1. (a) $y = \sqrt{24 - x^2}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}(24 - x^2)^{-\frac{1}{2}}(-2x) \qquad \text{1M}$$

$$= \frac{-x}{\sqrt{24 - x^2}}$$

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=3\sqrt{2}} = \frac{-3\sqrt{2}}{\sqrt{24 - (3\sqrt{2})^2}}$$

$$= -\sqrt{3}$$

The equation of L is

$$\frac{y - \sqrt{6}}{x - 3\sqrt{2}} = -3\sqrt{3} \qquad \text{1M}$$

$$y = -\sqrt{3}x + 4\sqrt{6} \qquad \text{1A}$$

(b) (i) $\left(3\sqrt{k - x^2}\right)^2 = \left(-\sqrt{3}x + 4\sqrt{6}\right)^2$ 1M

$$9(k - x^2) = 3x^2 - 24\sqrt{2}x + 96$$

$$0 = 12x^2 - 24\sqrt{2}x + (96 - 9k)$$

L is a tangent to C_1.

$$(24\sqrt{2})^2 - 4(12)(96 - 9k) = 0$$

$$432k - 3456 = 0$$

$$k = 8$$

Put $k = 8$.

$$0 = 12x^2 - 24\sqrt{2}x + 24$$

$$0 = 12(x - \sqrt{2})^2$$

$$x = \sqrt{2}$$

L is the tangent to C_1 at $x = \sqrt{2}$. 1

(ii) $(3\sqrt{8 - x^2})^2 = (\sqrt{24 - x^2})^2$ 1M

$$9(8 - x^2) = 24 - x^2$$

$$x^2 = 6$$

$$x = \sqrt{6} \quad \text{or} - \sqrt{6} \text{ (rejected)}$$

Required coordinates are $\left(\sqrt{6},\ 3\sqrt{2}\right)$. 1A

(iii) Required area

$$= \int_2^{\sqrt{6}} \left(-\sqrt{3}x + 4\sqrt{6} - 3\sqrt{8 - x^2}\right) \mathrm{d}x + \int_{\sqrt{6}}^{3\sqrt{2}} \left(-\sqrt{3}x + 4\sqrt{6} - \sqrt{24 - x^2}\right) \mathrm{d}x \qquad \text{1M+1A}$$

$$= \int_2^{3\sqrt{2}} \left(-\sqrt{3}x + 4\sqrt{6}\right) \mathrm{d}x - 3\int_2^{\sqrt{6}} \sqrt{8 - x^2}\, \mathrm{d}x - \int_{\sqrt{6}}^{3\sqrt{2}} \sqrt{24 - x^2}\, \mathrm{d}x$$

Consider $\int_2^{\sqrt{6}} \sqrt{8-x^2}\,\mathrm{d}x$. Let $x = \sqrt{8}\sin\theta$. Then $\mathrm{d}x = \sqrt{8}\cos\theta\,\mathrm{d}\theta$. 1M

$$\begin{aligned}\int_2^{\sqrt{6}} \sqrt{8-x^2}\,\mathrm{d}x &= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sqrt{8-8\sin^2\theta}\cdot\sqrt{8}\cos\theta\,\mathrm{d}\theta \\ &= 8\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \cos^2\theta\,\mathrm{d}\theta \\ &= 4\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (1+\cos 2\theta)\,\mathrm{d}\theta \qquad \text{1M} \\ &= 4\left[\theta + \frac{\sin 2\theta}{2}\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}} \\ &= -2+\sqrt{3}+\frac{\pi}{3}\end{aligned}$$

Consider $\int_{\sqrt{6}}^{3\sqrt{2}} \sqrt{24-x^2}\,\mathrm{d}x$.
Let $x = \sqrt{24}\sin\beta$. Then $\mathrm{d}x = \sqrt{24}\cos\beta\,\mathrm{d}\beta$.

$$\begin{aligned}\int_{\sqrt{6}}^{3\sqrt{2}} \sqrt{24-x^2}\,\mathrm{d}x &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sqrt{24-24\sin^2\beta}\cdot\sqrt{24}\cos\beta\,\mathrm{d}\beta \\ &= 24\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos^2\beta\,\mathrm{d}\beta \\ &= 12\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (1+\cos 2\beta)\,\mathrm{d}\beta \\ &= 12\left[\beta + \frac{\sin 2\beta}{2}\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} \\ &= 2\pi\end{aligned}$$

Required area

$$\begin{aligned}&= \left[-\frac{\sqrt{3}x^2}{2} + 4\sqrt{6}x\right]_2^{3\sqrt{2}} - 3\left(-2+\sqrt{3}+\frac{\pi}{3}\right) - 2\pi \\ &= 6 + 14\sqrt{3} - 8\sqrt{6} - 3\pi \qquad \text{1A}\end{aligned}$$

2. (a) $x^2 - x + \frac{1}{2} = \left(x - \frac{1}{2}\right)^2 + \frac{1}{4}$ 1M

Let $x - \frac{1}{2} = \frac{1}{2}\tan\theta$. Then $\mathrm{d}x = \frac{1}{2}\sec^2\theta\,\mathrm{d}\theta$

$$\begin{aligned}\int \frac{\mathrm{d}x}{x^2 - x + \frac{1}{2}} &= \frac{1}{2}\int \frac{\sec^2\theta}{\frac{1}{4}\tan^2\theta + \frac{1}{4}}\,\mathrm{d}\theta \qquad \text{1M} \\ &= 2\theta + \text{constant} \\ &= 2\tan^{-1}(2x-1) + \text{constant} \qquad \text{1A}\end{aligned}$$

(b) Let $u = \lambda - x$. Then $\mathrm{d}u = -\,\mathrm{d}x$.

$$\int_0^{\lambda} x f(x)\,\mathrm{d}x = -\int_{\lambda}^{0} (\lambda - u) f(\lambda - u)\,\mathrm{d}u \qquad \text{1M}$$

$$= \int_0^{\lambda} (\lambda - x) f(x)\,\mathrm{d}x \qquad \text{1M}$$

$$2\int_0^{\lambda} x f(x)\,\mathrm{d}x = \lambda \int_0^{\lambda} f(x)\,\mathrm{d}x$$

$$\int_0^{\lambda} x f(x)\,\mathrm{d}x = \frac{\lambda}{2}\int_0^{\lambda} f(x)\,\mathrm{d}x \qquad \text{1}$$

(c) Volume $= \pi \displaystyle\int_0^{\frac{\pi}{2}} \frac{x \sin x \cos x}{\sin^4 x + \cos^4 x}\,\mathrm{d}x$ 1M+1A

Let $f(x) = \dfrac{\sin x \cos x}{\sin^4 x + \cos^4 x}$.

$$f\left(\frac{\pi}{2} - x\right) = \frac{\sin\left(\frac{\pi}{2} - x\right)\cos\left(\frac{\pi}{2} - x\right)}{\sin^4\left(\frac{\pi}{2} - x\right) + \cos^4\left(\frac{\pi}{2} - x\right)} = \frac{\cos x \sin x}{\cos^4 x + \sin^4 x} = f(x).$$

Volume $= \dfrac{\pi^2}{4}\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x \cos x}{\sin^4 x + \cos^4 x}\,\mathrm{d}x$ 1M

Let $u = \sin^2 x$. Then $\mathrm{d}u = 2\sin x \cos x\,\mathrm{d}x$.

Volume $= \dfrac{\pi^2}{8}\displaystyle\int_0^1 \frac{\mathrm{d}u}{u^2 + (1-u)^2}$ 1M

$$= \frac{\pi^2}{16}\int_0^1 \frac{\mathrm{d}u}{u^2 - u + \frac{1}{2}}$$

$$= \frac{\pi^2}{8}\Big[\tan^{-1}(2u - 1)\Big]_0^1 \qquad \text{1M}$$

$$= \frac{\pi^3}{16} \qquad \text{1A}$$

3. (a) Let $u = x^2 - 2x + 4$. Then $\mathrm{d}u = (2x - 2)\,\mathrm{d}x$. 1M

$$\int_0^1 \frac{x - 1}{x^2 - 2x + 4}\,\mathrm{d}x = \frac{1}{2}\int_4^3 \frac{\mathrm{d}u}{u}$$

$$= \frac{1}{2}\Big[\ln|u|\Big]_4^3 \qquad \text{1A}$$

$$= \frac{1}{2}\ln\frac{3}{4} \qquad \text{1A}$$

(b) (i) $x^2 - 2x + 4 = (x - 1)^2 + 3$ 1A

(ii) Let $x - 1 = \sqrt{3}\tan\theta$. Then $\mathrm{d}x = \sqrt{3}\sec^2\theta\,\mathrm{d}\theta$. 1M

$$\int_0^1 \frac{1}{x^2 - 2x + 4}\,\mathrm{d}x = \int_{-\frac{\pi}{6}}^0 \frac{\sqrt{3}\sec^2\theta}{3\tan^2\theta + 3}\,\mathrm{d}\theta$$

$$= \frac{\sqrt{3}}{3}\int_{-\frac{\pi}{6}}^0 \mathrm{d}\theta \qquad \text{1M}$$

$$= \frac{\sqrt{3}}{3}\Big[\theta\Big]_{-\frac{\pi}{6}}^0$$

$$= \frac{\sqrt{3}\pi}{18} \qquad \text{1A}$$

(c) (i) $$\frac{A}{x+2} - \frac{B(x-4)}{x^2 - 2x + 4} = \frac{A(x^2 - 2x + 4) - B(x+2)(x-4)}{(x+2)(x^2 - 2x + 4)}$$

$$= \frac{(A - B)x^2 + 2(B - A)x + (4A + 8B)}{x^3 + 8}$$

We have $A - B = 0$, $B - A = 0$ and $4A + 8B = 12$.

Solving, we have $A = 1$ and $B = 1$. 1A

(ii) $$\text{Required volume} = \pi\int_{-1}^0 \left(\frac{1}{\sqrt{8 - x^3}}\right)^2 \mathrm{d}x \qquad \text{1M}$$

$$= \pi\int_{-1}^0 \frac{\mathrm{d}X}{8 - x^3}$$

Let $u = -x$. Then $\mathrm{d}u = -\,\mathrm{d}x$. 1M

$$\text{Required volume} = \pi\int_0^1 \frac{\mathrm{d}u}{u^3 + 8}$$

$$= \frac{\pi}{12}\int_0^1 \left(\frac{1}{u+2} - \frac{u-4}{u^2 - 2u + 4}\right)\mathrm{d}u \qquad \text{1M}$$

$$= \frac{\pi}{12}\int_0^1 \frac{\mathrm{d}u}{u+2} - \frac{\pi}{12}\left(\int_0^1 \frac{u-1}{u^2 - 2u + 4}\,\mathrm{d}u - \int_0^1 \frac{3}{u^2 - 2u + 4}\,\mathrm{d}u\right) \qquad \text{1M}$$

$$= \frac{\pi}{12}\Big[\ln|u+2|\Big]_0^1 - \frac{\pi}{12}\left(\frac{1}{2}\ln\frac{3}{4}\right) + \frac{\pi}{4}\left(\frac{\sqrt{3}\pi}{18}\right)$$

$$= \frac{\pi\ln 3}{24} + \frac{\sqrt{3}\pi^2}{24} \qquad \text{1A}$$

4. (a) (i) $$\cos^4 x\sin^2 x = \cos^2 x\left(\frac{2\sin x\cos x}{2}\right)^2$$

$$= \left(\frac{1 + \cos 2x}{2}\right)\left(\frac{\sin^2 2x}{4}\right) \qquad \text{1M}$$

$$= \frac{\sin^2 2x + \cos 2x\sin^2 2x}{8} \qquad \text{1}$$

(ii) Let $x = \frac{\pi}{2} - u$. Then $dx = -\,du$.

$$\int_0^{\frac{\pi}{2}} \sin^4 x \cos^2 x \, dx = -\int_{\frac{\pi}{2}}^{0} \sin^4\left(\frac{\pi}{2} - u\right) \cos^2\left(\frac{\pi}{2} - u\right) du \qquad \text{1M}$$

$$= \int_0^{\frac{\pi}{2}} \cos^4 u \sin^2 u \, du$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin^2 2u + \cos 2u \sin^2 2u}{8} \, du$$

$$= \frac{1}{16} \int_0^{\frac{\pi}{2}} (1 - \cos 4u) \, du + \frac{1}{16} \int_0^{\frac{\pi}{2}} \sin^2 2u \, d(\sin 2u) \qquad \text{1M+1M}$$

$$= \frac{1}{16} \left[u - \frac{\sin 4u}{4} \right]_0^{\frac{\pi}{2}} + \frac{1}{16} \left[\frac{\sin^3 2u}{3} \right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{32} \qquad \text{1A}$$

(b) (i) Let $x = k - u$. Then $dx = -\,du$.

$$\int_0^{k} f(x) \, dx = \int_0^{\frac{k}{2}} f(x) \, dx + \int_{\frac{k}{2}}^{k} f(x) \, dx$$

$$= \int_0^{\frac{k}{2}} f(x) \, dx - \int_{\frac{k}{2}}^{0} f(k - u) \, du \qquad \text{1M}$$

$$= \int_0^{\frac{k}{2}} f(x) \, dx + \int_0^{\frac{k}{2}} f(x) \, dx$$

$$= 2 \int_0^{\frac{k}{2}} f(x) \, dx \qquad \text{1}$$

(ii) Let $x = k - w$. Then $dx = -\,dw$.

$$\int_0^{k} x f(x) \, dx = -\int_{k}^{0} (k - w) f(k - w) \, dw \qquad \text{1M}$$

$$= k \int_0^{k} f(w) \, dw - \int_0^{k} w f(w) \, dw$$

$$= k \int_0^{k} f(x) \, dx - \int_0^{k} x f(x) \, dx$$

$$2 \int_0^{k} x f(x) \, dx = k \int_0^{k} f(x) \, dx$$

$$\int_0^{k} x f(x) \, dx = \frac{k}{2} \int_0^{k} f(x) \, dx$$

$$= k \int_0^{\frac{k}{2}} f(x) \, dx \qquad \text{1}$$

(iii) Let $f(x) = \cos^4 x \sin^2 x$.

Then $f(\pi - x) = \cos^4(\pi - x) \sin^2(\pi - x) = \cos^4 x \sin^2 x = f(x)$. 1M

$$\text{Volume} = \pi \int_0^{\pi} x \sin^4 x \cos^2 x \, dx \qquad \text{1M}$$

$$= \pi^2 \int_0^{\frac{\pi}{2}} \sin^4 x \cos^2 x \, dx$$

$$= \frac{\pi^3}{32} \qquad \text{1A}$$

5. (a) Required volume $= \pi \int_0^k (12y - y^2)\,\mathrm{d}y$ 1M+1A

$$= \pi \left[6y^2 - \frac{y^3}{3} \right]_0^k$$

$$= \pi \left(6k^2 - \frac{k^3}{3} \right)$$ 1

(b) (i) When $0 \le h \le$.

$$V = \pi \left[6(h+2)^2 - \frac{(h+2)^3}{3} \right] - \pi \left[6(2)^2 - \frac{2^3}{3} \right]$$ 1M

$$= \pi \left[6(h+2)^2 - \frac{1}{3}(h+2)^3 \right] - \frac{64\pi}{3}$$ 1

(ii) When $4 \le h \le 11$.

$$V = \pi(6)^2(h-4) + \pi \left[6(4+2)^2 - \frac{1}{3}(4+2)^3 \right] - \frac{64\pi}{3}$$ 1M

$$= 36\pi(h-4) + \frac{368\pi}{3}$$

$$\frac{\mathrm{d}V}{\mathrm{d}t} = 36\pi \cdot \frac{\mathrm{d}h}{\mathrm{d}t}$$ 1A

$$-3 = 36\pi \cdot \frac{\mathrm{d}h}{\mathrm{d}t}$$

$$\frac{\mathrm{d}h}{\mathrm{d}t} = -\frac{1}{12\pi}$$

Required rate is $\frac{1}{12\pi}$ cm/s. 1A

(iii) When $0 \le h \le 4$.

$$V = \pi \left[6(h+2)^2 - \frac{1}{3}(h+2)^3 \right] - \frac{64\pi}{3}$$

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \pi \left[12(h+2)\frac{\mathrm{d}h}{\mathrm{d}t} - (h+2)^2\frac{\mathrm{d}h}{\mathrm{d}t} \right]$$ 1A

When $h = 1$, $\frac{\mathrm{d}V}{\mathrm{d}t} = 5 - 3 = 2$.

$$2 = \pi \left[12(1+2)\frac{\mathrm{d}h}{\mathrm{d}t} - (1+2)^2\frac{\mathrm{d}h}{\mathrm{d}t} \right]$$ 1M

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{2}{27\pi}$$

Required rate is $\frac{2}{27\pi}$ cm/s. 1A

6. (a) Required volume $= \pi \int_0^h \left(\frac{y^2}{9} + r \right)^2 \mathrm{d}y$ 1M

$$= \pi \int_0^h \left(\frac{y^4}{81} + \frac{2ry^2}{9} + r^2 \right) \mathrm{d}y$$

$$= \pi \left[\frac{y^5}{5} + \frac{2ry^3}{27} + r^2 y \right]_0^h$$

$$= \pi \left(\frac{h^5}{405} + \frac{2rh^3}{27} + r^2 h \right)$$ 1

(b) (i) When $0 \leq h \leq 6$.

$$V = \pi\left[\frac{h^5}{405} + \frac{2(8)h^3}{27} + (8)^2 h\right]$$

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \pi\left(\frac{h^4}{81} + \frac{16h^2}{9} + 64\right)\frac{\mathrm{d}h}{\mathrm{d}t} \qquad \text{1A}$$

When $h = 3$.

$$4\pi = \pi\left[\frac{3^4}{81} + \frac{16(3)^2}{9} + 64\right]\frac{\mathrm{d}h}{\mathrm{d}t}$$

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{4}{81}$$

Required rate is $\frac{4}{81}$ cm/s. 1A

(ii) Base radius of the frustum $= \frac{6^2}{9} + 8$

$= 12$ cm 1A

Let H cm be the height of the frustum.

$$2H = \sqrt{20^2 - 12^2}$$

$$H = 8 \qquad \text{1A}$$

When $6 \leq h \leq T$.

$$V = \pi\left[\frac{6^5}{405} + \frac{16(6)^3}{27} + 64(6)\right] + \frac{1}{3}\pi(12)^2(16)\left[1 - \left(\frac{22-h}{16}\right)^3\right] \qquad \text{1M+1M+1A}$$

$$= \frac{6496}{5}\pi - \frac{3}{16}\pi(22-h)^3 \qquad \text{1}$$

(iii) When $6 \leq h \leq T$.

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -\frac{9}{16}\pi(22-h)^2(-1)\frac{\mathrm{d}h}{\mathrm{d}t}$$

$$= \frac{9}{16}\pi(22-h)^2\frac{\mathrm{d}h}{\mathrm{d}t} \qquad \text{1A}$$

When $h = 10$.

$$4\pi - k = \frac{9}{16}\pi(22-10)^2\left(\frac{1}{27}\right)$$

$$k = \pi \qquad \text{1A}$$

7. (a) (i) $h + 1 = x^2 + 1$

$x = \sqrt{h}$ or $-\sqrt{h}$ (rejected)

The coordinates of A are $\left(\sqrt{h},\ h+1\right)$. 1A

(ii) $y = x^2 + 1$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x \qquad \text{1A}$$

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=\sqrt{h}} = 2\sqrt{h}$$

The equation of L is

$$\frac{y-(h+1)}{x-\sqrt{h}} = 2\sqrt{h} \qquad \text{1M}$$

$$y = 2\sqrt{h}x - h + 1 \qquad \text{1}$$

(b) (i) Consider the capacity of the cup.

$$\pi\int_0^{h+1}\left(\frac{y+h-1}{2\sqrt{h}}\right)^2 \mathrm{d}y - \pi\int_1^{h+1}(y-1)\,\mathrm{d}y = \frac{31}{10}\pi \qquad \text{1M+1A}$$

$$\frac{\pi}{4h}\left[\frac{(y+h-1)^3}{3}\right]_0^{h+1} - \pi\left[\frac{(y-1)^2}{2}\right]_1^{h+1} = \frac{31}{10}\pi \qquad \text{1M}$$

$$\frac{\pi}{6}h^2 - \frac{\pi}{12h}(h-1)^3 = \frac{31}{10}\pi$$

$$10h^3 - 5(h^3 - 3h^2 + 3h - 1) = 186h$$

$$5h^3 + 15h^2 - 201h + 5 = 0$$

$$(h-5)(5h^2 + 40h - 1) = 0 \qquad \text{1A}$$

$$h = 5 \quad \text{or} \quad h = \frac{-40 \pm \sqrt{40^2 - 4(5)(-1)}}{2(5)}$$

$$h = 5 \quad \text{or} \quad h = \frac{-40 \pm \sqrt{1620}}{10} \text{ (rejected)} \qquad \text{1A}$$

(ii) Let V cm^3, A cm^2 and H cm be the volume of water in the cup, the area of the water surface and the depth of water at time t s respectively.

$$V = \pi\int_1^{H+1}(y-1)\,\mathrm{d}y \qquad \text{1M}$$

$$= \pi\left[\frac{(y-1)^2}{2}\right]_1^{H+1}$$

$$= \frac{\pi}{2}H^2$$

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \pi H\frac{\mathrm{d}H}{\mathrm{d}t} \qquad \text{1M}$$

When $H = 3$ and $\frac{\mathrm{d}V}{\mathrm{d}t} = 1$.

$$1 = \pi(3)\frac{\mathrm{d}H}{\mathrm{d}T}$$

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{1}{3\pi} \qquad \text{1A}$$

When $y = H + 1$.

$$H + 1 = x^2 + 1$$

$$x^2 = H$$

We have $A = \pi H$.

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \pi\frac{\mathrm{d}H}{\mathrm{d}t} \qquad \text{1M}$$

When $H = 3$, $\dfrac{dH}{dt} = \dfrac{1}{3\pi}$.

$$\frac{dA}{dt} = \pi\left(\frac{1}{3\pi}\right)$$
$$= \frac{1}{3}$$

Required rate is $\dfrac{1}{3}$ cm^2/s. 1A

8. (a) $\sin 16\theta \sec\theta \sec 2\theta \sec 4\theta \sec 8\theta$

$= (2\sin 8\theta \cos 8\theta)\sec\theta\sec 2\theta\sec 4\theta\sec 8\theta$ 1M

$= 2(2\sin 4\theta\cos 4\theta)\sec\theta\sec 2\theta\sec 4\theta$

$= 4(2\sin 2\theta\cos 2\theta)\sec\theta\sec 2\theta$

$= 8(2\sin\theta\cos\theta)\sec\theta$

$= 16\sin\theta$ 1

(b) Use the result of (a).

$$y = \sqrt{\sin 16x\sec^2 x\sec 2x\sec 4x\sec 8x}$$
$$= \sqrt{16\sin x\sec x} \quad \text{1M}$$
$$= 4\sqrt{\tan x}$$

Let $u = \cos x$. Then $du = -\sin x\,dx$. 1M

Required capacity $= \pi\displaystyle\int_0^d 16\tan x\,dx$ 1M+1A

$$= -16\pi\int_1^{\cos d}\frac{du}{u}$$
$$= -16\pi\Big[\ln|u|\Big]_1^{\cos d} \quad \text{1M}$$
$$= -16\pi\ln(\cos d) \quad \text{1}$$

(c) (i) Required capacity $= -16\pi\ln[\cos(0.1)]$

$\approx 0.252\ \text{m}^3$ 1A

(ii) Let p m be the depth of water when the volume of water in the container is $0.08\ \text{m}^3$.

$$-16\pi\ln(\cos p) = 0.08$$
$$p \approx 0.0564 \quad \text{1A}$$

Let k m and V m^3 be the depth and the volume of the water at time t s respectively.

$$V = -16\pi\ln(\cos k)$$
$$\frac{dV}{dt} = 16\pi\tan k\frac{dk}{dt} \quad \text{1M}$$
$$\left.\frac{dV}{dt}\right|_{k=p} = 16\pi\tan p\cdot\left.\frac{dk}{dt}\right|_{k=p}$$
$$\left.\frac{dk}{dt}\right|_{k=p} \approx 0.001\,06$$

Required rate is 0.001 06 m/s. 1A

9. (a) (i) $\dfrac{d}{dx}\sqrt{1-x^2} = \dfrac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x)$

$= -\dfrac{x}{\sqrt{1-x^2}}$ 1A

(ii) $y = \sin^{-1} x$

$\sin y = x$

$\cos y \dfrac{dy}{dx} = 1$ 1A+1A

$\dfrac{dy}{dx} = \dfrac{1}{\cos y}$

$\dfrac{d}{dx}\sin^{-1} x = \dfrac{1}{\sqrt{1-x^2}}$ 1

(b) Use the result of (a).

$\displaystyle\int (\sin^{-1} x)^2\,dx = x(\sin^{-1} x) - 2\int x\sin^{-1} x \cdot \frac{1}{\sqrt{1-x^2}}\,dx$ 1M

$\displaystyle = x(\sin^{-1} x)^2 + 2\sqrt{1-x^2}\sin^{-1} x - 2\int \sqrt{1-x^2}\cdot\frac{1}{\sqrt{1-x^2}}\,dx$ 1M

$= x(\sin^{-1} x)^2 + 2\sqrt{1-x^2}\sin^{-1} x - 2x + \text{constant}$ 1

(c) (i) Required capacity $\displaystyle = \pi\int_0^1 (\sin^{-1} y)^2\,dy$ 1M

$= \pi\left[y(\sin^{-1} y)^2 + 2\sqrt{1-y^2}\sin^{-1} - 2y\right]_0^1$

$= \dfrac{\pi^3}{4} - 2\pi$ 1A

(ii) Let h units and V cubic units be the depth and the volume of the wine respectively.

$\dfrac{\sqrt{3}}{2} = \sin x$

$x = \dfrac{\pi}{3}$ 1A

Note that $V = \pi\displaystyle\int_0^h (\sin^{-1} y)^2\,dy$.

$\dfrac{dV}{dt} = \dfrac{dV}{dh}\cdot\dfrac{dh}{dt}$

$= \pi(\sin^{-1} h)^2\cdot\dfrac{dh}{dt}$ 1M

$\dfrac{\pi^3}{3600} = \pi\left(\sin^{-1}\dfrac{\sqrt{3}}{2}\right)^2\cdot\left.\dfrac{dh}{dt}\right|_{h=\frac{\sqrt{3}}{2}}$ 1M

$= \pi\left(\dfrac{\pi}{3}\right)^2\cdot\left.\dfrac{dh}{dt}\right|_{h=\frac{\sqrt{3}}{2}}$

$\left.\dfrac{dh}{dt}\right|_{h=\frac{\sqrt{3}}{2}} = \dfrac{1}{400}$

Required rate is $\dfrac{1}{400}$ units/s. 1A

10. (a) For Γ, $\dfrac{dy}{dx}=\dfrac{1}{6}(12-x^2)^{-\frac{1}{2}}(-2x)=-\dfrac{x}{3}(12-x^2)^{-\frac{1}{2}}$ 1M

When $x=3$, $\dfrac{dy}{dx}=-\dfrac{3}{3}(12-9)^{-\frac{1}{2}}=-\dfrac{1}{\sqrt{3}}$ and $y=\dfrac{1}{3}\sqrt{12-9}=\dfrac{1}{\sqrt{3}}$.

The equation of L is

$$y-\frac{1}{\sqrt{3}}=-\frac{1}{\sqrt{3}}(x-3) \qquad \text{1M}$$

$$y=-\frac{x}{\sqrt{3}}+\frac{4}{\sqrt{3}} \qquad \text{1A}$$

(b) (i) $$-\frac{x}{\sqrt{3}}+\frac{4}{\sqrt{3}}=\sqrt{4-x^2} \qquad \text{1M}$$

$$\frac{1}{3}(x^2-8x+16)=4-x^2$$

$$\frac{4}{3}x^2-\frac{8}{3}x+\frac{4}{3}=0$$

$$x=1 \text{ (repeated)}$$

When $x=1$, $y=\sqrt{4-1}=\sqrt{3}$.

The point of contact of L and C is $\left(1,\ \sqrt{3}\right)$. 1A

(ii) $$\frac{1}{3}\sqrt{12-x^2}=\sqrt{4-x^2} \qquad \text{1M}$$

$$\frac{1}{9}(12-x^2)=4-x^2$$

$$8x^2=24$$

$$x=\sqrt{3} \quad \text{or} \quad -\sqrt{3}$$

When $x=\sqrt{3}$, $y=\sqrt{4-3}=1$

The point of intersection of C and Γ is $\left(\sqrt{3},\ 1\right)$. 1A

(iii) $$\text{Required area}=\int_1^3\left(\frac{-x}{\sqrt{3}}+\frac{4}{\sqrt{3}}\right)dx-\int_1^{\sqrt{3}}\sqrt{4-x^2}\,dx-\int_{\sqrt{3}}^3\frac{1}{3}\sqrt{12-x^2}\,dx \qquad \text{1M+1A}$$

$$=\frac{1}{\sqrt{3}}\left[-\frac{x^2}{2}+4x\right]_1^3-\int_1^{\sqrt{3}}\sqrt{4-x^2}\,dx-\frac{1}{\sqrt{3}}\int_{\sqrt{3}}^3\sqrt{4-\left(\frac{x}{\sqrt{3}}\right)^2}\,dx$$

$$=\frac{4}{\sqrt{3}}-2\int_1^{\sqrt{3}}\sqrt{4-x^2}\,dx$$

Let $x=2\sin\theta$. Then $dx=2\cos\theta\,d\theta$.

When $x=1$, $\theta=\dfrac{\pi}{6}$; when $x=\sqrt{3}$, $\theta=\dfrac{\pi}{3}$.

$$\text{Required area}=\frac{4}{\sqrt{3}}-2\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\sqrt{4\cos^2\theta}\cdot 2\cos\theta\,d\theta \qquad \text{1M}$$

$$=\frac{4}{\sqrt{3}}-4\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}(1+\cos 2\theta)\,d\theta \qquad \text{1M}$$

$$=\frac{4}{\sqrt{3}}-4\left[\theta+\frac{\sin 2\theta}{2}\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$=\frac{4}{\sqrt{3}}-\frac{2\pi}{3} \qquad \text{1A}$$

11. (a) $f'(x) = e^{-x^2} + xe^{-x^2}(-2x)$ 1M

$= e^{-x^2}(1-2x^2)$ 1A

$f''(x) = -2xe^{-x^2}(1-2x^2) + e^{-x^2}(-4x)$

$= 2xe^{-x^2}(2x^2-3)$ 1A

(b) When $f'(x) = 0$, $x = \pm\dfrac{1}{\sqrt{2}}$. 1M

We have $f''\left(\dfrac{1}{\sqrt{2}}\right) = -2\sqrt{2}e^{-\frac{1}{2}} < 0$ and $f''\left(-\dfrac{1}{\sqrt{2}}\right) = 2\sqrt{2}e^{-\frac{1}{2}} > 0$. 1M

The maximum point of G is $\left(\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2e}}\right)$.

The minimum point of G is $\left(-\dfrac{1}{\sqrt{2}}, -\dfrac{1}{\sqrt{2e}}\right)$. 1A

(c) (i) $f'(1) = -\dfrac{1}{e}$ 1M

Required equation is

$$y - \frac{1}{e} = -\frac{1}{e}(x-1)$$

$$y = -\frac{x}{e} + \frac{2}{e}$$ 1A

(ii) For $x \in (0,\ 1)$, $f''(x) = 2xe^{-x^2}(2x^2-3) < 0$.

G is concave downward in the interval $(0,\ 1)$.

Thus, G lies below L in the interval $(0,\ 1)$. 1

(iii) Let $u = x^2$. Then $\mathrm{d}u = 2x\,\mathrm{d}x$.

Required area $= \displaystyle\int_0^1 \left[-\frac{x}{e} + \frac{2}{e} - xe^{-x^2}\right] \mathrm{d}x$ 1M

$= \displaystyle\frac{1}{e}\int_0^1 (-x+2)\,\mathrm{d}x - \frac{1}{2}\int_0^1 e^{-u}\,\mathrm{d}u$ 1M

$= \displaystyle\frac{1}{e}\left[-\frac{x^2}{2} + 2x\right]_0^1 + \frac{1}{2}\left[e^{-u}\right]_0^1$

$= \dfrac{2}{e} - \dfrac{1}{2}$ 1A

12. (a) $1 - \cos 4\theta - 2\cos 2\theta \sin^2 2\theta = 2\sin^2 2\theta - 2\cos 2\theta \sin^2 2\theta$ 1M

$= 2\sin^2 2\theta(1-\cos 2\theta)$

$= 2(2\sin\theta\cos\theta)^2(2\sin^2\theta)$

$= 16\cos^2\theta\sin^4\theta$ 1

(b) $\displaystyle\int_0^{n\pi} \cos^2 x \sin^4 x\,\mathrm{d}x = \frac{1}{16}\int_0^{n\pi} (1 - \cos 4x - 2\cos 2x \sin^2 2x)\,\mathrm{d}x$

$= \displaystyle\frac{1}{16}\int_0^{n\pi} (1-\cos 4x)\,\mathrm{d}x - \frac{1}{16}\int_0^{n\pi} \sin^2 2x \cdot 2\cos 2x\,\mathrm{d}x$ 1M

Let $u = \sin 2x$. Then $\mathrm{d}u = 2\cos 2x\,\mathrm{d}x$.

When $x = 0$, $u = 0$; when $x = n\pi$, $u = 0$.

$\displaystyle\int_0^{n\pi} \cos^2 x \sin^4 x\,\mathrm{d}x = \frac{1}{16}\left[x - \frac{\sin 4x}{4}\right]_0^{n\pi} - 0$ 1M+1A

$= \dfrac{n\pi}{16}$ 1

(c) Let $u = k - x$. Then $\mathrm{d}u = -\,\mathrm{d}x$. 1M

When $x = 0$, $u = k$; when $x = k$, $u = 0$.

$$\int_0^k x f(x)\,\mathrm{d}x = -\int_k^0 (k-u) f(k-u)\,\mathrm{d}u$$

$$= \int_0^k (k-u) f(u)\,\mathrm{d}u \qquad \text{1M+1M}$$

$$2\int_0^k x f(x)\,\mathrm{d}x = k\int_0^k f(x)\,\mathrm{d}x$$

$$\int_0^k x f(x)\,\mathrm{d}x = \frac{k}{2}\int_0^k f(x)\,\mathrm{d}x \qquad \text{1}$$

(d) Required volume $= \pi \int_\pi^{2\pi} y\cos^2 y \sin^4 y\,\mathrm{d}y$ 1M

Let $f(y) = \cos^2 y \sin^4 y$.

For $r = 1$ or 2, $f(r\pi - y) = \cos^2(r\pi - y)\sin^4(r\pi - y) = \cos^2 y \sin^4 y = f(y)$. 1M

Required volume $= \pi \int_0^{2\pi} y\cos^2 y\sin^4 y\,\mathrm{d}y - \pi\int_0^\pi y\cos^2 y \sin^4 y\,\mathrm{d}y$

$$= \pi\cdot\frac{2\pi}{2}\int_0^{2\pi}\cos^2 y\sin^4 y\,\mathrm{d}y - \pi\cdot\frac{\pi}{2}\int_0^\pi \cos^2 y\sin^4 y\,\mathrm{d}y \qquad \text{1M}$$

$$= \pi^2\times\frac{2\pi}{16} - \frac{\pi^2}{2}\times\frac{\pi}{16}$$

$$= \frac{3\pi^3}{32} \qquad \text{1A}$$

13. (a) (i) $\int \sin^4 x\,\mathrm{d}x = \int (\sin x)\sin^3 x\,\mathrm{d}x$

$$= -\cos x\sin^3 x + \int \cos x(3\sin^2 x\cos x)\,\mathrm{d}x \qquad \text{1M}$$

$$= -\cos x\sin^3 x + 3\int (1-\sin^2 x)\sin^2 x\,\mathrm{d}x$$

$$4\int \sin^4 x\,\mathrm{d}x = -\cos x\sin^3 x + 3\int \sin^2 x\,\mathrm{d}x$$

$$\int \sin^4 x\,\mathrm{d}x = -\frac{\cos x\sin^3 x}{4} + \frac{3}{4}\int \sin^2 x\,\mathrm{d}x \qquad \text{1}$$

(ii) $\int_0^\pi \sin^4 x\,\mathrm{d}x = -\frac{1}{4}\Big[\cos x\sin^3 x\Big]_0^\pi + \frac{3}{4}\int_0^\pi \sin^2 x\,\mathrm{d}x$

$$= 0 + \frac{3}{8}\int_0^\pi (1-\cos 2x)\,\mathrm{d}x \qquad \text{1M}$$

$$= \frac{3}{8}\left[x - \frac{\sin 2x}{2}\right]_0^\pi \qquad \text{1M}$$

$$= \frac{3\pi}{8} \qquad \text{1A}$$

(b) (i) Let $u = \beta - x$. Then $\mathrm{d}u = -\,\mathrm{d}x$.

$$\int_0^{\beta} x f(x)\,\mathrm{d}x = -\int_{\beta}^{0} (\beta - u) f(\beta - u)\,\mathrm{d}u$$

$$= \beta \int_0^{\beta} f(u)\,\mathrm{d}u - \int_0^{\beta} u f(u)\,\mathrm{d}u \qquad \text{1M}$$

$$2\int_0^{\beta} x f(x)\,\mathrm{d}x = \beta \int_0^{\beta} f(x)\,\mathrm{d}x$$

$$\int_0^{\beta} x f(x)\,\mathrm{d}x = \frac{\beta}{2}\int_0^{\beta} f(x)\,\mathrm{d}x \qquad \text{1}$$

(ii) $\sin^4 x$ is continuous on $\mathbb{R}$ and $\sin^4(\pi - x) = \sin^4 x$ 1M

$$\int_0^{\pi} x \sin^4 x\,\mathrm{d}x = \frac{\pi}{2}\int_0^{\pi} \sin^4 x\,\mathrm{d}x \qquad \text{1M}$$

$$= \frac{\pi}{2}\cdot\frac{3\pi}{8}$$

$$= \frac{3\pi^2}{16} \qquad \text{1M}$$

(c) Required volume $= \pi \int_{\pi}^{2\pi} (\sqrt{x}\sin^2 x)^2\,\mathrm{d}x$ 1M

$$= \pi \int_{\pi}^{2\pi} x \sin^4 x\,\mathrm{d}x$$

Let $x = u + \pi$. Then $\mathrm{d}u = \mathrm{d}x$.

Volume $= \pi \int_0^{\pi} (u + \pi)\sin^4(u + \pi)\,\mathrm{d}u$ 1M

$$= \pi \int_0^{\pi} u \sin^4 u\,\mathrm{d}u + \pi^2 \int_0^{\pi} \sin^4 u\,\mathrm{d}u$$

$$= \pi \cdot \frac{3\pi^2}{16} + \pi^2 \cdot \frac{3\pi}{8}$$

$$= \frac{9\pi^3}{16} \qquad \text{1A}$$

14. (a) $x^2 - 4(9 - 3x) + 8 = 0$

$x^2 + 12x - 28 = 0$ 1M

$x = 2$ or -14 (rejected)

The coordinates of B are $(2,\ 3)$. 1A

(b) Required capacity

$$= \pi \int_0^3 \left(\frac{9 - y}{3}\right)^2 \mathrm{d}y + \pi \int_3^h (4y - 8)\,\mathrm{d}y \qquad \text{2M+1A}$$

$$= \pi \int_0^3 \left(9 - 2y + \frac{y^2}{9}\right) \mathrm{d}y + \pi \int_3^h (4y - 8)\,\mathrm{d}y$$

$$= \pi \left[9y - y^2 + \frac{y^3}{27}\right]_0^3 + \pi \left[2y^2 - 8y\right]_3^h \qquad \text{1M}$$

$$= \pi(2h^2 - 8h + 25) \qquad \text{1}$$

(c) (i) Put $x = 6$ in $x^2 - 4y + 8 = 0$, we have $y = 11$. 1M

The rqeuired capacity $= \pi[2(11)^2 - 8(11) + 25]$

$= 179\pi \text{ cm}^3$ 1A

(ii) Let h cm be the depth of water in the cup at time t s.

Let p cm be the depth of water when the volume of water in the cup is $35\pi \text{ cm}^3$.

Since the volume of frustum (the lower part) is $19\pi \text{ cm}^3$, $p > 3$. 1M

By (a)(ii), we have

$$\pi(2p^2 - 8x + 25) = 35\pi$$

$$p^2 - 4p - 5 = 0 \qquad \text{1M}$$

$$p = 5 \quad \text{or} \quad -1 \text{ (rejected)}$$

Let $V \text{ cm}^3$ be the volume of water in the cup at time t s.

For $3 < h \le 11$, we have $V = \pi(2h^2 - 8h + 25)$, and $\dfrac{dV}{dt} = \pi(4h - 8)\dfrac{dh}{dt}$ 1M

When $h = 5$,

$$24\pi = \pi[4(5) - 8]\frac{dh}{dt}$$

$$\frac{dh}{dt} = 2 \qquad \text{1A}$$

Required rate of change is 2 cm/s.

15. (a) (i) $\displaystyle\int_0^{\pi} [f(x) + f(\pi - x)]\, dx = \int_0^{\pi} [x \sin x + (\pi - x)\sin(\pi - x)]\, dx$ 1M

$\displaystyle = \int_0^{\pi} [x \sin x + (\pi - x)\sin x]\, dx$ 1A

$\displaystyle = \pi \int_0^{\pi} \sin x\, dx$ 1A

$\displaystyle = \pi\Big[-\cos x\Big]_0^{\pi}$ 1A

$= 2\pi$ 1

(ii) By consider the symmetry of the graphs,

$$\int_0^{\pi} f(x)\, dx = \int_0^{\pi} f(\pi - x)\, dx$$

$$\int_0^{\pi} f(x)\, dx = \frac{1}{2}\int_0^{\pi} [f(x) + f(\pi - x)]\, dx$$

$$= \pi \qquad \text{1A}$$

(b) (i) $\dfrac{d}{dx}\left(x^2 \sin x\right) = x^2 \cos x + 2x \sin x$ 1A

$\displaystyle\int_0^{\pi} x^2 \cos x\, dx = \Big[x^2 \sin x\Big]_0^{\pi} - 2\int_0^{\pi} x \sin x\, dx$ 1M

$= -2\pi$ 1A

(ii) Volume $= \pi \int_0^{\pi} x^2 \sin^2 \frac{x}{2}\, dx$ 1M

$= \frac{\pi}{2} \int_0^{\pi} x^2(1 - \cos x)\, dx$

$= \frac{\pi}{2} \int_0^{\pi} x^2\, dx - \frac{\pi}{2} \int_0^{\pi} x^2 \cos x\, dx$

$= \frac{\pi}{2} \left[\frac{x^3}{3}\right]_0^{\pi} - \frac{\pi}{2}(-2\pi)$ 1M

$= \frac{\pi^4}{6} + \pi^2$ 1A

16. (a) $\frac{dy}{dx} = \frac{1}{\sqrt{x}} - 1$ 1A

Therefore, $\frac{1}{\sqrt{r}} - 1 = 0$ which gives $r = 1$. 1

(b) (i) Area under $C_1 = \int_0^1 (2\sqrt{x} - x)\, dx$ 1A

$= \left[\frac{4}{3}x^{\frac{3}{2}} - \frac{x^2}{2}\right]_0^1$

$= \frac{5}{6}$

Area under $C_2 = \int_2^3 (2\sqrt{3-x} - (3-x))\, dx$ 1M

$= \left[-\frac{4}{3}(3-x)^{\frac{3}{2}} - 3x + \frac{x^2}{2}\right]_2^3$

$= \frac{5}{6}$

Area of $S = 4 \times \frac{5}{6} + 2(1 \times 1) = \frac{16}{3}$ 1A

(ii) Volume $= \frac{\pi}{2} \int_0^1 (2\sqrt{x} - x)^2\, dx + (1)^2\pi(1) + \frac{\pi}{2} \int_2^3 [2\sqrt{3-x} - (3-x)]^2\, dx$ 1M+1M

$= \frac{\pi}{2} \int_0^1 (4x - 4x^{\frac{3}{2}} + x^2)\, dx + \frac{\pi}{2} + \frac{\pi}{2} \int_2^3 [4(3-x) - 4(3-x)^{\frac{3}{2}} + (3-x)^2]\, dx$

$= \frac{\pi}{2}\left[2x^2 - \frac{8}{5}x^{\frac{5}{2}} + \frac{x^3}{3}\right]_0^1 + \frac{\pi}{2} + \frac{\pi}{2}\left[-2(3-x)^2 + \frac{8}{5}(3-x)^{\frac{5}{2}} - \frac{(3-x)^3}{3}\right]_2^3$ 1A

$= \frac{37\pi}{30}$

(iii) Middle part is a half cylinder. Area of $S_1 = \frac{1}{2}\pi(1)^2 = \frac{\pi}{2}$.

Length of the middle part $= \frac{37\pi}{90} \div \frac{\pi}{2} = \frac{37}{45}$ 1M

$OQ = \frac{1}{2}\left(3 - \frac{37}{45}\right) = \frac{49}{45}$

Therefore, $OQ : OP = \frac{49}{45} : 3 = 49 : 135$ 1A

17. (a) Required volume $= \pi \int_{r-h}^{r} (r^2 - y^2)\,\mathrm{d}y$ 1M

$$= \pi \left[r^2 y - \frac{y^3}{3} \right]_{r-h}^{r}$$ 1A

$$= \pi r^2 h - \frac{\pi}{3} h[r^2 + r(r-h) + (r-h)^2]$$

$$= \pi r h^2 - \frac{\pi h^3}{3}$$ 1

(b) (i) $\pi(4)^2(10) = \pi(4)^2(H) + \left[\pi(3)h^2 - \frac{\pi h^3}{3} \right]$ 1M+1A

$$H = \frac{1}{48}(h^3 - 9h^2 + 480)$$ 1

(ii) (1) $\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{1}{48}(3h^2 - 18h)\frac{\mathrm{d}h}{\mathrm{d}t}$ 1M

When $h = 3$, $\frac{\mathrm{d}H}{\mathrm{d}t} = -\frac{9}{16}\frac{\mathrm{d}h}{\mathrm{d}t}$ 1A

We have $\frac{\mathrm{d}}{\mathrm{d}t}(H + h) = \frac{1}{4} = \frac{\mathrm{d}H}{\mathrm{d}t} + \frac{\mathrm{d}h}{\mathrm{d}t}$. 1A

Solving, we have $\frac{\mathrm{d}H}{\mathrm{d}t} = -\frac{9}{28}$ and $\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{4}{7}$.

Required rate is $-\frac{9}{28}$ cm/s. 1A

(2) Required rate is $\frac{4}{7}$ cm/s 1A

18. (a) Required volume $= \pi \int_{0}^{k} (2ay - y^2)\,\mathrm{d}y$ 1M+1A

$$= \pi \left[ay^2 - \frac{y^3}{3} \right]_{0}^{k}$$

$$= \pi \left(ak^2 - \frac{k^3}{3} \right)$$ 1

(b) (i) Required volume $= \pi \left[(9)(15)^2 - \frac{(15)^3}{3} \right] - 2\pi \left[(9)(3)^2 - \frac{(3)^3}{3} \right]$ 1M

$$= 756\pi \text{ cm}^3$$ 1A

(ii) (1) Let V_A cm^3 and V_B cm^3 be the volume of water in containers A and B respectively. When container A is just full,

$$909\pi - 756\pi = \pi \left(18h^2 - \frac{h^3}{3} \right)$$ 1M

$$h^3 - 54h^2 + 459 = 0$$ 1

(2) $$h^3 - 54h^2 + 459 = 0$$

$$(h-3)(h^2 - 51h - 153) = 0$$

$$h = 3 \quad \text{or} \quad \frac{51 \pm 3\sqrt{357}}{2} \text{ (rejected)}$$ 1A

$$\frac{\mathrm{d}V_B}{\mathrm{d}t} = \pi(36h - h^2)\frac{\mathrm{d}h}{\mathrm{d}t}$$ 1A

$$\frac{\mathrm{d}V_A}{\mathrm{d}t} = -\frac{\mathrm{d}V_B}{\mathrm{d}t} = \pi(h^2 - 36h)\frac{\mathrm{d}h}{\mathrm{d}t}$$ 1A

When container A is just full,

$$45\pi = \pi(3^2 - 36(3))\frac{dh}{dt} \qquad \text{1M}$$

$$\frac{dh}{dt} = -\frac{5}{11}$$

The rate of decrease of water level in container B is $\frac{5}{11}$ cm/s. 1A

19. (a) Volume $= \pi\int_0^p (6y - y^2)\,dy$ 1M

$$= \pi\left[3y^2 - \frac{y^3}{3}\right]_0^p \qquad \text{1A}$$

$$= \frac{\pi p^2(9-p)}{3} \qquad \text{1}$$

(b) (i) Volume of water $= \pi(6)^2(6) - 2\times\frac{4}{3}\pi(3)^3 = 144\pi\text{ cm}^3$ 1A

(ii) $\pi(6)^2h - \frac{\pi h^2(9-h)}{3} - \frac{\pi k^2(9-k)}{3} = 144\pi$ 1M

$$108h - 9h^2 + h^3 - 9k^2 + k^3 = 432$$

$$k^3 - 9k^2 + h^3 - 9h^2 + 108h - 432 = 0 \qquad \text{1}$$

(iii) When $h = 5$,

$$k^3 - 9k^2 + 8 = 0$$

$$(k-1)(k^2 - 8k - 8) = 0$$

$$k = 1 \quad \text{or} \quad 4 \pm 2\sqrt{6}\text{ (rejected)} \qquad \text{1A}$$

From (b)(ii),

$$k^3 - 9k^2 + h^3 - 9h^2 + 108h - 432 = 0$$

$$3k^2\frac{dk}{dt} - 18k\frac{dk}{dt} + 3h^2\frac{dh}{dt} - 18h\frac{dh}{dt} + 108\frac{dh}{dt} = 0 \qquad \text{1M}$$

When $h = 5$ and $\frac{dh}{dt} = -5$, $k = 1$ and

$$-15\frac{dk}{dt} + (93)(-5) = 0$$

$$\frac{dk}{dt} = -31 \qquad \text{1A}$$

$$v = \frac{d}{dt}(h-k) = -5 - (-31) = 26 \qquad \text{1M+1A}$$

END OF PAPER