REG-AOT-2324-ASM-SET 5-MATH
Suggested solutions

Conventional Questions

1. (a) Area of $\triangle ACD = \frac{1}{2}(20)(24)\sin 60°$ 1M

$= 120\sqrt{3}\text{ cm}^2$ 1A

(b) Let M be the foot of perpendicular from A to CD.

Required angle is $\angle AMB$. 1A

In $\triangle ACD$,

$$CD^2 = 20^2 + 24^2 - 2(20)(24)\cos 60° \quad \text{1M}$$

$$CD \approx 22.3\text{ cm}$$

$$20^2 = CD^2 + 24^2 - 2(CD)(24)\cos\angle ADC$$

$$\angle ADC \approx 51.1°$$

In $\triangle ADM$, $AM = AD\sin\angle ADC \approx 18.7\text{ cm}$ 1M

In $\triangle ABM$,

$$\sin\angle AMB = \frac{18}{AM}$$

$$\angle AMB \approx 74.7° \quad \text{1A}$$

Required angle is $74.7°$.

(c) AB is the shortest distance from A to the plane BCD.

Let h be the shortest distance from B to the plane ACD.

By considering volume of the tetrahedron,

$$\frac{1}{3}(\text{area of }\triangle ACD)(h) = \frac{1}{3}(\text{area of }\triangle BCD)(AB)$$

$$\frac{1}{3}\times\frac{1}{2}(CD)(AM)(h) = \frac{1}{3}\times\frac{1}{2}(CD)(BM)(AB)$$

$$h = \frac{BM}{AM}(AB) \quad \text{1M}$$

Since $AM > BM$, $h < AB$.

The claim is agreed. 1A

2. (a) $\dfrac{18}{\sin\angle ABD} = \dfrac{22}{\sin 80°}$ 1M

$\angle ABD \approx 53.7°$ or $126°$ (rejected) 1A

(b) (i) $25^2 = 22^2 + 18^2 - 2(22)(18)\cos\angle ADC$ 1M

$\angle ADC \approx 76.6°$ 1A

(ii) Let E be a point on BD such that $AE \perp BD$.

Let F be a point on CD such that $EF \perp BD$.

Required angle is $\angle AEF$. 1M

$AE = 18\sin 80° \approx 17.7\text{ cm}$ 1M

$DE = 18\cos 80° \approx 3.13\text{ cm}$

$EF = DE\tan\angle CDB = DE\tan\angle ABD \approx 4.25\text{ cm}$

$DF = \sqrt{DE^2 + EF^2} \approx 5.28\text{ cm}$

Consider $\triangle ACD$.

$$AF^2 = 18^2 + DF^2 - 2(18)(DF)\cos\angle ADC$$

$$AF \approx 17.5\text{ cm}$$

Consider $\triangle AEF$.

$$AF^2 = AE^2 + EF^2 - 2(AE)(EF)\cos\angle AEF$$

$$\angle AEF \approx 80.7°$$

$$> 80°$$

The claim is agreed. 1A

3. (a) Let $OP = h$ m.

Then $AO = \dfrac{h}{\tan 16.5°}$, $BO = \dfrac{h}{\tan 20°}$ and $CO = \dfrac{h}{\tan 18°}$.

In $\triangle ABO$,

$$AO^2 = BO^2 + 340^2 - 2(BO)(340)\cos\angle ABO \qquad \text{1M}$$

$$\cos\angle ABO = \frac{340^2 + BO^2 - AO^2}{2(BO)(340)}$$

In $\triangle BOC$,

$$CO^2 = BO^2 + 430^2 - 2(BO)(430)\cos\angle OBC$$

$$\cos\angle OBC = \frac{BO^2 + 430^2 - CO^2}{2(BO)(430)}$$

Since $\angle ABO = 180° - \angle OBC$,

$$\frac{340^2 + BO^2 - AO^2}{2(BO)(340)} = -\frac{BO^2 + 430^2 - CO^2}{2(BO)(430)} \qquad \text{1M}$$

$$43\left[(340^2) + \left(\frac{h}{\tan 20°}\right)^2 - \left(\frac{h}{\tan 16.5°}\right)^2\right] = -34\left[\left(\frac{h}{\tan 20°}\right)^2 + (430)^2 - \left(\frac{h}{\tan 18°}\right)^2\right]$$

$$h^2\left[\frac{43+34}{\tan^2 20°} - \frac{43}{\tan^2 16.5°} - \frac{34}{\tan^2 18°}\right] = -34(430)^2 - 43(340)^2 \qquad \text{1M}$$

$$h \approx 221 \qquad \text{1A}$$

(b) Let H be a point on AC such that $PH \perp AC$ and $OH \perp AC$. Required angle is $\angle PHO$. 1A

Using the value of h, $AO \approx 745$ m, $BO \approx 607$ m and $CO \approx 680$ m.

$$AO^2 = CO^2 + (340+430)^2 - 2(CO)(340+430)\cos\angle OCA \qquad \text{1M}$$

$$\angle OCA \approx 61.5°$$

$OH = CO \sin\angle OCA \approx 597$ m 1A

$\angle PHO = \tan^{-1}\dfrac{h}{OH} \approx 20.3°$ 1A

4. (a) $MS = 2\sin 60° = \sqrt{3}$ cm

$PS = \sqrt{2^2 + 2^2} = 2\sqrt{2}$ cm

$\sin \angle MPS = \dfrac{\sqrt{3}}{2\sqrt{2}}$ 1M

$\angle MPS \approx 37.8°$ 1A

(b) Let A be a point on PS such that $MA \perp PS$.

Let B be a point on ST such that $AB \perp PS$.

Required angle is $\angle MAB$. 1A

$MA = MP \sin \angle MPS \approx 1.37$ cm 1M

$PA = MP \cos \angle MPS \approx 1.77$ cm

$AS = PS - PA \approx 1.06$ cm

$AB = AS \tan 45° \approx 1.06$ cm

$BS = \dfrac{AS}{\cos 45°} = 1.5$ cm

$BM^2 = BS^2 + MS^2 - 2(BS)(MS)\cos 30°$ 1M

$BM \approx 0.866$ cm

$BM^2 = AB^2 + MA^2 - 2(AB)(MA)\cos \angle MAB$

$\angle MAB \approx 39.2°$ 1A

Required angle is 39.2°.

(c) Let E be a point on WS such that $DE \perp WS$.

Let F be a point on RQ such that $DF \perp RQ$.

Area of $\triangle DSW = \dfrac{(WS)(DE)}{2} = \dfrac{WS\sqrt{DF^2 + EF^2}}{2}$ 1M

Since WS and EF are constant, area of $\triangle DSW$ is minimum when DF is minimum, i.e., $DF = 0$.

When D is at P, $DF \neq 0$. The area is therefore not minimum when D is at P. 1A

5. (a) $AC = \sqrt{78^2 + 104^2}$

$= 130\text{ cm}$

$AB = \sqrt{130^2 - 120^2}$ 1M

$= 50\text{ cm}$ 1A

$\angle BAD = \tan^{-1}\dfrac{120}{50} + \tan^{-1}\dfrac{104}{78}$

$\approx 121°$

$BD^2 = 50^2 + 78^2 - 2(50)(78)\cos\angle BAD$ 1M

$BD = 112\text{ cm}$ 1A

(b) (i) Note that $GA = GB = GC = GD = \dfrac{AC}{2} = 65\text{ cm}$.

$VG = \sqrt{169^2 - 65^2}$

$= 156\text{ cm}$ 1A

(ii) Let H be the projection of G on the plane VBD.

Let h cm be the required distance.

Consider the area of $\triangle BDG$.

Let $s = \dfrac{65 + 65 + 112}{2} = 121\text{ cm}$.

Area $= \sqrt{121(121 - 65)^2(121 - 112)}$ 1M

$= 1848\text{ cm}^2$

Consider the area of $\triangle VBD$.

Let $s = \dfrac{169 + 169 + 112}{2} = 225\text{ cm}$.

Area $= \sqrt{225(225 - 169)^2(225 - 112)}$

$= 840\sqrt{113}\text{ cm}$

Consider the volume of the tetrahedron $VBGD$.

$\dfrac{1}{3}(1848)(156) = \dfrac{1}{3}(840\sqrt{113})(h)$ 1M

$h \approx 32.3$ 1A

Required distance is 32.3 cm.

6. (a) $\angle BAC = 180° - 56° - 82° = 42°$

$$\frac{BC}{\sin 42°} = \frac{20}{\sin 56°}$$ 1M

$BC \approx 16.1$ cm 1A

(b) (i) $10^2 = BC^2 + 20^2 - 2(BC)(20)\cos\angle ABC$ 1M

$\angle ABC \approx 29.8°$ 1A

(ii) Let E be a point on BD such that $CE \perp BD$.

Let F be a point on AB such that $EF \perp BD$.

Required angle is $\angle CEF$. 1M

$BF = BC \approx 16.1$ cm

$CE = FE = BC\sin\frac{82°}{2} \approx 10.6$ cm

$CF^2 = BC^2 + BF^2 - 2(BC)(BF)\cos\angle ABC$ 1M

$CF \approx 8.29$ cm

$CF^2 = EF^2 + CE^2 - 2(EF)(CE)\cos\angle CEF$ 1M

$\angle CEF \approx 46.1° > 45°$

The claim is agreed. 1A

7. (a) $AD = \dfrac{\sqrt{30^2+40^2}}{2} = 25\,\text{cm}$

$\angle BAC = \cos^{-1}\dfrac{30}{50} = \cos^{-1}\dfrac{3}{5}$

$BD^2 = 25^2 + 30^2 - 2(25)(30)\cos\angle BAC$ 1M

$BD = 25\,\text{cm}$ 1A

$\angle ABD = \angle BAD = \cos^{-1}\dfrac{3}{5}$

$AF = 30\sin\angle ABD = 24\,\text{cm}$ 1A

$\cos\angle BAF = \dfrac{24}{30} = \dfrac{4}{5}$

$FE = \dfrac{30}{\cos\angle BAF} - 24 = 13.5\,\text{cm}$ 1A

(b) (i) Required angle is $\angle AFE$.

$\cos\angle AFE = \dfrac{13.5}{24}$ 1M

$\angle AFE \approx 55.8°$ 1A

(ii) Since $AF \perp BD$ and $FE \perp BD$,

the claim is agreed. 1A

(iii) Area of $\triangle BCD = \dfrac{\text{area of } \triangle ABC}{2}$

$= \dfrac{(30)(40)}{2\times 2}$

$= 300\,\text{cm}^2$

Required volume $= \dfrac{1}{3}(300)(24\sin\angle AFE)$ 1M

$\approx 1984\,\text{cm}^3$ 1A

(iv) Area of $\triangle ABD$ is equal to the area of $\triangle BCD$, i.e. $300\,\text{cm}^2$.

By considering the volume of the tetrahedron,

$\dfrac{1}{3}(300)(AB\sin\alpha) = \dfrac{1}{3}(300)(BC\sin\beta)$ 1M

$\dfrac{\sin\alpha}{\sin\beta} = \dfrac{40}{30} > 1$

So, $\sin\alpha > \sin\beta$ and therefore $\alpha > \beta$. 1A

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