M2REG-DEFINT-2324-ASM-SET 2-MATH
Suggested solutions

1. Let $u = 3 + 2\ln x$. Then $\mathrm{d}u = \dfrac{2}{x}\,\mathrm{d}x$. 1M

$$\begin{aligned}\int_1^e \frac{3+\ln x^2}{x}\,\mathrm{d}x &= \int_1^e \frac{3+2\ln x}{x}\,\mathrm{d}x \\ &= \frac{1}{2}\int_3^5 u\,\mathrm{d}u && \text{1A} \\ &= \frac{1}{4}\Big[u^2\Big]_3^5 \\ &= 4 && \text{1A}\end{aligned}$$

2. Let $u = e^x - e^{-x}$. Then $\mathrm{d}u = (e^x + e^{-x})\,\mathrm{d}x$. 1M

$$\begin{aligned}\int_{\ln 2}^{\ln 4} \frac{e^x + e^{-x}}{\sqrt{e^x - e^{-x}}}\,\mathrm{d}x &= \int_{\frac{3}{2}}^{\frac{15}{4}} \frac{\mathrm{d}u}{\sqrt{u}} && \text{1A} \\ &= \Big[2u^{\frac{1}{2}}\Big]_{\frac{3}{2}}^{\frac{15}{4}} \\ &= \sqrt{15} - \sqrt{6} && \text{1A}\end{aligned}$$

3. Let $u = x^3 + 1$. Then $\mathrm{d}u = 3x^2\,\mathrm{d}x$. 1M

$$\begin{aligned}\int_0^{\sqrt[3]{7}} \frac{x^5}{\sqrt[3]{x^3+1}}\,\mathrm{d}x &= \frac{1}{3}\int_0^{\sqrt[3]{7}} \frac{x^3(3x^2)}{\sqrt[3]{x^3+1}}\,\mathrm{d}x \\ &= \frac{1}{3}\int_1^8 \frac{u-1}{\sqrt[3]{u}}\,\mathrm{d}u && \text{1A} \\ &= \frac{1}{3}\int_1^8 (u^{\frac{2}{3}} - u^{-\frac{1}{3}})\,\mathrm{d}u \\ &= \frac{1}{3}\left[\frac{3u^{\frac{5}{3}}}{5} - \frac{3u^{\frac{2}{3}}}{2}\right]_1^8 \\ &= \frac{47}{10} && \text{1A}\end{aligned}$$

4. Let $u = \sqrt{x} + 4$. Then $\mathrm{d}u = \dfrac{1}{2\sqrt{x}}\,\mathrm{d}x$. 1M

$$\int_0^4 \frac{\sqrt{x}-4}{\sqrt{x}+4}\,\mathrm{d}x = \int_0^4 \frac{2\sqrt{x}(\sqrt{x}-4)}{\sqrt{x}+4}\cdot\frac{1}{2\sqrt{x}}\,\mathrm{d}x$$

$$= \int_4^6 \frac{2(u-4)(u-8)}{u}\,\mathrm{d}u \qquad \text{1A}$$

$$= \int_4^6 \left(2u - 24 + \frac{64}{u}\right)\mathrm{d}u$$

$$= \Big[u^2 - 24u + 64\ln|u|\Big]_4^6$$

$$= -28 + 64\ln\frac{3}{2} \qquad \text{1A}$$

5. Let $u = 1 - x^2$. Then $\mathrm{d}u = -2x\,\mathrm{d}x$. 1M

$$\int_0^1 x^3\cdot\sqrt{1-x^2}\,\mathrm{d}x = -\frac{1}{2}\int_1^0 (1-u)\sqrt{u}\,\mathrm{d}u \qquad \text{1A}$$

$$= \frac{1}{2}\int_0^1 (u^{\frac{1}{2}} - u^{\frac{3}{2}})\,\mathrm{d}u$$

$$= \frac{1}{2}\left[\frac{2}{3}u^{\frac{3}{2}} - \frac{2}{5}u^{\frac{5}{2}}\right]_0^1$$

$$= \frac{2}{15} \qquad \text{1A}$$

6. Let $u = e^{3x} + 3x + 1$. Then $\mathrm{d}u = (3e^{3x} + 3)\,\mathrm{d}x$. 1M

$$\int_0^3 \frac{x}{e^{3x}+3x+1}\,\mathrm{d}x = \frac{1}{3}\int_0^3 \left(1 - \frac{e^{3x}+1}{e^{3x}+3x+1}\right)\mathrm{d}x \qquad \text{1A}$$

$$= \frac{1}{3}\int_0^3 \mathrm{d}x - \frac{1}{9}\int_2^{e^9+10}\frac{\mathrm{d}u}{u} \qquad \text{1A}$$

$$= \frac{1}{3}\Big[x\Big]x_0^3 - \frac{1}{9}\Big[\ln|u|\Big]_2^{e^9+10}$$

$$= 1 - \frac{1}{9}\ln\frac{e^9+10}{2} \qquad \text{1A}$$

7. (a) Let $u = 1 - e^{-x}$. Then $\mathrm{d}u = e^{-x}\,\mathrm{d}x$. 1M

$$\int_{\ln 2}^{\ln 3} \frac{e^{-x}}{1 - e^{-x}}\,\mathrm{d}x = \int_{\frac{1}{2}}^{\frac{2}{3}} \frac{\mathrm{d}u}{u} \qquad \text{1A}$$

$$= \Big[\ln|u|\Big]_{\frac{1}{2}}^{\frac{1}{3}}$$

$$= \ln\frac{4}{3} \qquad \text{1A}$$

(b) $$\frac{A}{u} + \frac{B}{u-1} = \frac{A(u-1) + Bu}{u(u-1)}$$

$$= \frac{(A+B)u - A}{u(u-1)}$$

We have $A + B = 0$ and $-A = 1$. 1M

Solving, we have $A = -1$ and $B = 1$. 1A

(c) $$\int_{\ln 2}^{\ln 3} \frac{1}{e^x(e^x - 1)}\,\mathrm{d}x = \int_{\ln 2}^{\ln 3} \left(-\frac{1}{e^x} + \frac{1}{e^x - 1}\right)\mathrm{d}x \qquad \text{1M}$$

$$= -\int_{\ln 2}^{\ln 3} e^{-x}\,\mathrm{d}x + \int_{\ln 2}^{\ln 3} \frac{1}{e^x - 1}\,\mathrm{d}x$$

$$= -\int_{\ln 2}^{\ln 3} e^{-x}\,\mathrm{d}x + \int_{\ln 2}^{\ln 3} \frac{e^{-x}}{1 - e^{-x}}\,\mathrm{d}x$$

$$= \Big[e^{-x}\Big]_{\ln 2}^{\ln 3} + \ln\frac{4}{3} \qquad \text{1M}$$

$$= -\frac{1}{6} + \ln\frac{4}{3} \qquad \text{1A}$$

8. (a) $\dfrac{x}{x^2+4} - \dfrac{1}{x+4} = \dfrac{x(x+4)-(x^2+4)}{(x^2+4)(x+4)}$ 1M

$= \dfrac{4x-4}{(x^2+4)(x+4)}$

$= \dfrac{4(x-1)}{(x^2+4)(x+4)}$ 1

(b) Let $t = 2\tan\theta$. Then $\mathrm{d}t = 2\sec^2\theta\,\mathrm{d}\theta$.

$$\int_0^{\frac{\pi}{4}} \frac{2\tan\theta-1}{\tan\theta+2}\,\mathrm{d}\theta = \int_0^{\frac{\pi}{4}} \frac{2\tan\theta-1}{2\sec^2\theta(\tan\theta+2)}\cdot 2\sec^2\theta\,\mathrm{d}\theta$$

$$= \int_0^{2} \frac{t-1}{2\left(1+\frac{t^2}{4}\right)\left(\frac{t}{2}+2\right)}\,\mathrm{d}t \quad \text{1A}$$

$$= \int_0^{2} \frac{4(t-1)}{(t^2+4)(t+4)}\,\mathrm{d}t$$

$$= \int_0^{2} \left(\frac{t}{t^2+4} - \frac{1}{t+4}\right)\mathrm{d}t \quad \text{1M}$$

Let $u = t^2+4$. Then $\mathrm{d}u = 2t\,\mathrm{d}t$. 1M

$$\int_0^{\frac{\pi}{4}} \frac{2\tan\theta-1}{\tan\theta+2}\,\mathrm{d}\theta = \frac{1}{2}\int_4^{8} \frac{\mathrm{d}u}{u} - \int_0^{2} \frac{\mathrm{d}t}{t+4}$$

$$= \frac{1}{2}\Big[\ln|u|\Big]_4^8 - \Big[\ln|t+4|\Big]_0^2 \quad \text{1A}$$

$$= \frac{1}{2}(\ln 8 - \ln 4) - (\ln 6 - \ln 2)$$

$$= \frac{5}{2}\ln 2 - \ln 6 \quad \text{1A}$$

9. Let $x = \sqrt{3}\tan\theta$. Then $\mathrm{d}x = \sqrt{3}\sec^2\theta\,\mathrm{d}\theta$. 1M

$$\int_{-3}^{3} \frac{1}{(x^2+3)^{\frac{3}{2}}}\,\mathrm{d}x = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{\sqrt{3}\sec^2\theta}{(3\tan^2\theta+3)^{\frac{3}{2}}}\,\mathrm{d}\theta \quad \text{1A}$$

$$= \frac{1}{3}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{\sec^2\theta}{\sec^3\theta}\,\mathrm{d}\theta$$

$$= \frac{1}{3}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \cos\theta\,\mathrm{d}\theta$$

$$= \frac{1}{3}\Big[\sin\theta\Big]_{-\frac{\pi}{3}}^{\frac{\pi}{3}}$$

$$= \frac{\sqrt{3}}{3} \quad \text{1A}$$

10. Let $x = \sqrt{8}\sin\theta$. Then $\mathrm{d}x = \sqrt{8}\cos\theta\,\mathrm{d}\theta$. 1M

$$\int_{\sqrt{2}}^{2} \frac{x^4}{\sqrt{8-x^2}}\,\mathrm{d}x = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{64\sin^4\theta}{\sqrt{8-8\sin^2\theta}}\cdot\sqrt{8}\cos\theta\,\mathrm{d}\theta \quad \text{1A}$$

$$= 64\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sin^4\theta\,\mathrm{d}\theta$$

$$= 64\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \left(\frac{1-\cos 2\theta}{2}\right)^2 \mathrm{d}\theta \quad \text{1M}$$

$$= 16\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} (1-2\cos 2\theta+\cos^2 2\theta)\,\mathrm{d}\theta$$

$$= 16\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \left(1-2\cos 2\theta+\frac{1+\cos 4\theta}{2}\right)\mathrm{d}\theta \quad \text{1A}$$

$$= 16\left[\frac{3\theta}{2}-\sin 2\theta+\frac{\sin 4\theta}{8}\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$$

$$= 2\pi-16+7\sqrt{3} \quad \text{1A}$$

11. (a) $x^2-x+1 = \left(x-\frac{1}{2}\right)^2+\frac{3}{4}$ 1A

Let $x-\frac{1}{2} = \frac{\sqrt{3}}{2}\tan\theta$. Then $\mathrm{d}x = \frac{\sqrt{3}}{2}\sec^2\theta\,\mathrm{d}\theta$. 1M

$$\int_1^2 \frac{1}{x^2-x+1}\,\mathrm{d}x = \int_1^2 \frac{1}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}\,\mathrm{d}x$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\frac{\sqrt{3}}{2}\sec^2\theta}{\frac{3}{4}(\tan^2\theta+1)}\,\mathrm{d}\theta \quad \text{1A}$$

$$= \frac{2\sqrt{3}}{3}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \mathrm{d}\theta$$

$$= \frac{2\sqrt{3}}{3}\Big[\theta\Big]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= \frac{\sqrt{3}\pi}{9} \quad \text{1A}$$

(b) $$\int_1^2 \frac{x^2+x+1}{x^3+1}\,\mathrm{d}x = \int_1^2 \left(\frac{x^2}{x^3+1}+\frac{x+1}{x^3+1}\right)\mathrm{d}x$$

$$= \int_1^2 \frac{x^2}{x^3+1}\,\mathrm{d}x + \int_1^2 \frac{x+1}{x^2-x+1}\,\mathrm{d}x \quad \text{1A}$$

Let $u = x^3+1$. Then $\mathrm{d}u = 3x^2\,\mathrm{d}x$. 1M

$$\int_1^2 \frac{x^2+x+1}{x^3+1}\,\mathrm{d}x = \frac{1}{3}\int_2^9 \frac{\mathrm{d}u}{u} + \int_1^2 \frac{1}{x^2-x+1}\,\mathrm{d}x$$

$$= \frac{1}{3}\Big[\ln|u|\Big]_2^9 + \frac{\sqrt{3}\pi}{9} \quad \text{1M}$$

$$= \frac{1}{3}\ln\frac{9}{2}+\frac{\sqrt{3}\pi}{9} \quad \text{1A}$$

12. (a) $x^2 - x + 1 = \left(x - \frac{1}{2}\right)^2 + \frac{3}{4}$ 1A

Let $x - \frac{1}{2} = \frac{\sqrt{3}}{2}\tan\theta$. Then $\mathrm{d}x = \frac{\sqrt{3}}{2}\sec^2\theta\,\mathrm{d}\theta$. 1M

$$\int_1^2 \frac{1}{x^2 - x + 1}\,\mathrm{d}x = \int_1^2 \frac{1}{\left(x - \frac{1}{2}\right)^2 + \frac{3}{4}}\,\mathrm{d}x$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\frac{\sqrt{3}}{2}\sec^2\theta}{\frac{3}{4}(\tan^2\theta + 1)}\,\mathrm{d}\theta \qquad \text{1A}$$

$$= \frac{2\sqrt{3}}{3}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \mathrm{d}\theta$$

$$= \frac{2\sqrt{3}}{3}\Big[\theta\Big]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= \frac{\sqrt{3}\pi}{9} \qquad \text{1A}$$

(b) $$\int_1^2 \frac{x^2 + x + 1}{x^3 + 1}\,\mathrm{d}x = \int_1^2 \left(\frac{x^2}{x^3 + 1} + \frac{x + 1}{x^3 + 1}\right)\mathrm{d}x$$

$$= \int_1^2 \frac{x^2}{x^3 + 1}\,\mathrm{d}x + \int_1^2 \frac{x + 1}{x^2 - x + 1}\,\mathrm{d}x \qquad \text{1A}$$

Let $u = x^3 + 1$. Then $\mathrm{d}u = 3x^2\,\mathrm{d}x$. 1M

$$\int_1^2 \frac{x^2 + x + 1}{x^3 + 1}\,\mathrm{d}x = \frac{1}{3}\int_2^9 \frac{\mathrm{d}u}{u} + \int_1^2 \frac{1}{x^2 - x + 1}\,\mathrm{d}x$$

$$= \frac{1}{3}\Big[\ln|u|\Big]_2^9 + \frac{\sqrt{3}\pi}{9} \qquad \text{1M}$$

$$= \frac{1}{3}\ln\frac{9}{2} + \frac{\sqrt{3}\pi}{9} \qquad \text{1A}$$

13. (a) $2x^2 - x + 2 = 2\left(x - \frac{1}{4}\right)^2 + \frac{15}{8}$ 1A

Let $x - \frac{1}{4} = \frac{\sqrt{15}}{4}\tan\theta$. Then $\mathrm{d}x = \frac{\sqrt{15}}{4}\sec^2\theta\,\mathrm{d}\theta$. 1M

$$\int \frac{1}{2x^2 - x + 2}\,\mathrm{d}x = \int \frac{1}{2\left(x - \frac{1}{4}\right)^2 + \frac{15}{8}}\,\mathrm{d}x$$

$$= \int \frac{\frac{\sqrt{15}}{4}\sec^2\theta}{\frac{15}{8}\tan^2\theta + \frac{15}{8}}\,\mathrm{d}\theta$$ 1A

$$= \frac{2\sqrt{15}}{15}\int \mathrm{d}\theta$$

$$= \frac{2\sqrt{15}}{15}\theta + \text{constant}$$

$$= \frac{2\sqrt{15}}{15}\tan^{-1}\frac{4x - 1}{\sqrt{15}} + \text{constant}$$ 1A

(b) Let $u = \frac{1}{x}$. Then $\mathrm{d}u = -\frac{1}{x^2}\,\mathrm{d}x$.

$$\int_{\frac{1}{2}}^{2} \frac{1}{(x^5 + 1)(2x^2 - x + 2)}\,\mathrm{d}x = \int_{\frac{1}{2}}^{2} \frac{x^2}{(x^5 + 1)(2x^2 - x + 2)}\,\mathrm{d}x$$

$$= -\int_{2}^{\frac{1}{2}} \frac{\frac{1}{u^2}}{\left(\frac{1}{u^5} + 1\right)\left(\frac{2}{u^2} - \frac{1}{u} + 2\right)}\,\mathrm{d}u$$ 1A

$$= \int_{\frac{1}{2}}^{2} \frac{u^5}{(u^5 + 1)(2u^2 - u + 2)}\,\mathrm{d}u$$

$$= \int_{\frac{1}{2}}^{2} \frac{x^5}{(x^5 + 1)(2x^2 - x + 2)}\,\mathrm{d}x$$

$$\int_{\frac{1}{2}}^{2} \frac{1}{(x^5 + 1)(2x^2 - x + 2)}\,\mathrm{d}x$$

$$= \frac{1}{2}\left[\int_{\frac{1}{2}}^{2} \frac{1}{(x^5 + 1)(x^2 - x + 2)}\,\mathrm{d}x + \int_{\frac{1}{2}}^{2} \frac{x^5}{(x^5 + 1)(x^2 - x + 2)}\,\mathrm{d}x\right]$$

$$= \frac{1}{2}\int_{\frac{1}{2}}^{2} \frac{1 + x^5}{(x^5 + 1)(x^2 - x + 2)}\,\mathrm{d}x$$

$$= \frac{1}{2}\int_{\frac{1}{2}}^{2} \frac{1}{2x^2 - x + 2}\,\mathrm{d}x$$ 1A

$$= \frac{\sqrt{15}}{15}\left[\tan^{-1}\frac{4x - 1}{\sqrt{15}}\right]_{\frac{1}{2}}^{2}$$ 1M

$$= \frac{\sqrt{15}}{15}\left(\tan^{-1}\frac{7\sqrt{15}}{15} - \tan^{-1}\frac{\sqrt{15}}{15}\right)$$ 1A

14. (a) Let $x = \sin\theta$. Then $\mathrm{d}x = \cos\theta\,\mathrm{d}\theta$

When $x = 0$, $\theta = 0$; when $x = 1$, $\theta = \dfrac{\pi}{2}$.

$$\int_0^1 \sqrt{1-x^2}\,\mathrm{d}x = \int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2\theta}\cos\theta\,\mathrm{d}\theta \qquad \text{1M}$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} (1+\cos 2\theta)\,\mathrm{d}\theta \qquad \text{1M}$$

$$= \frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}} \qquad \text{1A}$$

$$= \frac{1}{2}\left(\frac{\pi}{2} + 0 - 0 - 0\right)$$

$$= \frac{\pi}{4} \qquad \text{1A}$$

(b) $$\frac{\mathrm{d}}{\mathrm{d}x}[(1-x^2)^{\frac{3}{2}}x] = (1-x^2)^{\frac{3}{2}} + \frac{3}{2}(1-x^2)^{\frac{1}{2}}(-2x)(x) \qquad \text{1M}$$

$$= \sqrt{1-x^2}[(1-x^2) - 3x^2]$$

$$= \sqrt{1-x^2} - 4x^2\sqrt{1-x^2} \qquad \text{1}$$

(c) By (b), $4x^2\sqrt{1-x^2} = \sqrt{1-x^2} - \dfrac{\mathrm{d}}{\mathrm{d}x}[(1-x^2)^{\frac{3}{2}}x]$

$$\int_0^1 x^2\sqrt{1-x^2}\,\mathrm{d}x = \frac{1}{4}\int_0^1 \sqrt{1-x^2}\,\mathrm{d}x - \frac{1}{4}\left[(1-x^2)^{\frac{3}{2}}x\right]_0^1 \qquad \text{1M}$$

$$= \frac{1}{4}\times\frac{\pi}{4} - \frac{1}{4}(0-0)$$

$$= \frac{\pi}{16} \qquad \text{1A}$$

15. (a) (i) $\tan\alpha = \dfrac{\cos\frac{2\pi}{11} - 1}{\sin\frac{2\pi}{11}}$

$= \dfrac{1 - 2\sin^2\frac{\pi}{11} - 1}{2\sin\frac{\pi}{11}\cos\frac{\pi}{11}}$ 1M

$= -\tan\dfrac{\pi}{11}$

$= \tan\left(-\dfrac{\pi}{11}\right)$

$\alpha = -\dfrac{\pi}{11}$ 1

(ii) $\tan\beta = \dfrac{\cos\frac{2\pi}{11} + 1}{\sin\frac{2\pi}{11}}$

$= \dfrac{2\cos^2\frac{\pi}{11} - 1 + 1}{2\sin\frac{\pi}{11}\cos\frac{\pi}{11}}$ 1M

$= \cot\dfrac{\pi}{11}$

$= \tan\left(\dfrac{\pi}{2} - \dfrac{\pi}{11}\right)$

$\beta = \dfrac{9\pi}{22}$ 1A

(b) (i) $x^2 + 2x\cos\dfrac{2\pi}{11} + 1 = x^2 + 2x\cos\dfrac{2\pi}{11} + \cos^2\dfrac{2\pi}{11} + \sin^2\dfrac{2\pi}{11}$

$= \left(x + \cos\dfrac{2\pi}{11}\right)^2 + \sin^2\dfrac{2\pi}{11}$ 1A

(ii) Let $x + \cos\dfrac{2\pi}{11} = \sin\dfrac{2\pi}{11}\tan\theta$. Then $dx = \sin\dfrac{2\pi}{11}\sec^2\theta\, d\theta$. 1M

When $x = -1$ and when $x = 1$,

$\tan\theta = \dfrac{-1 + \cos\frac{2\pi}{11}}{\sin\frac{2\pi}{11}}$ and $\tan\theta = \dfrac{1 + \cos\frac{2\pi}{11}}{\sin\frac{2\pi}{11}}$

$\theta = -\dfrac{\pi}{11}$ $\theta = \dfrac{9\pi}{22}$ 1M

$\displaystyle\int_{-1}^{1} \frac{\sin\frac{2\pi}{11}}{x^2 + 2x\cos\frac{2\pi}{11} + 1}\, dx$

$\displaystyle= \int_{-1}^{1} \frac{\sin\frac{2\pi}{11}}{\left(x + \cos\frac{2\pi}{11}\right)^2 + \sin^2\frac{2\pi}{11}}\, dx$

$\displaystyle= \int_{-\frac{\pi}{11}}^{\frac{9\pi}{22}} \frac{\sin^2\frac{2\pi}{11}\sec^2\theta}{\sin^2\frac{2\pi}{1}\tan^2\theta + \sin^2\frac{2\pi}{11}}\, d\theta$ 1A

$\displaystyle= \int_{-\frac{\pi}{11}}^{\frac{9\pi}{22}} d\theta$

$= \Big[\theta\Big]_{-\frac{\pi}{11}}^{\frac{9\pi}{22}}$

$= \dfrac{\pi}{2}$ 1A

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