M2REG-DEFINT-2324-ASM-SET 6-MATH
Suggested solutions

1. (a) Let $u = a - x$. Then $\mathrm{d}u = -\,\mathrm{d}x$. 1M

$$\int_0^a f(a-x)\,\mathrm{d}x = -\int_a^0 f(u)\,\mathrm{d}u$$
$$= \int_0^a f(u)\,\mathrm{d}u \qquad \text{1M}$$
$$= \int_0^a f(x)\,\mathrm{d}x \qquad \text{1}$$

We have $\int_0^a f(x)\,\mathrm{d}x = \int_0^a f(a-x)\,\mathrm{d}x$.

$$\int_0^a f(x)\,\mathrm{d}x = \frac{1}{2}\left[\int_0^a f(x)\,\mathrm{d}x + \int_0^a f(a-x)\,\mathrm{d}x\right]$$
$$= \frac{1}{2}\int_0^a [f(x) + f(a-x)]\,\mathrm{d}x \qquad \text{1}$$

(b) $\displaystyle\int_0^3 \frac{\mathrm{d}x}{e^{3-2x}+1}$

$$= \frac{1}{2}\int_0^3 \left[\frac{1}{e^{3-2x}+1} + \frac{1}{e^{3-2(3-x)}+1}\right]\mathrm{d}x \qquad \text{1M}$$
$$= \frac{1}{2}\int_0^3 \left[\frac{1}{e^{3-2x}+1} + \frac{1}{e^{2x-3}+1} \times \frac{e^{3-2x}}{e^{3-2x}}\right]\mathrm{d}x$$
$$= \frac{1}{2}\int_0^3 \frac{1+e^{3-2x}}{e^{3-2x}+1}\,\mathrm{d}x \qquad \text{1M}$$
$$= \frac{1}{2}\Big[x\Big]_0^3$$
$$= \frac{3}{2} \qquad \text{1A}$$

2. (a) (i) $\cos\theta(2\cos 4\theta - 2\cos 2\theta + 1) = 2\cos\theta\cos 4\theta - 2\cos\theta\cos 2\theta + \cos\theta$

$= (\cos 5\theta + \cos 3\theta) - (\cos 3\theta + \cos\theta) + \cos\theta$ 1M

$= \cos 5\theta$

Thus, $\dfrac{\cos 5\theta}{\cos\theta} = 2\cos 4\theta - 2\cos 2\theta + 1$. 1

(ii) $\dfrac{\cos 5\left(\frac{\pi}{4} - t\right)}{\cos\left(\frac{\pi}{4} - t\right)} = 2\cos\left(\dfrac{\pi}{4} - t\right) - 2\cos 2\left(\dfrac{\pi}{4} - t\right) + 1$

$\dfrac{\cos\frac{5\pi}{4}\cos 5t + \sin\frac{5\pi}{4}\sin 5t}{\cos\frac{\pi}{4}\cos t + \sin\frac{\pi}{4}\sin t} = -2\cos 4t - 2\sin 2t + 1$ 1M

$\dfrac{-\frac{\sqrt{2}}{2}\cos 5t - \frac{\sqrt{2}}{2}\sin 5t}{\frac{\sqrt{2}}{2}\cos t + \frac{\sqrt{2}}{2}\sin t} = -2\cos 4t - 2\sin 2t + 1$

$\dfrac{\cos 5t + \sin 5t}{\cos t + \sin t} = 2\cos 4t + 2\sin 2t - 1$ 1

(b) (i) Let $u = a - x$. Then $\mathrm{d}u = -\,\mathrm{d}x$. 1M

$\displaystyle\int_0^a f(a-x)\,\mathrm{d}x = -\int_a^0 f(u)\,\mathrm{d}u$

$\displaystyle = \int_0^a f(u)\,\mathrm{d}u$ 1M

$\displaystyle = \int_0^a f(x)\,\mathrm{d}x$ 1

(ii) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos 5x}{\cos x + \sin x}\,\mathrm{d}x = \int_0^{\frac{\pi}{2}} \frac{\cos 5\left(\frac{\pi}{2} - x\right)}{\cos\left(\frac{\pi}{2} - x\right) + \sin\left(\frac{\pi}{2} - x\right)}\,\mathrm{d}x$ 1M

$\displaystyle = \int_0^{\frac{\pi}{2}} \frac{\sin 5x}{\cos x + \sin x}\,\mathrm{d}x$ 1

(c) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos 5x}{\cos x + \sin x}\,\mathrm{d}x$

$\displaystyle = \frac{1}{2}\left[\int_0^{\frac{\pi}{2}} \frac{\cos 5x}{\cos x + \sin x}\,\mathrm{d}x + \int_0^{\frac{\pi}{2}} \frac{\sin 5x}{\cos x + \sin x}\,\mathrm{d}x\right]$ 1M

$\displaystyle = \frac{1}{2}\int_0^{\frac{\pi}{2}} (2\cos 4x + 2\sin 2x - 1)\,\mathrm{d}x$ 1M

$\displaystyle = \frac{1}{2}\left[\frac{\sin 4x}{2} - \cos 2x - x\right]_0^{\frac{\pi}{2}}$

$\displaystyle = 1 - \frac{\pi}{4}$ 1A

3. (a) Let $u = a - x$. Then $\mathrm{d}u = -\,\mathrm{d}x$. 1M

$$\int_0^a f(a-x)\,\mathrm{d}x = -\int_a^0 f(u)\,\mathrm{d}u$$

$$= \int_0^a f(u)\,\mathrm{d}u \qquad \text{1M}$$

$$= \int_0^a f(x)\,\mathrm{d}x \qquad \text{1}$$

(b) $$\int_0^{\frac{\pi}{4}} \ln(\cos x)\,\mathrm{d}x = \int_0^{\frac{\pi}{4}} \ln\left[\cos\left(\frac{\pi}{4} - x\right)\right]\mathrm{d}x \qquad \text{1M}$$

$$= \int_0^{\frac{\pi}{4}} \ln\left(\cos\frac{\pi}{4}\cos x + \sin\frac{\pi}{4}\sin x\right)\mathrm{d}x \qquad \text{1M}$$

$$= \int_0^{\frac{\pi}{4}} \ln\frac{\cos x + \sin x}{\sqrt{2}}\,\mathrm{d}x \qquad \text{1}$$

(c) $$\int_0^{\frac{\pi}{4}} \ln(\cos x)\,\mathrm{d}x = \int_0^{\frac{\pi}{4}} \ln\frac{\cos x + \sin x}{\sqrt{2}}\,\mathrm{d}x$$

$$= \int_0^{\frac{\pi}{4}} \ln(\cos x + \sin x)\,\mathrm{d}x - \int_0^{\frac{\pi}{4}} \sqrt{2}\,\mathrm{d}x \qquad \text{1M}$$

$$\int_0^{\frac{\pi}{4}} \ln\frac{\cos x}{\cos x + \sin x}\,\mathrm{d}x = -\ln\sqrt{2}\Big[x\Big]_0^{\frac{\pi}{4}} \qquad \text{1M}$$

$$\int_0^{\frac{\pi}{4}} \frac{1}{1 + \tan x}\,\mathrm{d}x = -\frac{\pi\ln 2}{8} \qquad \text{1}$$

(d) Note that $\dfrac{\mathrm{d}}{\mathrm{d}x}\,[\ln(1+\tan x)] = \dfrac{\sec^2 x}{1+\tan x}$.

$$\int_0^{\frac{\pi}{4}} \frac{x\sec^2 x}{1+\tan x}\,\mathrm{d}x = \Big[x\ln(1+\tan x)\Big]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,\mathrm{d}x \qquad \text{1M}$$

$$= \frac{\pi\ln 2}{4} + \int_0^{\frac{\pi}{4}} \ln\frac{1}{1+\tan x}\,\mathrm{d}x$$

$$= \frac{\pi\ln 2}{4} + \left(-\frac{\pi\ln 2}{8}\right) \qquad \text{1M}$$

$$= \frac{\pi\ln 2}{8} \qquad \text{1A}$$

4. (a) Let $u = \pi - x$. Then $\mathrm{d}u = -\,\mathrm{d}x$. 1M

$$\int_{\frac{\pi}{2}}^{\pi} e^{\frac{1}{1+\sin x}} f(x)\,\mathrm{d}x = -\int_{\frac{\pi}{2}}^{0} e^{\frac{1}{1+\sin(\pi-u)}} f(\pi-u)\,\mathrm{d}u$$

$$= \int_{0}^{\frac{\pi}{2}} e^{\frac{1}{1+\sin u}} [-f(u)]\,\mathrm{d}u \qquad \text{1M}$$

$$= -\int_{0}^{\frac{\pi}{2}} e^{\frac{1}{1+\sin x}} f(x)\,\mathrm{d}x \qquad 1$$

$$\int_{0}^{\pi} e^{\frac{1}{1+\sin x}} f(x)\,\mathrm{d}x = \int_{0}^{\frac{\pi}{2}} e^{\frac{1}{1+\sin x}} f(x)\,\mathrm{d}x + \int_{\frac{\pi}{2}}^{\pi} e^{\frac{1}{1+\sin x}} f(x)\,\mathrm{d}x$$

$$= \int_{0}^{\frac{\pi}{2}} e^{\frac{1}{1+\sin x}} f(x)\,\mathrm{d}x - \int_{0}^{\frac{\pi}{2}} e^{\frac{1}{1+\sin x}} f(x)\,\mathrm{d}x$$

$$= 0 \qquad 1$$

(b) Let $v = \pi - x$. Then $\mathrm{d}v = -\,\mathrm{d}x$. 1M

$$\int_{0}^{\pi} \sin x g(x)\,\mathrm{d}x = \int_{0}^{\frac{\pi}{2}} \sin x g(x)\,\mathrm{d}x + \int_{\frac{\pi}{2}}^{\pi} \sin x g(x)\,\mathrm{d}x$$

$$= \int_{0}^{\frac{\pi}{2}} \sin x g(x)\,\mathrm{d}x - \int_{\frac{\pi}{2}}^{0} \sin(\pi-u) g(\pi-u)\,\mathrm{d}u$$

$$= \int_{0}^{\frac{\pi}{2}} \sin x g(x)\,\mathrm{d}x + \int_{0}^{\frac{\pi}{2}} \sin u g(u)\,\mathrm{d}u \qquad \text{1M}$$

$$= \int_{0}^{\frac{\pi}{2}} \sin x g(x)\,\mathrm{d}x + \int_{0}^{\frac{\pi}{2}} \sin x g(x)\,\mathrm{d}x$$

$$= 2\int_{0}^{\frac{\pi}{2}} \sin x g(x)\,\mathrm{d}x \qquad 1$$

(c) (i) Put $f(x) = \cos x \sin x$.

$f(\pi - x) = \cos(\pi - x)\sin(\pi - x) = -\cos x \sin x = -f(x)$ 1M

$$\int_{0}^{\pi} e^{\frac{1}{1+\sin x}} \cos x \sin x\,\mathrm{d}x = 0 \qquad \text{1A}$$

Let $w = \cos x$. Then $\mathrm{d}w = -\sin x\,\mathrm{d}x$. 1M

$$\int_{0}^{\frac{\pi}{2}} \sin x \cos^2 x\,\mathrm{d}x = -\int_{1}^{0} u^2\,\mathrm{d}u$$

$$= -\frac{1}{3}\left[u^3\right]_{1}^{0} \qquad \text{1A}$$

$$= \frac{1}{3} \qquad \text{1A}$$

(ii) $$\int_{0}^{\pi} \sin 2x(\cos x + e^{\frac{1}{1+\sin x}})\,\mathrm{d}x = \int_{0}^{\pi} (\sin 2x \cos x + e^{\frac{1}{1+\sin x}} \sin 2x)\,\mathrm{d}x$$

$$= 2\int_{0}^{\pi} \sin x \cos^2 x\,\mathrm{d}x + 2\int_{0}^{\pi} e^{\frac{1}{1+\sin x}} \sin x \cos x\,\mathrm{d}x$$

$$= 2\int_{0}^{\pi} \sin x \cos^2 x\,\mathrm{d}x \qquad \text{1M}$$

Put $g(x) = \cos^2 x$. $g(\pi - x) = \cos^2(\pi - x) = \cos^2 x = g(x)$. 1M

$$\begin{aligned}\int_0^{\pi} \sin 2x(\cos x + e^{\frac{1}{1+\sin x}})\,\mathrm{d}x &= 2\int_0^{\pi} \sin x \cos^2 x\,\mathrm{d}x \\ &= 4\int_0^{\frac{\pi}{2}} \sin x \cos^2 x\,\mathrm{d}x \\ &= 4\left(\frac{1}{3}\right) \\ &= \frac{4}{3}\end{aligned}$$ 1A

5. (a) Let $u = x^3$. Then $\mathrm{d}u = 3x^2\,\mathrm{d}x$. 1M

$$\begin{aligned}\int x^2 \sin x^3\,\mathrm{d}x &= \frac{1}{3}\int \sin u\,\mathrm{d}u \\ &= -\frac{\cos u}{3} + \text{constant} \\ &= -\frac{\cos x^3}{3} + \text{constant}\end{aligned}$$ 1A

(i) Let $u = a + b - x$. Then $\mathrm{d}u = -\,\mathrm{d}x$. 1M

$$\int_a^b \frac{f(x)}{f(a+b-x)+f(x)}\,\mathrm{d}x$$

$$= -\int_b^a \frac{f(a+b-u)}{f(u)+f(a+b-u)}\,\mathrm{d}u$$

$$= \int_a^b \frac{f(a+b-u)}{f(u)+f(a+b-u)}\,\mathrm{d}u$$ 1M

$$= \int_a^b \frac{f(a+b-x)}{f(x)+f(a+b-x)}\,\mathrm{d}x$$ 1A

$$\int_a^b \frac{f(x)}{f(a+b-x)+f(x)}\,\mathrm{d}x$$

$$= \frac{1}{2}\left[\int_a^b \frac{f(x)}{f(a+b-x)+f(x)}\,\mathrm{d}x + \int_a^b \frac{f(a+b-x)}{f(a+b-x)+f(x)}\,\mathrm{d}x\right]$$ 1M

$$= \frac{1}{2}\int_a^b \mathrm{d}x$$ 1A

$$= \frac{1}{2}\Big[x\Big]_a^b$$

$$= \frac{b-a}{2}$$ 1

(ii) Let $u = x^3$. Then $\mathrm{d}u = 3x^2\,\mathrm{d}x$. 1M

$$\int_{\sqrt[3]{\log 5}}^{\sqrt[3]{\log 6}} \frac{x^2 \sin x^3}{\sin(\log 30 - x^3) + \sin x^3}\,\mathrm{d}x$$

$$= \frac{1}{3}\int_{\log 5}^{\log 6} \frac{\sin u}{\sin(\log 5 + \log 6 - u) + \sin u}\,\mathrm{d}u$$ 1M

$$= \frac{1}{3} \cdot \frac{\log 6 - \log 5}{2}$$ 1M

$$= \frac{1}{6}\log\frac{6}{5}$$ 1A

6. (a) Let $u = \pi - x$. Then $\mathrm{d}u = -\,\mathrm{d}x$. 1M

$$\int_0^{\pi} xf(\sin x)\,\mathrm{d}x = -\int_{\pi}^{0} (\pi - u)f(\sin(\pi - u))\,\mathrm{d}u$$

$$= \int_0^{\pi} (\pi - u)f(\sin u)\,\mathrm{d}u \quad \text{1M}$$

$$= \pi\int_0^{\pi} f(\sin u)\,\mathrm{d}u - \int_0^{\pi} uf(\sin u)\,\mathrm{d}u$$

$$= \pi\int_0^{\pi} f(\sin x)\,\mathrm{d}x - \int_0^{\pi} xf(\sin x)\,\mathrm{d}x$$

$$2\int_0^{\pi} xf(\sin x)\,\mathrm{d}x = \pi\int_0^{\pi} f(\sin x)\,\mathrm{d}x$$

$$\int_0^{\pi} xf(\sin x)\,\mathrm{d}x = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\,\mathrm{d}x \quad \text{1}$$

(b) (i) Let $v = -x$. Then $\mathrm{d}v = -\,\mathrm{d}x$. 1M

$$\int_{-\pi}^{0} \frac{g(x)}{1+e^x}\,\mathrm{d}x = -\int_{\pi}^{0} \frac{g(-v)}{1+e^{-v}}\,\mathrm{d}v$$

$$= \int_0^{\pi} \frac{e^v g(v)}{e^v(1+e^{-v})}\,\mathrm{d}v$$

$$= \int_0^{\pi} \frac{e^v g(v)}{1+e^v}\,\mathrm{d}v$$

$$= \int_0^{\pi} \frac{e^x g(x)}{1+e^x}\,\mathrm{d}x \quad \text{1}$$

(ii) $$\int_{-\pi}^{\pi} \frac{g(x)}{1+e^x}\,\mathrm{d}x = \int_{-\pi}^{0} \frac{g(x)}{1+e^x}\,\mathrm{d}x + \int_0^{\pi} \frac{g(x)}{1+e^x}\,\mathrm{d}x$$

$$= \int_0^{\pi} \frac{e^x g(x)}{1+e^x}\,\mathrm{d}x + \int_0^{\pi} \frac{g(x)}{1+e^x}\,\mathrm{d}x$$

$$= \int_0^{\pi} \frac{(1+e^x)g(x)}{1+e^x}\,\mathrm{d}x$$

$$= \int_0^{\pi} g(x)\,\mathrm{d}x \quad \text{1}$$

(c) Put $g(x) = \dfrac{\sqrt{2+\cos x}}{\sqrt{2+\cos x}+\sqrt{2-\cos x}}$.

$$g(-x) = \frac{\sqrt{2+\cos(-x)}}{\sqrt{2+\cos(-x)}+\sqrt{2-\cos(-x)}} = \frac{\sqrt{2+\cos x}}{\sqrt{2+\cos x}+\sqrt{2-\cos x}} = g(x) \qquad \text{1M}$$

$g(x)$ is a continuous even function on $[-\pi,\ \pi]$.

$$\int_{-\pi}^{\pi} \frac{\sqrt{2+\cos x}}{(1+e^x)(\sqrt{2+\cos x}+\sqrt{2-\cos x})}\,\mathrm{d}x$$

$$= \int_0^{\pi} g(x)\,\mathrm{d}x$$

$$= \Big[xg(x)\Big]_0^{\pi} - \int_0^{\pi} xg'(x)\,\mathrm{d}x \qquad \text{1M}$$

$$g'(x) = \frac{\frac{-\sin x}{2\sqrt{2+\cos x}}(\sqrt{2+\cos x}+\sqrt{2-\cos x}) - (\sqrt{2+\cos x})\left(\frac{-\sin x}{2\sqrt{2+\cos x}}+\frac{\sin x}{2\sqrt{2-\cos x}}\right)}{(\sqrt{2+\cos x}+\sqrt{2-\cos x})^2} \qquad \text{1M}$$

$$= \frac{-\sin x}{2\sqrt{3+\sin^2 x}+(3+\sin^2 x)} \qquad \text{1A}$$

Take $f(x) = \dfrac{-x}{2\sqrt{3+x^2}+3+x^2}$ such that it is continuous on $[0,\ 1]$ and $g'(x) = f(\sin x)$.

$$\int_{-\pi}^{\pi} \frac{\sqrt{2+\cos x}}{(1+e^x)(\sqrt{2+\cos x}+\sqrt{2-\cos x})}\,\mathrm{d}x$$

$$= \Big[xg(x)\Big]_0^{\pi} - \int_0^{\pi} xf(\sin x)\,\mathrm{d}x$$

$$= \pi g(\pi) - \frac{\pi}{2}\int_0^{\pi} f(\sin x)\,\mathrm{d}x \qquad \text{1M}$$

$$= \pi g(\pi) - \frac{\pi}{2}\int_0^{\pi} g'(x)\,\mathrm{d}x$$

$$= \pi g(\pi) - \frac{\pi}{2}\Big[g(x)\Big]_0^{\pi}$$

$$= \frac{\pi}{2}\left(\frac{\sqrt{2-1}}{\sqrt{2-1}+\sqrt{2-1}} + \frac{\sqrt{2+1}}{\sqrt{2+1}+\sqrt{2-1}}\right)$$

$$= \frac{\pi}{2} \qquad \text{1A}$$

7. Cancelled

END OF PAPER