M2REG-AOI-2324-ASM-SET 2-MATH
Suggested solutions

1. (a) Put $x = 2y - 2$ into $y^2 = mx - 3$,

$$y^2 = m(2y-2) - 3 \qquad \text{1M}$$

$$y^2 - 2my + (2m+3) = 0$$

$$\Delta = (2m)^2 - 4(1)(2m+3) = 0 \qquad \text{1M}$$

$$4m^2 - 8m - 12 = 0$$

$$m = 3 \quad \text{or} \quad -1 \text{ (rejected)} \qquad \text{1A}$$

(b) $y^2 - 2(3)y + [2(3)+3] = 0$

$$y^2 - 6y + 9 = 0$$

$$y = 3$$

When $y = 3$, $x = 2(3) - 2 = 4$. The coordinates of P are $(4,\ 3)$. 1A

(c) $$\text{Area} = \int_{-2}^{4} \frac{x+2}{2}\,\mathrm{d}x - \int_{1}^{4} \sqrt{3x-3}\,\mathrm{d}x \qquad \text{1M+1M}$$

$$= \left[\frac{x^2}{4} + x\right]_{-2}^{4} - \frac{1}{3}\left[\frac{2}{3}(3x-3)^{\frac{3}{2}}\right]_{1}^{4} \qquad \text{1M}$$

$$= 3 \qquad \text{1A}$$

2. $$\text{Required area} = -\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos y\,\mathrm{d}y \qquad \text{1M}$$

$$= -\Big[\sin y\Big]_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \qquad \text{1M}$$

$$= 2 \qquad \text{1A}$$

3. (a) $$\int \ln y\,\mathrm{d}y = y\ln y - \int \mathrm{d}y \qquad \text{1M}$$

$$= y\ln y - y + \text{constant} \qquad \text{1A}$$

(b) $$\text{Required area} = -\int_{1}^{e^2} (-\ln y)\,\mathrm{d}y \qquad \text{1M+1M}$$

$$= \Big[y\ln y - y\Big]_{1}^{e^2} \qquad \text{1M}$$

$$= e^2 + 1 \qquad \text{1A}$$

4. (a) $y^2 = 4x - 12$

$2y\dfrac{dy}{dx} = 4$ 1M

$\dfrac{dy}{dx} = \dfrac{2}{y}$

Slope of tangent at $P = \dfrac{2}{-2} = -1$

Required equation is

$y + 2 = -1(x - 4)$ 1M

$y = -x + 2$ 1A

(b) Area $= \displaystyle\int_{-2}^{0} \left[\frac{y^2 + 12}{4} - (-y + 2)\right] dy$ 1M

$= \displaystyle\int_{-2}^{0} \left[\frac{y^2}{4} + y + 1\right] dy$

$= \left[\dfrac{y^3}{12} + \dfrac{y^2}{2} + y\right]_{-2}^{0}$ 1M

$= \dfrac{2}{3}$ 1A

5. The coordinates of A, B and C are $(4,\ 0)$, $(0,\ 2)$ and $(9,\ 5)$ respectively. 1A

Area $= \displaystyle\int_{0}^{2} [(4 + y) - (4 - 2y)]\, dy + \int_{2}^{5} [(4 + y) - (y^2 - 4y + 4)]\, dy$ 1M+1M

$= \left[\dfrac{3y^2}{2}\right]_0^2 + \left[\dfrac{-y^3}{3} + \dfrac{5y^2}{2}\right]_2^5$ 1M

$= \dfrac{39}{2}$ 1A

6. (a) $\displaystyle\int u(5^u)\, du = \frac{u(5^u)}{\ln 5} - \frac{1}{\ln 5}\int 5^u\, du$ 1M

$= \dfrac{u(5^u)}{\ln 5} - \dfrac{5^u}{(\ln 5)^2} + C$ 1A

(b) Required area $= \displaystyle\int_0^1 x(5^{2x})\, dx$ 1M

Let $u = 2x$. Then $du = 2\, dx$.

Required area $= \dfrac{1}{4}\displaystyle\int_0^2 u(5^u)\, du$ 1M

$= \dfrac{1}{4}\left[\dfrac{u(5^u)}{\ln 5} - \dfrac{5^u}{(\ln 5)^2}\right]_0^2$ 1M

$= \dfrac{1}{4}\left[\left(\dfrac{2(25)}{\ln 5} - \dfrac{25}{(\ln 5)^2}\right) - \left(0 - \dfrac{1}{(\ln 5)^2}\right)\right]$

$= \dfrac{25}{2\ln 5} - \dfrac{6}{(\ln 5)^2}$ 1A

7. (a) $\dfrac{dy}{dx} = \dfrac{1}{x+2}$ 1A

Slope of $L = \dfrac{1}{h+2}$

Equation of L is

$$y - \ln(h+2) = \frac{1}{h+2}(x-h)$$

$$y = \frac{x-h}{h+2} + \ln(h+2) \qquad \text{1A}$$

$$A = \int_0^h \left(\frac{x-h}{h+2} + \ln(h+2) - \ln(x+2)\right) dx \qquad \text{1M}$$

$$= \left[\frac{x^2-2hx}{2(h+2)} + \ln(h+2)x\right]_0^h - \int_0^h \ln(x+2)\, dx$$

$$= \frac{-h^2}{2h+4} + h\ln(h+2) - \Big[x\ln(x+2)\Big]_0^h + \int_0^h \frac{x}{x+2}\, dx \qquad \text{1M}$$

$$= \frac{-h^2}{2h+4} + h\ln(h+2) - h\ln(h+2) + \int_0^h \left(1 - \frac{2}{x+2}\right) dx$$

$$= \frac{-h^2}{2h+4} + \Big[x - 2\ln|x+2|\Big]_0^h$$

$$= \frac{h^2+4h}{2h+4} - 2\ln(h+2) + 2\ln 2 \qquad \text{1}$$

(b) $A = \dfrac{h}{2} + 1 - \dfrac{2}{h+2} - 2\ln(h+2) + 2\ln 2$

$$\frac{dA}{dt} = \frac{dA}{dh} \cdot \frac{dh}{dt} \qquad \text{1M}$$

$$= \left(\frac{1}{2} + \frac{2}{(h+2)^2} - \frac{2}{h+2}\right) \cdot 3^{-t}(-\ln 3) \qquad \text{1A}$$

When $t = 1$, $h = 3^{-1} = \dfrac{1}{3}$,

$$\frac{dA}{dt} = \left(\frac{1}{2} + \frac{2}{\left(\frac{1}{3}+2\right)^2} - \frac{2}{\frac{1}{3}+2}\right) \cdot \frac{1}{3} \cdot (-\ln 3)$$

$$= -\frac{\ln 3}{294}$$

Required rate is $-\dfrac{\ln 3}{294}$ square units per second. 1A

8. (a) $\int (\ln x)^2 \, dx = x(\ln x)^2 - 2\int \ln x \, dx$ 1M

$= x(\ln x)^2 - 2x\ln x + 2\int dx$ 1M

$= x(\ln x)^2 - 2x\ln x + 2x + \text{constant}$ 1A

(b) Volume $= \pi \int_0^1 \left(\sqrt{x}\ln\left(x^2+1\right)\right)^2 dx$ 1M

$= \pi \int_0^1 x\left(\ln\left(x^2+1\right)\right)^2 dx$

Let $u = x^2 + 1$. Then $du = 2x\,dx$.

Volume $= \frac{\pi}{2}\int_1^2 (\ln u)^2 \, du$ 1M

$= \frac{\pi}{2}\left[u(\ln u)^2 - 2u\ln u + 2u\right]_1^2$ 1M

$= \pi((\ln 2)^2 - 2\ln 2 + 1)$ 1A

9. (a) The x-coordinate of A is 1. 1A

$$x^2 - 1 = \frac{12}{x^2}$$

$$x^4 - x^2 - 12 = 0$$

$$x^2 = 4 \quad \text{or} \quad -3 \text{ (rejected)}$$

$$x = 2 \quad \text{or} \quad -2$$

The x-coordinate of B is 2. 1A

(b) Required area

$= -\int_0^1 (x^2-1)\,dx + \int_1^2 (x^2-1)\,dx + \int_2^3 \frac{12}{x^2}\,dx$ 1M+1A

$= -\left[\frac{x^3}{3} - x\right]_0^1 + \left[\frac{x^3}{3} - x\right]_1^2 + \left[-\frac{12}{x}\right]_2^3$ 1M

$= 4$ 1A

(c) Required volume

$= \pi\int_0^2 (x^2-1)^2\,dx + \pi\int_2^3 \left(\frac{12}{x^2}\right)^2 dx$ 1M+1A

$= \pi\int_0^2 (x^4 - 2x^2 + 1)\,dx + \pi\int_2^3 \frac{144}{x^4}\,dx$

$= \pi\left[\frac{x^5}{5} - \frac{2x^3}{3} + x\right]_0^2 + \pi\left[-\frac{48}{x^3}\right]_2^3$ 1M

$= \frac{328\pi}{45}$ 1A

10. (a) $g'(x) = e^x \cos x - e^x \sin x$ 1M

When $g'(x) = 0$,

$$e^x(\cos x - \sin x) = 0$$
$$\tan x = 1$$
$$x = \frac{\pi}{4}$$

x	$0 < x < \frac{\pi}{4}$	$\frac{\pi}{4} < x < \pi$
$g'(x)$	$+$	$-$

1M

Thus, G attains its maximum value only at $x = \frac{\pi}{4}$, and G has only one maximum point. 1

(b) $\int e^{2x} \cos 2x \, dx = \frac{1}{2}e^{2x} \cos 2x + \int e^{2x} \sin 2x \, dx$ 1M

$= \frac{1}{2}e^{2x} \cos 2x + \frac{1}{2}e^{2x} \sin 2x - \int e^{2x} \cos 2x \, dx$ 1M

$2\int e^{2x} \cos 2x \, dx = \frac{1}{2}e^{2x} \cos 2x + \frac{1}{2}e^{2x} \sin 2x + \text{constant}$

$\int e^{2x} \cos 2x \, dx = \frac{1}{4}e^{2x} \cos 2x + \frac{1}{4}e^{2x} \sin 2x + \text{constant}$ 1A

(c) Volume $= \pi \int_0^{\frac{\pi}{4}} e^{2x} \cos^2 x \, dx$ 1M

$= \frac{\pi}{2} \int_0^{\frac{\pi}{4}} e^{2x}(\cos 2x + 1) \, dx$

$= \frac{\pi}{2}\left[\frac{1}{4}e^{2x} \cos 2x + \frac{1}{4}e^{2x} \sin 2x\right]_0^{\frac{\pi}{4}} + \frac{\pi}{2}\left[\frac{1}{2}e^{2x}\right]_0^{\frac{\pi}{4}}$

$= \frac{3\pi}{8}(e^{\frac{\pi}{2}} - 1)$ 1A

11. (a) $\int (\ln x)^2 \, dx = x(\ln x)^2 - 2\int \ln x \, dx$ 1M

$= x(\ln x)^2 - 2x \ln x + 2\int dx$ 1M

$= x(\ln x)^2 - 2x \ln x + 2x + \text{constant}$ 1A

(b) Required volume $= \pi \int_1^{e^3} (\ln y)^2 \, dy$ 1M

$= \pi\left[y(\ln y)^2 - 2y \ln y + 2y\right]_1^{e^3}$ 1M

$= \pi(5e^3 - 2)$ 1A

12. (a) (i) $f(x) = \dfrac{162(x-1)}{(x+1)^3}$

$$= \frac{162}{(x+1)^2} - \frac{324}{(x+1)^3}$$

$$f'(x) = -\frac{324}{(x+1)^3} + \frac{972}{(x+1)^4}$$

$$= \frac{324(2-x)}{(x+1)^4} \qquad \text{1A}$$

$$f''(x) = \frac{972}{(x+1)^4} - \frac{3888}{(x+1)^5}$$

$$= \frac{972(x-3)}{(x+1)^5} \qquad \text{1A}$$

(ii) When $f'(x) = 0$, $x = 2$.

x	$x < -1$	$-1 < x < 2$	$x > 2$
$f'(x)$	+	+	−

1M

The maximum point is $(2,\ 6)$. 1A

No minimum point.

When $f''(x) = 0$, $x = 3$.

x	$x < -1$	$-1 < x < 3$	$x > 3$
$f''(x)$	+	−	+

1M

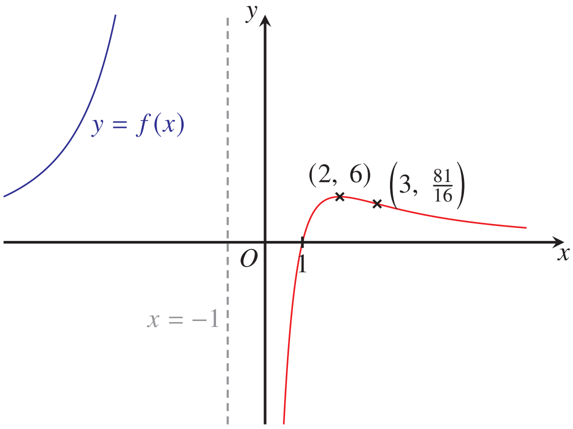
The point of inflexion is $\left(3,\ \dfrac{81}{16}\right)$. 1A

(b) Vertical asymptote is $x = -1$. 1A

Horizontal asymptote is $y = 0$. 1A

(c) (Shape, extrema and point of inflexion) 1A

(All correct) 1A



(d) Let $u = x + 1$. Then $\mathrm{d}u = \mathrm{d}x$.

$$\text{Volume} = \pi \int_0^1 \left[\frac{162(x-1)}{(x+1)^3} \right]^2 \mathrm{d}x \qquad \text{1M}$$

$$= 26\,244\pi \int_1^2 \frac{(u-2)^2}{u^6} \, \mathrm{d}u$$

$$= 26\,244\pi \int_1^2 (u^{-4} - 4u^{-5} + 4u^{-6}) \, \mathrm{d}u \qquad \text{1M}$$

$$= 26\,244 \left[\frac{u^{-3}}{-3} + u^{-4} - \frac{4u^{-5}}{5} \right]_1^2$$

$$= \frac{67\,797\pi}{20} \qquad \text{1A}$$

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