M2SUM-DRILL-2425-ASM-SET 4-MATH
Suggested solutions

1. (a) Vertical asymptote is $x = 1$. 1A

$$f(x) = \frac{x^2(x+5)}{(x-1)^2}$$

$$= x + 7 + \frac{13}{x-1} + \frac{6}{(x-1)^2}$$ 1M

Oblique asymptote is $y = x + 7$. 1A

(b) $$f(x) = x + 7 + \frac{13}{x-1} + \frac{6}{(x-1)^2}$$

$$f'(x) = 1 - \frac{13}{(x-1)^2} - \frac{12}{(x-1)^3}$$ 1M

$$f''(x) = \frac{26}{(x-1)^3} + \frac{36}{(x-1)^4}$$

$$= \frac{2(13x+5)}{(x-1)^4}$$ 1A

(c) Note that $f'(x) = \dfrac{x(x-5)(x+2)}{(x-1)^3}$.

When $f'(x) = 0$, $x = 5$ or 0 or -2. 1A

x	$x < -2$	$-2 < x < 0$	$0 < x < 1$	$1 < x < 5$	$x > 5$
$f'(x)$	$+$	$-$	$+$	$-$	$+$

1M

Maximum point is $\left(-2, \dfrac{4}{3}\right)$. 1A

(d) When $f''(x) = 0$, $x = \dfrac{-5}{13}$.

x	$x < -\frac{5}{13}$	$-\frac{5}{13} < x < 1$	$x > 1$
$f''(x)$	$-$	$+$	$+$

1M

G has one point of inflexion at $x = -\dfrac{5}{13}$.

The claim is disagreed. 1A

(e) Let $u = x - 1$. Then $\mathrm{d}u = \mathrm{d}x$.

$$
\begin{aligned}
\text{Area} &= \int_{-5}^{0} f(x)\,\mathrm{d}x && \text{1M} \\
&= \int_{-5}^{0} \left(x + 7 + \frac{13x - 7}{(x-1)^2} \right) \mathrm{d}x \\
&= \left[\frac{x^2}{2} + 7x \right]_{-5}^{0} + \int_{-6}^{-1} \frac{13(u+1) - 7}{u^2}\,\mathrm{d}u \\
&= \frac{45}{2} + \int_{-6}^{-1} (13u^{-1} + 6u^{-2})\,\mathrm{d}u \\
&= \frac{45}{2} + \left[13 \ln |u| - 6u^{-1} \right]_{-6}^{-1} && \text{1M} \\
&= \frac{55}{2} - 13 \ln 6 && \text{1A}
\end{aligned}
$$

2. (a) $y = e^{-3x}$

$\frac{dy}{dx} = -3e^{-3x}$ 1M

Let $(a,\ 0)$ and $(0,\ b)$ be the coordinates of Q and R.

$\frac{e^{-3r} - 0}{r - a} = -3e^{-3r}$ 1M

$1 = -3(r - a)$

$a = \frac{1 + 3r}{3}$ 1A

The x-coordinate of Q is $\frac{1 + 3r}{3}$.

$\frac{e^{-3r} - b}{r - 0} = -3e^{-3r}$

$e^{-3r} - b = -3re^{-3r}$

$b = (1 + 3r)e^{-3r}$ 1A

The y-coordinate of R is $(1 + 3r)e^{-3r}$.

(b) Let A square units be the area of $\triangle OQR$.

$A = \frac{1}{2} \cdot \frac{1 + 3r}{3} \cdot (1 + 3r)e^{-3r}$ 1M

$= \frac{1}{6}(1 + 3r)^2 e^{-3r}$ 1A

$\frac{dA}{dr} = \frac{1}{6}(1 + 3r)^2 e^{-3r}(-3) + (1 + 3r)e^{-3r}$ 1M

$= \frac{1}{2}(1 + 3r)(1 - 3r)e^{-3r}$

When $\frac{dA}{dr} = 0$, $r = \frac{1}{3}$ or $-\frac{1}{3}$ (rejected).

r	$0 < r < \frac{1}{3}$	$r > \frac{1}{3}$
$\frac{dA}{dr}$	$+$	$-$

1M

A is the greatest when $r = \frac{1}{3}$.

Required area $= \frac{1}{6}\left[1 + 3\left(\frac{1}{3}\right)\right]^2 e^{-3\left(\frac{1}{3}\right)}$

$= \frac{2}{3e}$ square units 1A

(c) $OR = (1 + 3r)e^{-3r}$

$\frac{dOR}{dt} = [3e^{-3r} - 3(1 + 3r)e^{-3r}]\frac{dr}{dt}$ 1M

$= -9re^{-3r}\frac{dr}{dt}$

$$\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt}$$

$$= \left[\frac{1}{2}(1+3r)(1-3r)e^{-3r}\right]\left[\frac{1}{-9re^{-3r}}\frac{dOR}{dt}\right]$$

$$= -\frac{(1+3r)(1-3r)}{18r} \cdot \frac{dOR}{dt}$$

When $r = \frac{1}{6}$.

$$\frac{dA}{dt} = -\frac{\left(1+\frac{1}{2}\right)\left(1-\frac{1}{2}\right)}{3} \cdot \frac{dOR}{dt}$$ 1M

$$= -\frac{1}{4}\frac{dOR}{dt}$$

Note that $0 \le -\frac{dOR}{dt} \le 1$.

$$0 \le \frac{dA}{dt} \le \frac{1}{4} < \frac{1}{3}$$

The claim is agreed. 1A

3. (a) Note that $x^2 - x + 1 = \left(x - \frac{1}{2}\right)^2 + \frac{3}{4}$.

(b) Let $x - \frac{1}{2} = \frac{\sqrt{3}}{2}\tan\theta$. Then $\mathrm{d}x = \frac{\sqrt{3}}{2}\sec^2\theta\,\mathrm{d}\theta$. 1M

$$\int_{-1}^{1} \frac{\mathrm{d}x}{x^2 - x + 1} = \int_{-\frac{\pi}{3}}^{\frac{\pi}{6}} \frac{\frac{\sqrt{3}}{2}\sec^2\theta}{\frac{3}{4}\tan^2\theta + \frac{3}{4}}\,\mathrm{d}\theta \quad \text{1M}$$

$$= \frac{2\sqrt{3}}{3}\int_{-\frac{\pi}{3}}^{\frac{\pi}{6}} \mathrm{d}\theta$$

$$= \frac{2\sqrt{3}}{3}\Big[\theta\Big]_{-\frac{\pi}{3}}^{\frac{\pi}{6}}$$

$$= \frac{\sqrt{3}\pi}{3} \quad \text{1A}$$

(c) Let $u = a - x$. Then $\mathrm{d}u = -\,\mathrm{d}x$.

$$\int_0^a f(x)\,\mathrm{d}x = \int_0^{\frac{a}{2}} f(x)\,\mathrm{d}x + \int_{\frac{a}{2}}^{a} f(x)\,\mathrm{d}x \quad \text{1M}$$

$$= \int_0^{\frac{a}{2}} f(x)\,\mathrm{d}x - \int_{\frac{a}{2}}^{0} f(a-u)\,\mathrm{d}u$$

$$= \int_0^{\frac{a}{2}} f(x)\,\mathrm{d}x + \int_0^{\frac{a}{2}} f(u)\,\mathrm{d}u$$

$$= 2\int_0^{\frac{a}{2}} f(x)\,\mathrm{d}x \quad \text{1}$$

(d) (i) $\sin 2x = 2\sin x\cos x$

$$= \frac{2\cdot\frac{\sin x}{\cos x}}{\sec^2 x}$$

$$= \frac{2\tan x}{1 + \tan^2 x}$$

$$= \frac{2t}{1 + t^2} \quad \text{1}$$

(ii) Let $u = x - \frac{\pi}{4}$. Then $\mathrm{d}u = \mathrm{d}x$. 1M

$$\int_0^{\frac{\pi}{2}} \frac{1}{2 + \cos 2x}\,\mathrm{d}x = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2 + \cos\left(\frac{\pi}{2} + 2u\right)}\,\mathrm{d}u$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2 - \sin 2u}\,\mathrm{d}u$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2 - \sin 2x}\,\mathrm{d}x \quad \text{1}$$

(iii) Let $t = \tan x$. Then $\mathrm{d}t = \sec^2 x \,\mathrm{d}x = (1 + t^2)\,\mathrm{d}x$.

$$\int_0^{\frac{\pi}{2}} \frac{1}{2+\cos 2x}\,\mathrm{d}x = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2-\sin 2x}\,\mathrm{d}x$$

$$= \int_{-1}^{1} \frac{1}{2-\frac{2t}{1+t^2}} \cdot \frac{1}{1+t^2}\,\mathrm{d}t \qquad \text{1M}$$

$$= \frac{1}{2}\int_{-1}^{1} \frac{1}{t^2-t+1}\,\mathrm{d}t$$

$$= \frac{\sqrt{3}\pi}{6} \qquad \text{1M}$$

(e) Put $f(x) = \dfrac{1}{2+\cos 2x}$.

$$f(\pi - x) = \frac{1}{2+\cos(2\pi - 2x)} = \frac{1}{2+\cos 2x} = f(x) \qquad \text{1M}$$

Use the result of (b) and (c)(iii).

$$\int_0^{\pi} \frac{1}{2+\cos 2x}\,\mathrm{d}x = 2\int_0^{\frac{\pi}{2}} \frac{1}{2+\cos 2x}\,\mathrm{d}x$$

$$= 2\left(\frac{\sqrt{3}\pi}{6}\right) \qquad \text{1M}$$

$$= \frac{\sqrt{3}\pi}{3}$$

Note that $f(x+n\pi) = \dfrac{1}{2+\cos(2x+2n\pi)} = \dfrac{1}{2+\cos 2x} = f(x)$ for all positive integers n.

$$\int_0^{2022\pi} \frac{1}{2+\cos 2x}\,\mathrm{d}x$$

$$= \int_0^{\pi} f(x)\,\mathrm{d}x + \int_{\pi}^{2\pi} f(x)\,\mathrm{d}x + \ldots + \int_{2021\pi}^{2022\pi} f(x)\,\mathrm{d}x$$

$$= \int_0^{\pi} f(x)\,\mathrm{d}x + \int_{\pi}^{2\pi} f(x+\pi)\,\mathrm{d}x + \ldots + \int_{2021\pi}^{2022\pi} f(x+2021\pi)\,\mathrm{d}x$$

$$= 2022\int_0^{\pi} f(x)\,\mathrm{d}x$$

$$= 2022\left(\frac{\sqrt{3}\pi}{3}\right)$$

$$= 674\sqrt{3}\pi \qquad \text{1A}$$

4. (a) $\displaystyle\int_0^{\pi} \cos^4 x \, dx$

$$= \int_0^{\pi} \frac{1}{4}(1 + \cos 2x)^2 \, dx \qquad \text{1M}$$

$$= \int_0^{\pi} \frac{1}{4}(1 + 2\cos 2x + \cos^2 2x) \, dx$$

$$= \int_0^{\pi} \frac{1}{4}\left[1 + 2\cos 2x + \frac{1}{2}(1 + \cos 4x)\right] dx \qquad \text{1M}$$

$$= \int_0^{\pi} \left(\frac{3}{8} + \frac{\cos 2x}{2} + \frac{\cos 4x}{8}\right) dx$$

$$= \left[\frac{3x}{8}] + \frac{\sin 2x}{4} + \frac{\sin 4x}{32}\right]_0^{\pi}$$

$$= \frac{3\pi}{8} \qquad \text{1A}$$

(b) Let $u = k - x$. Then $du = -dx$. 1M

$$\int_0^k f(x)g(x) \, dx = -\int_k^0 f(k-u)g(k-u) \, du$$

$$= \int_0^k f(u)g(k-u) \, du$$

$$= \int_0^k f(x)g(k-x) \, dx$$

$$= \frac{1}{2}\left[\int_0^k f(x)g(x) \, dx + \int_0^k f(x)g(k-x) \, dx\right] \qquad \text{1M}$$

$$= \frac{1}{2}\int_0^k f(x)[g(x) + g(k-x)] \, dx$$

$$= \frac{a}{2}\int_0^k f(x) \, dx \qquad \text{1}$$

(c) (i) Put $f(x) = \cos^4 x$, $g(x) = \dfrac{1}{1 + e^{\sin x}}$ and $k = 2\pi$.

$f(2\pi - x) = \cos^4(2\pi - x) = \cos^4 x = f(x)$

$$g(x) + g(2\pi - x) = \frac{1}{1 + e^{\sin x}} + \frac{1}{1 + e^{\sin(2\pi - x)}}$$

$$= \frac{1}{1 + e^{\sin x}} + \frac{1}{1 + e^{-\sin x}} \cdot \frac{e^{\sin x}}{e^{\sin x}}$$

$$= 1 \qquad \text{1M}$$

Use the result of (b).

$$\int_0^{2\pi} \frac{\cos^4 x}{1 + e^{\sin x}} \, dx = \frac{1}{2}\int_0^{2\pi} \cos^4 x \, dx \qquad \text{1M}$$

$$= \frac{1}{2}\int_0^{\pi} \cos^4 x \, dx + \frac{1}{2}\int_{\pi}^{2\pi} \cos^4 x \, dx$$

Let $u = 2\pi - x$. Then $\mathrm{d}u = -\,\mathrm{d}x$. 1M

$$\begin{aligned}\int_0^{2\pi} \frac{\cos^4 x}{1 + e^{\sin x}}\,\mathrm{d}x &= \frac{1}{2}\int_0^{\pi} \cos^4 x\,\mathrm{d}x - \frac{1}{2}\int_{\pi}^{0} \cos^4(2\pi - u)\,\mathrm{d}u \\ &= \frac{1}{2}\int_0^{\pi} \cos^4 x\,\mathrm{d}x + \frac{1}{2}\int_0^{\pi} \cos^4 u\,\mathrm{d}u \\ &= \int_0^{\pi} \cos^4 x\,\mathrm{d}x \\ &= \frac{3\pi}{8}\end{aligned}$$ 1A

(ii) $$\begin{aligned}\int_0^{2\pi} \frac{e^{\sin x}\cos^4 x}{1 + e^{\sin x}}\,\mathrm{d}x &= \int_0^{2\pi} \frac{(1 + e^{\sin x} - 1)\cos^4 x}{1 + e^{\sin x}}\,\mathrm{d}x \\ &= \int_0^{2\pi} \cos^4 x\,\mathrm{d}x - \int_0^{2\pi} \frac{\cos^4 x}{1 + e^{\sin x}}\,\mathrm{d}x \quad \text{1M} \\ &= 2\left(\frac{3\pi}{8}\right) - \frac{3\pi}{8} \\ &= \frac{3\pi}{8}\end{aligned}$$ 1A

5. (a) (i) Let $x = 2\tan\theta$. Then $\mathrm{d}x = 2\sec^2\theta\,\mathrm{d}\theta$. 1M

$$\int \frac{1}{x^2+4}\,\mathrm{d}x = \int \frac{2\sec^2\theta}{4\tan^2\theta+4}\,\mathrm{d}\theta$$

$$= \frac{1}{2}\int \mathrm{d}\theta \quad \text{1M}$$

$$= \frac{1}{2}\theta + \text{constant}$$

$$= \frac{1}{2}\tan^{-1}\frac{x}{2} + \text{constant} \quad \text{1A}$$

(ii) $$\int \ln\left(x^2+4\right)\mathrm{d}x = x\ln\left(x^2+4\right) - \int \frac{2x^2}{x^2+4}\,\mathrm{d}x \quad \text{1M}$$

$$= x\ln\left(x^2+4\right) - 2\int\left(1-\frac{4}{x^2+4}\right)\mathrm{d}x$$

$$= x\ln\left(x^2+4\right) - 2x + 8\int\frac{\mathrm{d}x}{x^2+4}$$

$$= x\ln\left(x^2+4\right) - 2x + 4\tan^{-1}\frac{x}{2} + \text{constant} \quad \text{1A}$$

(b) Let $u = -x$. Then $\mathrm{d}u = -\,\mathrm{d}x$. 1M

$$\int_{-a}^{a} g(x)h(x)\,\mathrm{d}x = \int_{-a}^{0} g(x)h(x)\,\mathrm{d}x + \int_{0}^{a} g(x)h(x)\,\mathrm{d}x \quad \text{1M}$$

$$= -\int_{a}^{0} g(-u)h(-u)\,\mathrm{d}u + \int_{0}^{a} g(x)h(x)\,\mathrm{d}x$$

$$= \int_{0}^{a} g(-u)h(u)\,\mathrm{d}u + \int_{0}^{a} g(x)h(x)\,\mathrm{d}x$$

$$= \int_{0}^{a} [g(-x)+g(x)]h(x)\,\mathrm{d}x$$

$$= \int_{0}^{a} h(x)\,\mathrm{d}x \quad \text{1}$$

(c) Let $g(x) = \dfrac{7^x}{7^x+7^{-x}}$, then $g(x)+g(-x) = \dfrac{7^x}{7^x+7^{-x}} + \dfrac{7^{-x}}{7^{-x}+7^x} = 1$. 1M

Let $h(x) = \ln\left(x^2+4\right)$, then $h(-x) = \ln\left((-x)^2+4\right) = \ln\left(x^2+4\right) = h(x)$. 1M

$$\int_{-2}^{2} \frac{7^x\ln\left(x^2+4\right)}{7^x+7^{-x}}\,\mathrm{d}x = \int_{0}^{2}\ln\left(x^2+4\right)\mathrm{d}x \quad \text{1M}$$

$$= \left[x\ln\left(x^2+4\right) - 2x + 4\tan^{-1}\frac{x}{2}\right]_0^2$$

$$= 6\ln 2 - 4 + \pi \quad \text{1A}$$

6. (a) (i) The vertical asymptote is $x = -1$. 1A

The horizontal asymptote is $y = 0$. 1A

(ii) $$y = \frac{36x}{(x+1)^2}$$
$$= \frac{36}{x+1} - \frac{36}{(x+1)^2}$$
$$\frac{dy}{dx} = -\frac{36}{(x+1)^2} + \frac{72}{(x+1)^3} \quad \text{1M}$$
$$= \frac{36(1-x)}{(x+1)^3}$$

When $\frac{dy}{dx} = 0$, $x = 1$.

x	$x < -1$	$-1 < x < 1$	$x > 1$
$\frac{dy}{dx}$	$-$	$+$	$-$

1M

Local maximum point of G is $(1,\ 9)$. 1A

(b) (i) $$V = \pi \int_0^k \left(\frac{36x}{(x+1)^2}\right) dx \quad \text{1M}$$
$$= 1296\pi \int_0^k \frac{x^2}{(x+1)^4}\, dx$$

Let $u = x + 1$. Then $du = dx$. 1M

$$V = 1296\pi \int_1^{k+1} \frac{(u-1)^2}{u^4}\, du$$
$$= 1296\pi \int_1^{k+1} \left(\frac{1}{u^2} - \frac{2}{u^3} + \frac{1}{u^4}\right) du$$
$$= 1296\pi \left[-\frac{1}{u} + \frac{1}{u^2} - \frac{1}{3u^3} \right]_1^{k+1} \quad \text{1M}$$
$$= 1296\pi \left[-\frac{1}{k+1} + \frac{1}{(k+1)^2} - \frac{1}{3(k+1)^3} + \frac{1}{3} \right]$$
$$= 432\pi \cdot \frac{-3(k+1)^2 + 3(k+1) - 1 + (k+1)^3}{(k+1)^3}$$
$$= 432\pi \cdot \frac{[(k+1)-1]^3}{(k+1)^3}$$
$$= \frac{432k^3\pi}{(k+1)^3} \quad \text{1}$$

(ii) $$\frac{dV}{dt} = 432\pi \cdot \frac{3k^2(k+1)^3 - 3k^3(k+1)^2}{(k+1)^6} \cdot \frac{dk}{dt} \quad \text{1M}$$
$$= \frac{1296k^2\pi}{(k+1)^4} \cdot \frac{dk}{dt}$$
$$16 = \frac{1296(2)^2\pi}{(2+1)^4} \cdot \frac{dk}{dt} \quad \text{1M}$$
$$\frac{dk}{dt} = \frac{1}{4\pi}$$
$$< 0.08$$

The claim is incorrect. 1A

END OF PAPER