M2INT-INT-2324-ASM-SET 2-MATH
Suggested solutions

1. $$\int (1+x)\ln x\,\mathrm{d}x = \left(x+\frac{x^2}{2}\right)\ln x - \int \left(x+\frac{x^2}{2}\right)\times\frac{1}{x}\,\mathrm{d}x \qquad \text{1M+1A}$$
$$= \left(x+\frac{x^2}{2}\right)\ln x - \int \left(1+\frac{x}{2}\right)\mathrm{d}x$$
$$= \left(x+\frac{x^2}{2}\right)\ln x - x - \frac{x^2}{4} + \text{constant} \qquad \text{1A}$$

2. Let $u = 2-5e^x$. Then $\mathrm{d}u = -5e^x\,\mathrm{d}x$. 1M
$$\int e^x \ln(2-5e^x)\,\mathrm{d}x = -\frac{1}{5}\int \ln u\,\mathrm{d}u \qquad \text{1M}$$
$$= -\frac{u\ln u}{5} + \frac{1}{5}\int u\cdot\frac{1}{u}\,\mathrm{d}u \qquad \text{1M+1M}$$
$$= -\frac{u\ln u}{5} + \frac{u}{5} + C$$
$$= -\frac{(2-5e^x)\ln(2-5e^x)}{5} + \frac{2-5e^x}{5} + C \qquad \text{1A}$$

3. Let $u = \sin x$. Then $\mathrm{d}u = \cos x\,\mathrm{d}x$. 1M
$$\int e^{\sin x}\cos^3 x\,\mathrm{d}x = \int e^u(1-u^2)\,\mathrm{d}u \qquad \text{1M}$$
$$= e^u - \int u^2 e^u\,\mathrm{d}u$$
$$= e^u - \left[u^2e^u - \int (2u)e^u\,\mathrm{d}u\right] \qquad \text{1M+1M}$$
$$= e^u - u^2e^u + 2\left[ue^u - \int e^u\,\mathrm{d}u\right] \qquad \text{1M}$$
$$= e^u - u^2e^u + 2ue^u - 2e^u + C$$
$$= -\sin^2 x e^{\sin x} + 2\sin x e^{\sin x} - e^{\sin x} + C \qquad \text{1A}$$

4. $$\int e^x \sin 2x\,\mathrm{d}x = e^x\sin 2x - \int e^x(2\cos 2x)\,\mathrm{d}x \qquad \text{1M+1M}$$
$$= e^x\sin 2x - 2\left[e^x\cos 2x - \int e^x(-2\sin 2x)\,\mathrm{d}x\right] \qquad \text{1M}$$
$$5\int e^x\sin 2x\,\mathrm{d}x = e^x\sin 2x - 2e^x\cos 2x + C \qquad \text{1M}$$
$$\int e^x\sin 2x\,\mathrm{d}x = \frac{e^x\sin 2x}{5} - \frac{2e^x\cos 2x}{5} + C \qquad \text{1A}$$

5. (a) Let $x = \sin\theta$. Then $\mathrm{d}x = \cos\theta\,\mathrm{d}\theta$. 1M

$$\int \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \int \frac{\cos\theta\,\mathrm{d}\theta}{\sqrt{1-\sin^2\theta}} \quad \text{1M}$$
$$= \int \mathrm{d}\theta$$
$$= \theta + \text{constant}$$
$$= \sin^{-1} x + \text{constant} \quad \text{1A}$$

(b) From (a), $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\sin^{-1} x\right) = \dfrac{1}{\sqrt{1-x^2}}$

$$\int \sin^{-1} x\,\mathrm{d}x = x\sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x \quad \text{1M}$$

Let $u = 1 - x^2$. Then $\mathrm{d}u = -2x\,\mathrm{d}x$. 1M

$$\int \sin^{-1} x\,\mathrm{d}x = x\sin^{-1} x + \frac{1}{2}\int \frac{\mathrm{d}u}{\sqrt{u}} \quad \text{1M}$$
$$= x\sin^{-1} x + \sqrt{1-x^2} + \text{constant} \quad \text{1A}$$

6. $$\int \cos(\ln x)\,\mathrm{d}x = x\cos(\ln x) - \int x\left[-\sin(\ln x)\cdot\frac{1}{x}\right]\mathrm{d}x \quad \text{1M+1M}$$
$$= x\cos(\ln x) + \left[x\sin(\ln x) - \int x\left(\cos(\ln x)\cdot\frac{1}{x}\right)\mathrm{d}x\right] \quad \text{1M}$$
$$2\int \cos(\ln x)\,\mathrm{d}x = x\cos(\ln x) + x\sin(\ln x) + \text{constant} \quad \text{1M}$$
$$\int \cos(\ln x)\,\mathrm{d}x = \frac{x\cos(\ln x) + x\sin(\ln x)}{2} + \text{constant} \quad \text{1A}$$

7. $$\int_0^{\frac{\pi}{6}} x^2\cos 3x\,\mathrm{d}x = \left[x^2\left(\frac{\sin 3x}{3}\right)\right]_0^{\frac{\pi}{6}} - \int_0^{\frac{\pi}{6}} (2x)\left(\frac{\sin 3x}{3}\right)\mathrm{d}x \quad \text{1M+1M}$$
$$= \frac{\pi^2}{108} - 0 - \frac{2}{3}\left[\left[x\left(-\frac{\cos 3x}{3}\right)\right]_0^{\frac{\pi}{6}} - \int_0^{\frac{\pi}{6}}\left(-\frac{\cos 3x}{3}\right)\mathrm{d}x\right] \quad \text{1M}$$
$$= \frac{\pi^2}{108} + \frac{2}{9}(0-0) - \frac{2}{9}\left[\frac{\sin 3x}{3}\right]_0^{\frac{\pi}{6}} \quad \text{1A}$$
$$= \frac{\pi^2}{108} - \frac{2}{27} \quad \text{1A}$$

8. (a) $\displaystyle\int_0^{\frac{\pi}{2}} e^{\frac{x}{2}} \sin x \, dx = \left[2e^{\frac{x}{2}} \sin x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (2e^{\frac{x}{2}})(\cos x) \, dx$ 1M+1M

$\displaystyle = 2e^{\frac{\pi}{4}} - 2\left[\left[2e^{\frac{x}{2}} \cos x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (2e^{\frac{x}{2}})(-\sin x) \, dx\right]$ 1M

$\displaystyle = 2e^{\frac{\pi}{4}} - 4(0-1) - 4\int_0^{\frac{\pi}{2}} e^{\frac{x}{2}} \sin x \, dx$

$\displaystyle 5\int_0^{\frac{\pi}{2}} e^{\frac{x}{2}} \sin x \, dx = 2e^{\frac{\pi}{4}} + 4$ 1M

$\displaystyle \int_0^{\frac{\pi}{2}} e^{\frac{x}{2}} \sin x \, dx = \frac{2}{5}(e^{\frac{\pi}{4}} + 2)$ 1A

(b) $\displaystyle\int_0^{\frac{\pi}{4}} e^x (\sin x + \cos x)^2 \, dx = \int_0^{\frac{\pi}{4}} e^x (1 + \sin 2x) \, dx$

Let $u = 2x$. Then $du = 2\,dx$.

When $x = 0$, $u = 0$; when $x = \frac{\pi}{4}$, $u = \frac{\pi}{2}$.

$\displaystyle \int_0^{\frac{\pi}{4}} e^x (\sin x + \cos x)^2 \, dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} e^{\frac{u}{2}} (1 + \sin u) \, du$ 1M

$\displaystyle = \left[e^{\frac{u}{2}}\right]_0^{\frac{\pi}{2}} + \frac{1}{5}(e^{\frac{\pi}{4}} + 2)$ 1M

$\displaystyle = e^{\frac{\pi}{4}} - 1 + \frac{1}{5}(e^{\frac{\pi}{4}} + 2)$

$\displaystyle = \frac{1}{5}(6e^{\frac{\pi}{4}} - 3)$ 1A

9. (a) $\displaystyle\int_0^{\frac{\pi}{8}} x \cos 2x \, dx = \left[x \times \frac{\sin 2x}{2}\right]_0^{\frac{\pi}{8}} - \int_0^{\frac{\pi}{8}} \frac{\sin 2x}{2} \, dx$ 1M

$\displaystyle = \left(\frac{\pi}{8} \times \frac{\sqrt{2}}{4} - 0\right) + \left[\frac{\cos 2x}{4}\right]_0^{\frac{\pi}{8}}$ 1A

$\displaystyle = \frac{\pi\sqrt{2}}{32} + \frac{\sqrt{2}}{8} - \frac{1}{4}$ 1A

(b) $\displaystyle\int_0^{\frac{\pi}{8}} x^2 \sin 2x \, dx = \left[x^2\left(-\frac{\cos 2x}{2}\right)\right]_0^{\frac{\pi}{8}} - \int_0^{\frac{\pi}{8}} (2x)\left(-\frac{\cos 2x}{2}\right) dx$ 1M+1M

$\displaystyle = \left[\frac{\pi^2}{64} \times \left(-\frac{\sqrt{2}}{4}\right) - 0\right] + \int_0^{\frac{\pi}{8}} x \cos 2x \, dx$

$\displaystyle = -\frac{\pi^2\sqrt{2}}{256} + \frac{\pi\sqrt{2}}{32} + \frac{\sqrt{2}}{8} - \frac{1}{4}$ 1A

10. $$\int_0^3 \left(\frac{x}{e^{2x}}\right)^2 \mathrm{d}x = \int_0^3 x^2 e^{-4x}\,\mathrm{d}x$$

$$= -\frac{1}{4}\left[x^2 e^{-4x}\right]_0^3 + \frac{1}{2}\int_0^3 x e^{-4x}\,\mathrm{d}x \qquad \text{1M+1A}$$

$$= -\frac{9e^{-12}}{4} - \frac{1}{8}\left[x e^{-4x}\right]_0^3 + \frac{1}{8}\int_0^3 e^{-4x}\,\mathrm{d}x$$

$$= -\frac{9e^{-12}}{4} - \frac{3e^{-12}}{8} - \frac{1}{32}\left[e^{-4x}\right]_0^3$$

$$= -\frac{85}{32e^{12}} + \frac{1}{32} \qquad \text{1A}$$

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