

DAYM2-TRIG-2324-ASM-SET 1-MATH

建議題解

$$\begin{aligned}
 1. \quad (a) \quad & \frac{\cos A + \cos B}{\sin A + \sin B} \\
 &= \frac{2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}}{2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}} && 1M \\
 &= \frac{\cos \frac{A+B}{2}}{\sin \frac{A+B}{2}} \\
 &= \cot \frac{A+B}{2} && 1
 \end{aligned}$$

$$(b) \quad 5 \sin A - 3 \cos A = 3 \cos B - 5 \sin B$$

$$5(\sin A + \sin B) = 3(\cos A + \cos B)$$

$$\frac{5}{3} = \frac{\cos A + \cos B}{\sin A + \sin B} \quad (\sin A + \sin B \neq 0)$$

$$\cot \frac{A+B}{2} = \frac{5}{3}$$

$$\tan \frac{A+B}{2} = \frac{3}{5}$$

1A

$$\cot(A+B) = \frac{1 - \tan^2 \frac{A+B}{2}}{2 \tan \frac{A+B}{2}} \quad 1M$$

$$= \frac{1 - \left(\frac{3}{5}\right)^2}{2 \left(\frac{3}{5}\right)}$$

$$= \frac{8}{15}$$

1A

$$\begin{aligned}
2. \quad (a) \quad \cot 3x &= \frac{1}{\tan(2x+x)} \\
&= \frac{1 - \tan 2x \tan x}{\tan 2x + \tan x} & 1M \\
&= \frac{1 - \left(\frac{2 \tan x}{1 - \tan^2 x}\right) \tan x}{\frac{2 \tan x}{1 - \tan^2 x} + \tan x} \\
&= \frac{(1 - \tan^2 x) - 2 \tan^2 x}{2 \tan x + \tan x(1 - \tan^2 x)} \\
&= \frac{1 - 3 \tan^2 x}{3 \tan x - \tan^3 x} \\
&= \frac{1 - \frac{3}{\cot^2 x}}{\frac{3}{\cot x} - \frac{1}{\cot^3 x}} \times \frac{\cot^3 x}{\cot^3 x} & 1M \\
&= \frac{\cot^3 x - 3 \cot x}{3 \cot^2 x - 1} & 1
\end{aligned}$$

$$\begin{aligned}
(b) \quad \cot x - 3 \cot 3x &= 4 \\
\cot x - 3 \left(\frac{\cot^3 x - 3 \cot x}{3 \cot^2 x - 1} \right) &= 4 & 1M
\end{aligned}$$

$$\begin{aligned}
(3 \cot^3 x - \cot x) - 3(\cot^3 x - 3 \cot x) &= 4(3 \cot^2 x - 1) \\
-12 \cot^2 x + 8 \cot x + 4 &= 0 & 1M
\end{aligned}$$

$$\begin{aligned}
\cot x &= 1 \quad \text{或} \quad -\frac{1}{3} \\
x &= \frac{\pi}{4} & 1A
\end{aligned}$$

$$\begin{aligned}
3. \quad (a) \quad \cos \theta(x + \sec \theta)(x \cos 3\theta - 1) \\
&= (x \cos \theta + 1)(x \cos 3\theta - 1) \\
&= x^2 \cos 3\theta + x(\cos 3\theta - \cos \theta) - 1 \\
&= x^2 \cos 3\theta + x \left(-2 \sin \frac{3\theta + \theta}{2} \sin \frac{3\theta - \theta}{2} \right) - 1 & 1M \\
&= x^2 \cos 3\theta - 2x \sin 2\theta \sin \theta - 1 & 1
\end{aligned}$$

$$\begin{aligned}
(b) \quad \cos 4\theta + \cos 2\theta + 4 \sin^2 \theta \cos \theta &= \frac{1}{2} \\
2 \cos 3\theta \cos \theta + 2(2 \sin \theta \cos \theta) \cos \theta &= \frac{1}{2} & 1M \\
4 \cos 3\theta \cos \theta + 4 \sin 2\theta \sin \theta &= 1 \\
(-2)^2 \cos 3\theta \cos \theta - 2(-2) \sin 2\theta \sin \theta - 1 &= 0 \\
\cos \theta(-2 + \sec \theta)(-2 \cos 3\theta - 1) &= 0 & 1M
\end{aligned}$$

$$\begin{aligned}
\cos \theta &= 0 & \text{或} & \sec \theta = 2 & \text{或} & \cos 3\theta = -\frac{1}{2} \\
\theta &= \frac{\pi}{2} \quad (\text{捨去}) & \cos \theta &= \frac{1}{2} & 3\theta &= \frac{2\pi}{3} & \text{或} & \frac{4\pi}{3} \\
& & \theta &= \frac{\pi}{3} & \theta &= \frac{2\pi}{9} & \text{或} & \frac{4\pi}{9} & 1A+1A
\end{aligned}$$

$$4. \quad (a) \quad \tan 4x = \frac{2 \tan 2x}{1 - \tan^2 2x} \quad 1M$$

$$= \frac{2 \left(\frac{2 \tan x}{1 - \tan^2 x} \right)}{1 - \left(\frac{2 \tan x}{1 - \tan^2 x} \right)^2}$$

$$= \frac{4 \tan x (1 - \tan^2 x)}{(1 - \tan^2 x)^2 - (2 \tan x)^2}$$

$$= \frac{4 \tan x (1 - \tan^2 x)}{1 - 6 \tan^2 x + \tan^4 x} \quad 1$$

$$(b) \quad \cot^4 x + 4 \cot^3 x - 6 \cot^2 x - 4 \cot x + 1 = 0$$

$$1 + 4 \tan x - 6 \tan^2 x - 4 \tan^3 x + \tan^4 x = 0$$

$$4 \tan x - 4 \tan^3 x = -(1 - 6 \tan^2 x + \tan^4 x)$$

$$\frac{4 \tan x (1 - \tan^2 x)}{1 - 6 \tan^2 x + \tan^4 x} = -1$$

$$\tan 4x = -1 \quad 1M$$

$$4x = \frac{7\pi}{4} \quad \text{或} \quad \frac{11\pi}{4}$$

$$x = \frac{7\pi}{16} \quad \text{或} \quad \frac{11\pi}{16} \quad 1A+1A$$

5. (a) 當 $n = 1$ 時，

$$\begin{aligned}\text{左式} &= \sin \theta \left[\sin \frac{\pi}{4} + \sin \left(\frac{\pi}{4} + 2\theta \right) \right] \\ &= \sin \theta \left[2 \sin \left(\frac{\pi}{4} + \theta \right) \cos \theta \right] \\ &= 2 \sin \theta \sin \left(\frac{\pi}{4} + \theta \right) \cos \theta \\ \text{右式} &= \sin 2\theta \sin \left(\frac{\pi}{4} + \theta \right) \\ &= 2 \sin \theta \sin \left(\frac{\pi}{4} + \theta \right) \cos \theta \\ &= \text{左式}\end{aligned}$$

對 $n = 1$ ，命題為真。

1

假設 $\sin \theta \sum_{k=0}^p \sin \left(\frac{\pi}{4} + 2k\theta \right) = \sin(p+1)\theta \sin \left(\frac{\pi}{4} + p\theta \right)$ ，其中 $p \in \mathbb{Z}^+$ 。

1

$$\begin{aligned}\sin \theta \sum_{k=0}^{p+1} \sin \left(\frac{\pi}{4} + 2k\theta \right) &= \sin \theta \sum_{k=0}^p \sin \left(\frac{\pi}{4} + 2k\theta \right) + \sin \theta \sin \left(\frac{\pi}{4} + 2(p+1)\theta \right) \\ &= \sin(p+1)\theta \sin \left(\frac{\pi}{4} + p\theta \right) + \sin \theta \sin \left(\frac{\pi}{4} + 2(p+1)\theta \right) \\ &= \frac{1}{2} \left[\cos \left(\frac{\pi}{4} - \theta \right) - \cos \left(\frac{\pi}{4} + (2p+1)\theta \right) \right] + \frac{1}{2} \left[\cos \left(\frac{\pi}{4} + (2p+1)\theta \right) - \cos \left(\frac{\pi}{4} + (2p+3)\theta \right) \right] \\ &= \frac{1}{2} \left[\cos \left(\frac{\pi}{4} - \theta \right) - \cos \left(\frac{\pi}{4} + (2p+3)\theta \right) \right] \\ &= \sin(p+2)\theta \sin \left(\frac{\pi}{4} + (p+1)\theta \right)\end{aligned}$$

1M

對 $n = p+1$ ，命題為真。

1

藉數學歸納法， $\forall n \in \mathbb{Z}^+$ ，命題為真。

1

(b) 代 $\theta = \frac{\pi}{6}$ 及 $n = 33$ 。

$$\begin{aligned}\sin \frac{\pi}{6} \sum_{k=0}^{33} \sin \left[\frac{\pi}{4} + 2k \left(\frac{\pi}{6} \right) \right] &= \sin \frac{34\pi}{6} \sin \left(\frac{\pi}{4} + \frac{33\pi}{6} \right) \\ \frac{1}{2} \left[\sin \frac{\pi}{4} + \sin \frac{7\pi}{12} + \sin \frac{11\pi}{12} + \dots + \sin \frac{45\pi}{4} \right] &= \sin \frac{17\pi}{3} \sin \frac{23\pi}{4} \\ \sin \frac{\pi}{4} + \sin \frac{7\pi}{12} + \sin \frac{11\pi}{12} + \dots + \sin \frac{45\pi}{4} &= 2 \sin \left(6\pi - \frac{\pi}{3} \right) \sin \left(6\pi - \frac{\pi}{4} \right) \\ &= 2 \left(-\frac{\sqrt{3}}{2} \right) \left(-\frac{\sqrt{2}}{2} \right) \\ &= \frac{\sqrt{6}}{2}\end{aligned}$$

1M

1A

$$\begin{aligned}
6. \quad (a) \quad & \frac{\sin 3\theta}{\sin \theta} - \frac{\cos 3\theta}{\cos \theta} \\
&= \frac{\sin 3\theta \cos \theta - \cos 3\theta \sin \theta}{\sin \theta \cos \theta} \\
&= \frac{\sin(3\theta - \theta)}{\frac{1}{2} \sin 2\theta} \\
&= 2
\end{aligned}$$

1M
1

(b) 代 $\theta = \frac{\pi}{10}$ 。

$$\begin{aligned}
& \frac{\sin \frac{3\pi}{10}}{\sin \frac{\pi}{10}} - \frac{\cos \frac{3\pi}{10}}{\cos \frac{\pi}{10}} = 2 \\
& \frac{\sin \left(\frac{\pi}{2} - \frac{\pi}{5} \right)}{\sin \frac{\pi}{10}} - \frac{\cos \left(\frac{\pi}{2} - \frac{\pi}{5} \right)}{\cos \frac{\pi}{10}} = 2 \\
& \frac{1 - 2 \sin^2 \frac{\pi}{10}}{\sin \frac{\pi}{10}} - \frac{2 \sin \frac{\pi}{10} \cos \frac{\pi}{10}}{\cos \frac{\pi}{10}} = 2 \\
& 1 - 2 \sin^2 \frac{\pi}{10} - 2 \sin^2 \frac{\pi}{10} = 2 \sin \frac{\pi}{10} \\
& 4 \sin^2 \frac{\pi}{10} + 2 \sin \frac{\pi}{10} - 1 = 0
\end{aligned}$$

1M
1M

因此， $\sin \frac{\pi}{10}$ 為二次方程 $4x^2 + 2x - 1 = 0$ 的根。

$$\begin{aligned}
& 4x^2 + 2x - 1 = 0 \\
& x = \frac{-2 \pm \sqrt{2^2 - 4(4)(-1)}}{2(4)} \\
& = \frac{-1 \pm \sqrt{5}}{4}
\end{aligned}$$

由於 $0 < \frac{\pi}{10} < \frac{\pi}{2}$ ，可得 $\sin \frac{\pi}{10} > 0$ 。

因此， $\sin \frac{\pi}{10} = \frac{-1 + \sqrt{5}}{4}$ 。

1A

7. (a) 當 $n = 1$ 時，

$$\text{左式} = 2(1 - \cos x) \sin x = 2 \sin x - \sin 2x$$

$$\text{右式} = 2 \sin x - \sin 2x = \text{左式}$$

對 $n = 1$ ，命題為真。

1

$$\text{假設 } 2(1 - \cos x) \sum_{k=1}^p k \sin kx = (p+1) \sin px - p \sin(p+1)x, \text{ 其中 } p \in \mathbb{Z}^+.$$

1

$$2(1 - \cos x) \sum_{k=1}^{p+1} k \sin kx$$

$$= 2(1 - \cos x) \sum_{k=1}^p k \sin kx + 2(1 - \cos x)(p+1) \sin(p+1)x$$

$$= (p+1) \sin px - p \sin(p+1)x + 2(1 - \cos x)(p+1) \sin(p+1)x$$

1M

$$= (p+1) \sin px + (p+2) \sin(p+1)x - 2(p+1) \cos x \sin(p+1)x$$

$$= (p+1) \sin px + (p+2) \sin(p+1)x - (p+1) [\sin(p+2)x + \sin px]$$

$$= (p+2) \sin(p+1)x - (p+1) \sin(p+2)x$$

對 $n = p+1$ ，命題為真。

藉數學歸納法， $\forall n \in \mathbb{Z}^+$ ，命題為真。

1

(b) 代 $x = \frac{\pi}{2}$ 及 $n = 2018$ 。

$$2 \left(1 - \cos \frac{\pi}{2}\right) \sum_{k=1}^{2018} k \sin \frac{k\pi}{2} = (2018+1) \sin \frac{2018\pi}{2} - 2018 \sin \frac{2019\pi}{2}$$

1M

$$2 \sum_{k=1}^{2018} k \sin \frac{k\pi}{2} = 2018$$

$$4 \sum_{k=1}^{2018} k \sin \frac{k\pi}{4} \cos \frac{k\pi}{4} = 2018$$

1M

$$\sum_{k=1}^{2018} k \sin \frac{k\pi}{4} \cos \frac{k\pi}{4} = \frac{1009}{2}$$

1A

8. (a) 當 $n = 1$ 時，

$$\text{左式} = 2 \cos \theta [-\cos 2\theta + \cos 4\theta]$$

$$= -(\cos 3\theta \cos \theta) + (\cos 5\theta + \cos 3\theta)$$

$$= \cos 5\theta - \cos \theta$$

$$\text{右式} = \cos 5\theta - \cos \theta = \text{左式}$$

對 $n = 1$ ，命題為真。

1

假設 $2 \cos \theta \sum_{k=1}^{2p} (-1)^k \cos 2k\theta = \cos(4p+1)\theta - \cos \theta$ ，其中 $p \in \mathbb{Z}^+$ 。

1

$$2 \cos \theta \sum_{k=1}^{2p+2} (-1)^k \cos 2k\theta$$

$$= 2 \cos \theta \sum_{k=1}^{2p} (-1)^k \cos 2k\theta + 2 \cos \theta [(-1)^{2p+1} \cos(4p+2)\theta + (-1)^{2p+2} \cos(4p+4)\theta]$$

$$= \cos(4p+1)\theta - \cos \theta + 2 \cos \theta [-\cos(4p+2)\theta + \cos(4p+4)\theta] \quad 1M$$

$$= \cos(4p+1)\theta - \cos \theta - [\cos(4p+3)\theta + \cos(4p+1)\theta] + [\cos(4p+5)\theta + \cos(4p+3)\theta] \quad 1M$$

$$= \cos(4p+5)\theta - \cos \theta$$

對 $n = p+1$ ，命題為真。

藉數學歸納法， $\forall n \in \mathbb{Z}^+$ ，命題為真。

1

(b) 代 $\theta = \frac{\pi}{14}$ 及 $n = 64$ 。

$$2 \cos \frac{\pi}{14} \sum_{k=1}^{128} (-1)^k \cos \frac{2k\pi}{14} = \cos \frac{257\pi}{14} - \cos \frac{\pi}{14} \quad 1M$$

$$2 \cos \frac{\pi}{14} \sum_{k=1}^{128} (-1)^k \left(2 \cos^2 \frac{k\pi}{14} - 1 \right) = \cos \left(18\pi + \frac{\pi}{2} - \frac{\pi}{7} \right) - \cos \frac{\pi}{14} \quad 1M$$

$$2 \cos \frac{\pi}{14} \left[2 \sum_{k=1}^{128} (-1)^k \cos^2 \frac{k\pi}{14} - \sum_{k=1}^{128} (-1)^k \right] = 2 \sin \frac{\pi}{14} \cos \frac{\pi}{14} - \cos \frac{\pi}{14}$$

$$4 \sum_{k=1}^{128} (-1)^k \cos^2 \frac{k\pi}{14} - 0 = 2 \sin \frac{\pi}{14} - 1$$

$$\sum_{k=1}^{128} (-1)^k \cos^2 \frac{k\pi}{14} = \frac{1}{2} \sin \frac{\pi}{14} - \frac{1}{4}$$

$$\text{因此，} a = \frac{1}{2}、b = \frac{1}{14} \text{ 及 } c = -\frac{1}{4}。 \quad 1A$$

$$9. \sin(7x-1) - 3 \cos 3x + \sin(x-1) = 0$$

$$2 \sin(4x-1) \cos 3x - 3 \cos 3x = 0 \quad 1M$$

$$\cos 3x [2 \sin(4x-1) - 3] = 0$$

$$\cos 3x = 0 \quad \text{或} \quad \sin(4x-1) = \frac{3}{2} \quad (\text{捨去}) \quad 1M$$

$$3x = \frac{\pi}{2} \quad 1A$$

$$x = \frac{\pi}{6} \quad 1A$$