M2REG-TRIG-2324-ASM-SET 6-MATH

建議題解

1. θ 在第二象限或第三象限。

當 θ 在第二象限，

$\csc\theta = \dfrac{23}{\sqrt{23^2 - 5^2}}$ 1M

$= \dfrac{23}{6\sqrt{14}}$

$\cot\theta = -\dfrac{5}{6\sqrt{14}}$

$\csc\theta - \cot\theta = \dfrac{23}{6\sqrt{14}} - \left(-\dfrac{5}{6\sqrt{14}}\right)$

$= \dfrac{\sqrt{14}}{3}$ 1A

當 θ 在第三象限，

$\csc\theta = -\dfrac{23}{\sqrt{23^2 - 5^2}}$

$= -\dfrac{23}{6\sqrt{14}}$

$\cot\theta = \dfrac{5}{6\sqrt{14}}$

$\csc\theta - \cot\theta = -\dfrac{23}{6\sqrt{14}} - \dfrac{5}{6\sqrt{14}}$

$= -\dfrac{\sqrt{14}}{3}$ 1A

因此，$\csc\theta - \cot\theta = -\dfrac{\sqrt{14}}{3}$ 或 $\dfrac{\sqrt{14}}{3}$。

2. $\dfrac{(1 - \cot\theta)(1 + \cot\theta)}{2}$

$= \dfrac{(1 - \cot^2\theta) - 2}{2}$ 1M

$= \dfrac{-(\cot^2\theta + 1)}{2}$

$= \dfrac{-\csc^2\theta}{2}$ 1M

$= -\dfrac{1}{2\sin^2\theta}$

$= -\dfrac{1}{2a^2}$ 1A

3. (a) $(\csc x + \cot x)(\csc x - \cot x) = \csc^2 x - \cot^2 x$

$$\frac{3}{2}(\csc x - \cot x) = 1$$ 1M

$$\csc x - \cot x = \frac{2}{3}$$ 1

(b) $(\csc x + \cot x) + (\csc x - \cot x) = \frac{3}{2} + \frac{2}{3}$ 1M

$$\csc x = \frac{13}{12}$$ 1A

$$\cot x = \frac{3}{2} - \frac{13}{12} = \frac{5}{12}$$ 1A

4. $\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$ 1M

$$\frac{5}{7} = \frac{\tan A + \tan B}{1 - \left(-\frac{1}{6}\right)}$$

$\tan A + \tan B = \dfrac{5}{6}$ 1M

考慮兩根之和及兩根之積，

$\tan A$ 及 $\tan B$ 為方程 $6x^2 - 5x - 1 = 0$ 的根。 1M+1

5. (a) $\tan\left(2 \cdot \dfrac{\pi}{8}\right) = \dfrac{2\tan\frac{\pi}{8}}{1 - \tan^2\frac{\pi}{8}}$ 1M

$$1 = \frac{2\tan\frac{\pi}{8}}{1 - \tan^2\frac{\pi}{8}}$$

$$1 - \tan^2\frac{\pi}{8} = 2\tan\frac{\pi}{8}$$

$$\tan^2\frac{\pi}{8} + 2\tan\frac{\pi}{8} - 1 = 0$$

因此，$\tan\dfrac{\pi}{8}$ 為方程 $x^2 + 2x - 1 = 0$。 1

(b) $x^2 + 2x - 1 = 0$

$$x = \frac{-2 \pm \sqrt{2^2 - 4(1)(-1)}}{2(1)}$$ 1M

$$= -1 \pm \sqrt{2}$$

留意 $\tan\dfrac{\pi}{8} > 0 > -1 - \sqrt{2}$。 1M

因此，$\tan\dfrac{\pi}{8} = \sqrt{2} - 1$。 1A

6. $\sec 4A = 4$

$\cos 4A = \frac{1}{4}$

$2\cos^2 2A - 1 = \frac{1}{4}$ 1M

$\cos^2 2A = \frac{5}{8}$

$\sin^2 2A = 1 - \frac{5}{8}$ 1M

$= \frac{3}{8}$

$\cos^4 A + \sin^4 A = (\cos^2 A + \sin^2 A)^2 - 2\sin^2 A \cos^2 A$ 1M

$= 1^2 - 2\left(\frac{\sin 2A}{2}\right)^2$

$= 1 - \frac{1}{2}\sin^2 2A$

$= 1 - \frac{1}{2}\left(\frac{3}{8}\right)$

$= \frac{13}{16}$ 1A

7. (a) 當 $n = 1$，

左式 $= \cos x$

右式 $= \frac{\sin 2x}{2\sin x} = \frac{2\sin x \cos x}{2\sin x} = \cos x =$ 左式

對 $n = 1$，命題為真。 1

假設 $\cos x \cos 2x \cos 4x \ldots \cos\left(2^{k-1}x\right) = \frac{\sin\left(2^k x\right)}{2^k \sin x}$，其中 $\sin x \neq 0$ 及 $k \in \mathbb{Z}^+$。 1M

$\cos x \cos 2x \cos 4x \ldots \cos\left(2^{k-1}x\right)\cos\left(2^k x\right)$

$= \frac{\sin\left(2^k x\right)}{2^k \sin x}\cos\left(2^k x\right)$ 1M

$= \frac{1}{2^k \sin x}\left(\frac{2\sin\left(2^k x\right)\cos\left(2^k x\right)}{2}\right)$

$= \frac{\sin\left(2^{k+1}x\right)}{2^{k+1}\sin x}$ 1

對 $n = k + 1$，命題為真。

藉數學歸納法，$\forall n \in \mathbb{Z}^+$，命題為真。 1

(b) $\cos\frac{\pi}{3}\cos\frac{2\pi}{3}\cos\frac{4\pi}{3}\ldots\cos\frac{256\pi}{3}$

$= \cos\frac{\pi}{3}\cos\frac{2\pi}{3}\cos\frac{4\pi}{3}\ldots\cos\frac{2^{9-1}\pi}{3}$

$= \frac{\sin\frac{2^9\pi}{3}}{2^9\sin\frac{\pi}{3}}$ 1M

$= \frac{\sin\left(170\pi + \frac{2\pi}{3}\right)}{512\sin\frac{\pi}{3}}$

$= \frac{\sin\frac{2\pi}{3}}{512\sin\frac{\pi}{3}}$

$= \frac{1}{512}$ 1A

8. $\sin 3x + 2\sin 5x + \sin 7x = (\sin 3x + \sin 7x) + 2\sin 5x$

$= 2\sin 5x \cos 2x + 2\sin 5x$ 1M

$= 2\sin 5x(\cos 2x + 1)$

$= 2\sin 5x[(2\cos^2 x - 1) + 1]$ 1M

$= 4\cos^2 x \sin 5x$ 1

9. $\cos 4\theta \cos 2\theta - \sin 8\theta \sin 6\theta$

$= \frac{1}{2}(\cos 6\theta + \cos 2\theta) - \frac{1}{2}(\cos 2\theta - \cos 14\theta)$ 1M

$= \frac{1}{2}(\cos 6\theta + \cos 14\theta)$

$= \frac{1}{2} \cdot 2\cos \frac{6\theta + 14\theta}{2} \cos \frac{14\theta - 6\theta}{2}$ 1M

$= \cos 10\theta \cos 4\theta$ 1

10. (a) $\cos 3\theta \cos^3 \theta + \sin 3\theta \sin^3 \theta$

$= \cos 3\theta \cos \theta \cos^2 \theta + \sin 3\theta \sin \theta \sin^2 \theta$

$= \frac{1}{2}(\cos 4\theta + \cos 2\theta) \cdot \frac{1}{2}(1 + \cos 2\theta) + \frac{1}{2}(\cos 2\theta - \cos 4\theta) \cdot \frac{1}{2}(1 - \cos 2\theta)$ 1M

$= \frac{1}{4}[(\cos 4\theta + \cos 2\theta + \cos 4\theta \cos 2\theta + \cos^2 2\theta) - (\cos 4\theta - \cos 2\theta - \cos 4\theta \cos 2\theta + \cos^2 2\theta)]$

$= \frac{\cos 2\theta}{2}(1 + \cos 4\theta)$

$= \frac{\cos 2\theta}{2}\left[1 + (2\cos^2 2\theta - 1)\right]$ 1M

$= \cos^3 2\theta$ 1

(b) $\cos^3 \frac{2\pi}{12} = \cos \frac{3\pi}{12} \cos^3 \frac{\pi}{12} + \sin \frac{3\pi}{12} \sin^3 \frac{\pi}{12}$ 1M

$\left(\frac{\sqrt{3}}{2}\right)^3 = \frac{1}{\sqrt{2}} \cos^3 \frac{\pi}{12} + \frac{1}{\sqrt{2}} \sin^3 \frac{\pi}{12}$

$\frac{3\sqrt{3}}{8} = \frac{1}{\sqrt{2}}\left(\cos^3 \frac{\pi}{12} + \sin^3 \frac{\pi}{12}\right)$

$\frac{3\sqrt{6}}{8} = \left(\cos \frac{\pi}{12} + \sin \frac{\pi}{12}\right)\left(\cos^2 \frac{\pi}{12} - \cos \frac{\pi}{12} \sin \frac{\pi}{12} + \sin^2 \frac{\pi}{12}\right)$ 1M

$\frac{3\sqrt{6}}{8} = \left(\cos \frac{\pi}{12} + \sin \frac{\pi}{12}\right)\left(1 - \frac{1}{2}\sin \frac{\pi}{6}\right)$

$\frac{3\sqrt{6}}{8} = \left(\cos \frac{\pi}{12} + \sin \frac{\pi}{12}\right)\left[1 - \frac{1}{2}\left(\frac{1}{2}\right)\right]$

$\cos \frac{\pi}{12} + \sin \frac{\pi}{12} = \frac{\sqrt{6}}{2}$ 1A

11. (a) $\sin\frac{n\pi}{24}\sin\frac{\pi}{24} = -\frac{1}{2}\left[\cos\left(\frac{n\pi}{24}+\frac{\pi}{24}\right)-\cos\left(\frac{n\pi}{24}-\frac{\pi}{24}\right)\right]$ 1M

$= -\frac{1}{2}\left[\cos\frac{(n+1)\pi}{24}-\cos\frac{(n-1)\pi}{24}\right]$ 1

(b) $\sum_{k=1}^{5}\sin\frac{2k\pi}{24}\sin\frac{\pi}{24} = \sum_{k=1}^{5}\left[-\frac{1}{2}\left[\cos\frac{(2k+1)\pi}{24}-\cos\frac{(2k-1)\pi}{24}\right]\right]$ 1M

$= -\frac{1}{2}\left[\sum_{k=1}^{5}\cos\frac{(2k+1)\pi}{24}-\sum_{k=1}^{5}\cos\frac{(2k-1)\pi}{24}\right]$

$= -\frac{1}{2}\left(\cos\frac{11\pi}{24}-\cos\frac{\pi}{24}\right)$

$= -\frac{1}{2}\left[\cos\left(\frac{\pi}{2}-\frac{\pi}{24}\right)-\cos\frac{\pi}{24}\right]$

$= -\frac{1}{2}\left(\sin\frac{\pi}{24}-\cos\frac{\pi}{24}\right)$ 1A

$\sum_{k=1}^{5}\sin\frac{k\pi}{12} = \frac{\sum_{k=1}^{5}\sin\frac{2k\pi}{24}\sin\frac{\pi}{24}}{\sin\frac{\pi}{24}}$ 1M

$= \frac{-\frac{1}{2}\left(\sin\frac{\pi}{24}-\cos\frac{\pi}{24}\right)}{\sin\frac{\pi}{24}}$

$= -\frac{1}{2}+\frac{1}{2}\cot\frac{\pi}{24}$

$= -\frac{1}{2}+\frac{1}{2}(2+\sqrt{2}+\sqrt{3}+\sqrt{6})$

$= \frac{1+\sqrt{2}+\sqrt{3}+\sqrt{6}}{2}$ 1A