M2INT-INT-2324-ASM-SET 4-MATH
Suggested solutions

1. (a) Let $u = a - x$. Then $du = -dx$. 1M

$$\int_0^a f(a-x)\,dx = -\int_a^0 f(u)\,du$$
$$= \int_0^a f(u)\,du$$
$$= \int_0^a f(x)\,dx \quad 1$$

(b) $$\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx = \int_0^{\frac{\pi}{4}} \ln\left[1+\tan\left(\frac{\pi}{4}-x\right)\right]dx \quad 1M$$
$$= \int_0^{\frac{\pi}{4}} \ln\left[1+\frac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}\tan x}\right]dx \quad 1M$$
$$= \int_0^{\frac{\pi}{4}} \ln\left[\frac{(1+\tan x)+(1-\tan x)}{1+\tan x}\right]dx$$
$$= \int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan x}\right)dx \quad 1$$

(c) Using the result of (b), we have

$$\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx = \int_0^{\frac{\pi}{4}} \ln 2\,dx - \int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx \quad 1M$$
$$2\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx = \ln 2\Big[x\Big]_0^{\frac{\pi}{4}}$$
$$\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx = \frac{\pi\ln 2}{8} \quad 1A$$

Note that $\dfrac{d}{dx}[\ln(1+\tan x)] = \dfrac{\sec^2 x}{1+\tan x}$.

$$\int_0^{\frac{\pi}{4}} \frac{x\sec^2 x}{1+\tan x}\,dx = \Big[x\ln(1+\tan x)\Big]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx \quad 1M$$
$$= \frac{\pi}{4}\ln 2 - \frac{\pi\ln 2}{8}$$
$$= \frac{\pi\ln 2}{8} \quad 1A$$

(d) $\displaystyle\int_0^{\frac{\pi}{4}} \frac{x\tan x(\tan x-1)}{\tan x+1}\,\mathrm{d}x$

$\displaystyle= \int_0^{\frac{\pi}{4}} \frac{x\tan^2 x - x\tan x}{\tan x+1}\,\mathrm{d}x$

$\displaystyle= \int_0^{\frac{\pi}{4}} \frac{x(\sec^2 x-1) - x\tan x}{\tan x+1}\,\mathrm{d}x$ 1M

$\displaystyle= \int_0^{\frac{\pi}{4}} \frac{x\sec^2 x}{1+\tan x}\,\mathrm{d}x - \int_0^{\frac{\pi}{4}} x\,\mathrm{d}x$ 1M

$\displaystyle= \frac{\pi\ln 2}{8} - \left[\frac{x^2}{2}\right]_0^{\frac{\pi}{4}}$

$\displaystyle= \frac{\pi\ln 2}{8} - \frac{\pi^2}{32}$ 1A

2. (a) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[x^{n+\frac{1}{2}}(4-x)^{\frac{3}{2}}\right] = \left(n+\dfrac{1}{2}\right)x^{n-\frac{1}{2}}(4-x)^{\frac{3}{2}} + x^{n+\frac{1}{2}}\cdot\dfrac{3}{2}(4-x)^{\frac{1}{2}}(-1)$ 1M

$= x^{n-\frac{1}{2}}(4-x)^{\frac{1}{2}}\left[\left(n+\dfrac{1}{2}\right)(4-x)-\dfrac{3}{2}x\right]$ 1M

$= x^{n-\frac{1}{2}}(4-x)^{\frac{1}{2}}(-nx-2x+4n+2)$ 1

(b) By (a),

$$\left[x^{n+\frac{1}{2}}(4-x)^{\frac{3}{2}}\right]_0^2 = \int_0^2 x^{n-\frac{1}{2}}(4-x)^{\frac{1}{2}}(-nx-2x+4n+2)\,\mathrm{d}x \quad \text{1M}$$

$$2^{n+\frac{1}{2}}\cdot 2^{\frac{3}{2}} = \int_0^2 x^{n-1}\sqrt{4x-x^2}[-(n+2)x+(4n+2)]\,\mathrm{d}x$$

$$2^{n+2} = -(n+2)\int_0^2 x^n\sqrt{4x-x^2}\,\mathrm{d}x + (4n+2)\int_0^2 x^{n-1}\sqrt{4x-x^2}\,\mathrm{d}x \quad \text{1M}$$

$$2^{n+2} = -(n+2)I_n + 2(2n+1)I_{n-1}$$

$$I_n = \frac{2(2n+1)}{n+2}I_{n-1} - \frac{2^{n+2}}{n+2} \quad \text{1}$$

(c) $I_0 = \displaystyle\int_0^2 \sqrt{4x-x^2}\,\mathrm{d}x = \int_0^2 \sqrt{4-(x-2)^2}\,\mathrm{d}x$ 1M

Let $x-2 = 2\sin\theta$. Then $\mathrm{d}x = 2\cos\theta\,\mathrm{d}\theta$. 1M

When $x=0$, $\theta = -\dfrac{\pi}{2}$; when $x=2$, $\theta = 0$.

$$I_0 = \int_{-\frac{\pi}{2}}^0 \sqrt{4-4\sin^2\theta}\cdot 2\cos\theta\,\mathrm{d}\theta \quad \text{1M}$$

$$= 4\int_{-\frac{\pi}{2}}^0 \cos^2\theta\,\mathrm{d}\theta$$

$$= 2\int_{-\frac{\pi}{2}}^0 (1+\cos 2\theta)\,\mathrm{d}\theta \quad \text{1M}$$

$$= 2\left[\theta + \frac{\sin 2\theta}{2}\right]_{-\frac{\pi}{2}}^0 \quad \text{1A}$$

$$= \pi \quad \text{1A}$$

(d) $I_2 = \dfrac{2(4+1)}{2+2}I_1 - \dfrac{2^2+2}{2+2}$ 1M

$= \dfrac{5}{2}\left[\dfrac{2(2+1)}{1+2}I_0 - \dfrac{2^{1+2}}{1+2}\right] - \dfrac{3}{2}$ 1M

$= 5\pi - \dfrac{32}{3}$ 1A

3. (a) $\dfrac{d}{dx}\left(1-x^2\right)^{\frac{3}{2}} = \dfrac{3}{2}(1-x^2)^{\frac{1}{2}}(-2x)$ 1M

$= -3x(1-x^2)^{\frac{1}{2}}$ 1A

(b) $I_{2n} = \left[x^{2n-1} \times \dfrac{(1-x^2)^{\frac{3}{2}}}{-3}\right]_0^1 - \displaystyle\int_0^1 (2n-1)x^{2n-2}\left(\dfrac{(1-x^2)^{\frac{3}{2}}}{-3}\right) dx$ 1M+1M

$= 0 + \dfrac{2n-1}{3}\displaystyle\int_0^1 x^{2n-2}(1-x^2)^{\frac{1}{2}}(1-x^2)\, dx$ 1M

$= \dfrac{2n-1}{3}\displaystyle\int_0^1 x^{2n-2}(1-x^2)^{\frac{1}{2}}\, dx - \dfrac{2n-1}{3}\displaystyle\int_0^1 x^{2n}(1-x^2)^{\frac{1}{2}}\, dx$ 1M

$= \dfrac{2n-1}{3}I_{2n-2} - \dfrac{2n-1}{3}I_{2n}$

$\dfrac{2n+2}{3}I_{2n} = \dfrac{2n-1}{3}I_{2n-2}$ 1M

$I_{2n} = \dfrac{2n-1}{2n+2}I_{2n-2}$ 1

(c) Let $x = \sin\theta$. Then $dx = \cos\theta\, d\theta$. 1M

When $x = 0$, $\theta = 0$; when $x = 1$, $\theta = \dfrac{\pi}{2}$.

$I_0 = \displaystyle\int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2\theta}\cos\theta\, d\theta$ 1M

$= \dfrac{1}{2}\displaystyle\int_0^{\frac{\pi}{2}} (1+\cos 2\theta)\, d\theta$ 1M

$= \dfrac{1}{2}\left[\theta + \dfrac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}}$ 1A

$= \dfrac{\pi}{4}$ 1A

(d) When $n = 1$,

L.H.S. $= I_2 = \dfrac{2-1}{2+2}I_0 = \dfrac{1}{4} \times \dfrac{\pi}{4} = \dfrac{\pi}{16}$ 1M

R.H.S. $= \dfrac{\pi}{2^{2+1}(2)} = \dfrac{\pi}{16} =$ L.H.S.

It is true for $n = 1$. 1

Assume $I_{2k} = \dfrac{(2k-1)!\pi}{2^{2k-1}(k-1)!(k+1)!}$, where $k \in \mathbb{Z}^+$. 1

$I_{2k+2} = \dfrac{2k+2-1}{2k+2+2}I_{2k}$

$= \dfrac{2k+1}{2k+4} \times \dfrac{(2k-1)!\pi}{2^{2k-1}(k-1)!(k+1)!}$ 1

$= \dfrac{\frac{(2k+1)!}{2k}\pi}{2^{2k}(k+2)(k-1)!(k+1)!}$

$= \dfrac{(2k+1)!\pi}{2^{2k+1}k!(k+2)!}$

= R.H.S. 1

It is true for $n = k+1$.

By M.I., it is true $\forall n \in \mathbb{Z}^+$. 1

(e) $I_{10} = \dfrac{9!\pi}{2^{10+1}(4!)(6!)}$ 1M

$= \dfrac{21\pi}{2048}$ 1A

4. (a) Let $u = a - x$. Then $\mathrm{d}u = -\,\mathrm{d}x$. 1M

When $x = 0$, $u = a$; when $x = a$, $u = 0$.

$$\int_0^a f(x)\,\mathrm{d}x = -\int_a^0 f(a-u)\,\mathrm{d}u$$

$$= \int_0^a f(a-u)\,\mathrm{d}u \qquad \text{1M}$$

$$= \int_0^a f(a-x)\,\mathrm{d}x \qquad \text{1}$$

(b) $\displaystyle\int_{-a}^a f(x)\,\mathrm{d}x = \int_{-a}^0 f(x)\,\mathrm{d}x + \int_0^a f(x)\,\mathrm{d}x.$ 1M

Let $u = -x$. Then $\mathrm{d}u = -\,\mathrm{d}x$.

When $x = -a$, $u = a$; when $x = 0$, $u = 0$.

$$\int_{-a}^a f(x)\,\mathrm{d}x = -\int_a^0 f(-u)\,\mathrm{d}u + \int_0^a f(x)\,\mathrm{d}x$$

$$= \int_0^a f(u)\,\mathrm{d}u + \int_0^a f(x)\,\mathrm{d}x \qquad \text{1M}$$

$$= 2\int_0^a f(x)\,\mathrm{d}x \qquad \text{1}$$

(c) $$\int_0^\pi \sin^n x\,\mathrm{d}x = \Big[\sin^{n-1} x(-\cos x)\Big]_0^\pi + \int_0^\pi (n-1)\sin^{n-2} x\cos^2 x\,\mathrm{d}x \qquad \text{1M+1M}$$

$$= 0 + (n-1)\int_0^\pi \sin^{n-2} x(1-\sin^2 x)\,\mathrm{d}x$$

$$n\int_0^\pi \sin^n x\,\mathrm{d}x = (n-1)\int_0^\pi \sin^{n-2} x\,\mathrm{d}x \qquad \text{1M}$$

$$\int_0^\pi \sin^n x\,\mathrm{d}x = \frac{n-1}{n}\int_0^\pi \sin^{n-2} x\,\mathrm{d}x \qquad \text{1}$$

(d) $(-x)\sin^{2019}(-x) = x\sin^{2019} x$ for all $x \in \mathbb{R}$. So, $x\sin^{2019} x$ is an even function.

By (b),

$$\int_{-\pi}^\pi x\sin^{2019} x\,\mathrm{d}x = 2\int_0^\pi x\sin^{2019} x\,\mathrm{d}x \qquad \text{1M}$$

$$= 2\int_0^\pi (\pi - x)\sin^{2019}(\pi - x)\,\mathrm{d}x \qquad \text{1M}$$

Therefore,

$$2\int_0^\pi x\sin^{2019} x\,\mathrm{d}x = 2\pi\int_0^\pi \sin^{2019} x\,\mathrm{d}x - 2\int_0^\pi x\sin^{2019} x\,\mathrm{d}x$$

$$2\int_0^\pi x\sin^{2019} x\,\mathrm{d}x = \pi\int_0^\pi \sin^{2019} x\,\mathrm{d}x \qquad \text{1M}$$

Therefore,

$$\begin{aligned}\int_{-\pi}^{\pi} x\sin^{2019} x\,\mathrm{d}x &= \pi\int_0^{\pi}\sin^{2019} x\,\mathrm{d}x \\ &= \pi\times\frac{2018}{2019}\int_0^{\pi}\sin^{2017} x\,\mathrm{d}x && \text{1M} \\ &= \pi\times\frac{2018\times 2016\times\ldots\times 2}{2019\times 2017\times\ldots\times 3}\int_0^{\pi}\sin x\,\mathrm{d}x && \text{1M} \\ &= \pi\times\frac{[2^{1009}(1009!)]^2}{2019!}\Big[-\cos x\Big]_0^{\pi} \\ &= \frac{2^{2019}(1009!)^2\pi}{2019!} && \text{1A}\end{aligned}$$

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