$= \int_{0}^{a} \ln\frac{\cos a\cos x + \sin a\sin x}{\cos x}\,\mathrm{d}x \qquad \text{1M}$$

$$= \int_{0}^{a} \ln(\cos a + \sin a\tan x)\,\mathrm{d}x \qquad 1$$

(b) (i) Let $x = \tan\theta$. Then $\mathrm{d}x = \sec^2\theta\,\mathrm{d}\theta$. 1M

$$\int_0^1 \frac{\tan^{-1}x}{1+x}\,\mathrm{d}x = \int_0^{\frac{\pi}{4}} \frac{\theta}{1+\tan\theta}\cdot\sec^2\theta\,\mathrm{d}\theta$$

$$= \Big[\theta\ln(1+\tan\theta)\Big]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \ln(1+\tan\theta)\,\mathrm{d}\theta \qquad \text{1M+1M}$$

$$= \frac{\pi}{4}\ln 2 - \int_0^{\frac{\pi}{4}} \ln(1+\tan\theta)\,\mathrm{d}\theta \qquad 1$$

(ii) Put $a = \dfrac{\pi}{4}$ such that $0 < a < \dfrac{\pi}{2}$,

$$\int_0^1 \frac{\tan^{-1}x}{1+x}\,\mathrm{d}x = \frac{\pi}{4}\ln 2 - \int_0^{\frac{\pi}{4}} \ln(1+\tan\theta)\,\mathrm{d}\theta$$

$$= \frac{\pi}{4}\ln 2 - \int_0^{\frac{\pi}{4}} \left[\ln\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\tan\theta\right) + \frac{1}{2}\ln 2\right]\mathrm{d}\theta \qquad \text{1M}$$

$$= \frac{\pi}{4}\ln 2 - \int_0^{\frac{\pi}{4}} \ln\left(\cos\frac{\pi}{4} + \sin\frac{\pi}{4}\tan\theta\right)\mathrm{d}\theta - \frac{1}{2}\ln 2\Big[\theta\Big]_0^{\frac{\pi}{4}} \qquad \text{1M}$$

$$= \frac{\pi}{8}\ln 2 \qquad \text{1A}$$

32. (a) (i) $\left(\sin\dfrac{x}{2} + \cos\dfrac{x}{2}\right)^2 = \sin^2\dfrac{x}{2} + 2\sin\dfrac{x}{2}\cos\dfrac{x}{2} + \cos^2\dfrac{x}{2}$

$$= 1 + \sin x \qquad 1$$

(ii) $$\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}x}{1+\sin x} = \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}x}{(\sin\frac{x}{2} + \cos\frac{x}{2})^2}$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sec^2\frac{x}{2}}{(\tan\frac{x}{2} + 1)^2}\,\mathrm{d}x$$

Let $u = \tan\dfrac{x}{2} + 1$. Then $\mathrm{d}u = \dfrac{1}{2}\sec^2\dfrac{x}{2}\,\mathrm{d}x$. 1M

When $x = 0$, $u = 1$; when $x = \dfrac{\pi}{2}$, $u = 2$.

$$\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}x}{1+\sin x} = 2\int_1^2 \frac{\mathrm{d}u}{u^2}$$

$$= 2\left[-\frac{1}{u}\right]_1^2$$

$$= 1 \qquad \text{1A}$$

(iii) $$\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\sin x}\,\mathrm{d}x = \int_0^{\frac{\pi}{2}} \left(1 - \frac{1}{1+\sin x}\right)\mathrm{d}x \qquad \text{1M}$$

$$= \Big[x\Big]_0^{\frac{\pi}{2}} - 1$$

$$= \frac{\pi}{2} - 1 \qquad \text{1A}$$

(b) $\displaystyle\int_0^{\pi} f(x)\,\mathrm{d}x = \int_0^{\frac{\pi}{2}} f(x)\,\mathrm{d}x + \int_{\frac{\pi}{2}}^{\pi} f(x)\,\mathrm{d}x.$

Let $u = \pi - x$. Then $\mathrm{d}u = -\,\mathrm{d}x$. 1M

When $x = \frac{\pi}{2}$, $u = \frac{\pi}{2}$; when $x = \pi$, $u = 0$.

$$\int_0^{\pi} f(x)\,\mathrm{d}x = \int_0^{\frac{\pi}{2}} f(x)\,\mathrm{d}x - \int_{\frac{\pi}{2}}^{0} f(\pi - u)\,\mathrm{d}u$$

$$= \int_0^{\frac{\pi}{2}} f(x)\,\mathrm{d}x + \int_0^{\frac{\pi}{2}} f(u)\,\mathrm{d}u \qquad \text{1M}$$

$$= 2\int_0^{\frac{\pi}{2}} f(x)\,\mathrm{d}x \qquad \text{1}$$

(c) Let $u = \pi - x$. Then $\mathrm{d}u = -\,\mathrm{d}x$.

When $x = 0$, $u = \pi$; when $x = \pi$, $u = 0$.

$$\int_0^{\pi} g(x)\ln(1 + e^{\cos x})\,\mathrm{d}x = -\int_{\pi}^{0} g(\pi - u)\ln\left(1 + e^{\cos(\pi - u)}\right)\mathrm{d}u$$

$$= -\int_0^{\pi} g(u)\ln(1 + e^{-\cos u})\,\mathrm{d}u$$

$$= -\int_0^{\pi} g(u)[\ln(e^{\cos u} + 1) - \ln e^{\cos u}]\,\mathrm{d}u \qquad \text{1M}$$

$$2\int_0^{\pi} g(x)\ln(1 + e^{\cos x})\,\mathrm{d}x = \int_0^{\pi} g(x)\cos x\,\mathrm{d}x \qquad \text{1M}$$

$$\int_0^{\pi} g(x)\ln(1 + e^{\cos x})\,\mathrm{d}x = \frac{1}{2}\int_0^{\pi} g(x)\cos x\,\mathrm{d}x \qquad \text{1}$$

(d) Let $g(x) = \dfrac{\cos x}{(1 + \sin x)^2}$.

Then $g(x)$ is continuous on $[0,\ \pi]$

$g(\pi - x) = \dfrac{\cos(\pi - x)}{(1 + \sin(\pi - x))^2} = -\dfrac{\cos x}{(1 + \sin x)^2} = -g(x)$.

By (c),

$$\int_0^{\pi} \frac{\cos x \ln(1 + e^{\cos x})}{(1 + \sin x)^2}\,\mathrm{d}x = \frac{1}{2}\int_0^{\pi} \frac{\cos^2 x}{(1 + \sin x)^2}\,\mathrm{d}x \qquad \text{1M}$$

$$= \frac{1}{2}\int_0^{\pi} \frac{1 - \sin^2 x}{(1 + \sin x)^2}\,\mathrm{d}x$$

$$= \frac{1}{2}\int_0^{\pi} \frac{1 - \sin x}{1 + \sin x}\,\mathrm{d}x \qquad \text{1M}$$

Let $f(x) = \dfrac{1 - \sin x}{1 + \sin x}$. So, $f(\pi - x) = \dfrac{1 - \sin(\pi - x)}{1 + \sin(\pi - x)} = \dfrac{1 - \sin x}{1 + \sin x} = f(x)$.

$$\int_0^{\pi} \frac{\cos x \ln(1 + e^{\cos x})}{(1 + \sin x)^2}\,\mathrm{d}x = \int_0^{\frac{\pi}{2}} \frac{1}{1 + \sin x}\,\mathrm{d}x - \int_0^{\frac{\pi}{2}} \frac{\sin x}{1 + \sin x}\,\mathrm{d}x \qquad \text{1M}$$

$$= 1 - \left(\frac{\pi}{2} - 1\right)$$

$$= 2 - \frac{\pi}{2} \qquad \text{1A}$$

END OF PAPER